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GLOBAL INTEGRABILITY FOR SOLUTIONS TO BOUNDARY VALUE PROBLEMS OF ANISOTROPIC FUNCTIONALS

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Abstract. This paper deals with solutions to boundary value problems of anisotropic integral functionals

$$\mathcal{I}(u)=\int_\Omega f(x,Du(x))dx,$$

with the energy $f(x,z)$ has growth p_i with respect to z_i, like in

$$\int_\Omega\left(\left(1+\sum_{j=1}^n|D_ju|^{p_j}\right)^{\frac{p_1-2}{p_1}}|D_1u|^2+\cdots+\left(1+\sum_{j=1}^n|D_ju|^{p_j}\right)^{\frac{p_n-2}{p_n}}|D_nu|^2\right)dx.$$

We show that higher integrability of the boundary datum u_* forces minimizers u to be more integrable. A similar result is obtained for obstacle problems.

1. Introduction and Statement of Main Result.

Let Ω be a bounded open subset of $\mathrm{R}^n, n\geq 2$. We consider anisotropic integral functionals

$$\mathcal{I}(u)=\int_\Omega f(x,Du(x))dx, \tag{1.1}$$

where $Du=(D_1u,\cdots,D_nu)=\left(\frac{\partial u}{\partial x_1},\cdots,\frac{\partial u}{\partial x_n}\right)$ is the gradient of u, and the energy $f(x,z)$ is supposed to be nonnegative.

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In a recent paper [7], Leonetti and Siepe considered functionals (1.1) with the energy density $f(x,z)$ that satisfies

$$\sum_{i=1}^{n} |z_i|^{p_i} \le f(x,z) \le c\left(1+\sum_{i=1}^{n}|z_i|^{p_i}\right), \tag{1.2}$$

where the component z_i of $z=(z_1,\cdots,z_n)$ has the exponent p_i that might be different from the exponent p_j of the component z_j, when $j \ne i$. This is suggested by the integral functional

$$\int_\Omega (|D_1u|^{p_1}+|D_2u|^{p_2}+\cdots+|D_nu|^{p_n})\,dx. \tag{1.3}$$

This anisotropic framework seems to be useful when dealing with some reinforced materials, see [10]. For some recent developments on anisotropic functionals and anisotropic elliptic equations, see [7, 1–6].

Another example in the anisotropic setting is given by

$$\int_\Omega \left(\left(1+\sum_{j=1}^{n}|D_ju|^{p_j}\right)^{\frac{p_1-2}{p_1}}|D_1u|^2+\cdots+\left(1+\sum_{j=1}^{n}|D_ju|^{p_j}\right)^{\frac{p_n-2}{p_n}}|D_nu|^2\right)dx. \tag{1.4}$$

Such an example suggests us to consider energies $f(x,z)$ where

$$\sum_{i=1}^{n}\left(1+\sum_{j=1}^{n}|z_j|^{p_j}\right)^{\frac{p_i-2}{p_i}}|z_i|^2 \le f(x,z) \le c\sum_{i=1}^{n}\left(1+\sum_{j=1}^{n}|z_j|^{p_j}\right)^{\frac{p_i-2}{p_i}}|z_i|^2. \tag{1.5}$$

The aim of the present paper is to consider boundary value problems of the integral functionals (1.1) with the energy $f(x,z)$ that satisfies (1.5). We will show that higher integrability of the boundary datum u_* forces minimizers u to have higher integrability as well. We should mention that the idea of the proof of the main theorem in this paper comes from [7, 1].

We now introduce some symbols and notations used in this paper.

Let $p_1,\cdots,p_n \in (1,+\infty)$, let $\bar p$ be the harmonic mean of $p_1,\cdots,p_n$, i.e. $\frac{1}{\bar p}=\frac{1}{n}\sum_{i=1}^{n}\frac{1}{p_i}$, and $p_m=\max_{1\le i\le n} p_i$. In this paper we assume $\bar p<n$, and introduce the Sobolev exponent $\bar p^*=\frac{n\bar p}{n-\bar p}$. The anisotropic Sobolev space $W^{1,(p_i)}(\Omega)$ is defined as usual by

$$W^{1,(p_i)}(\Omega)=\left\{v\in W^{1,1}(\Omega): D_iv\in L^{p_i}(\Omega) \text{ for every } i=1,\cdots,n\right\}$$

and $W_0^{1,(p_i)}(\Omega)$ is denoted to be the closure of $C_0^\infty(\Omega)$ in the norm of $W^{1,(p_i)}(\Omega)$.

Let the boundary datum $u_*:\Omega\to \mathrm{R}$ satisfying

$$u_*\in W^{1,(q_i)}(\Omega), \text{ with } q_i>p_i \text{ for every } i=1,\cdots,n. \tag{1.6}$$

The main result of this paper is the following theorem.

THEOREM 1.1. *Under the previous assumptions (1.5) and (1.6), let* $u \in u_* + W_0^{1,(p_i)}(\Omega)$ *minimize (1.1), that is,*

$$\int_\Omega f(x, Du(x))dx \le \int_\Omega f(x, Dw(x))dx, \quad \forall w \in u_* + W_0^{1,(p_i)}(\Omega). \tag{1.7}$$

Then we have

$$u \in u_* + L^t_{weak}(\Omega),$$

where

$$t = \frac{\bar{p}\bar{p}^*}{\bar{p} - b\bar{p}^*} > \bar{p}^*, \tag{1.8}$$

and b is any number such that

$$\begin{aligned} &0 < b \le \min_{1\le i\le n} \left(1 - \frac{p_i}{q_i}\right), \\ &b < \min_{1\le i\le n} \left(1 - \frac{p_i - 2}{p_i \min_{1\le j\le n} \left(\frac{q_j}{p_j}\right)} - \frac{2}{q_j}\right) \quad \textit{and} \quad b < \frac{\bar{p}}{\bar{p}^*}. \end{aligned} \tag{1.9}$$

REMARK 1.2. We should compare (1.5) with (1.2) and Theorem 1.1 in this paper with [7, Theorem 2.1]. Note that

$$|z_i|^2 = (|z_i|^{p_i})^{\frac{2}{p_i}} \le \left(1 + \sum_{j=1}^n |z_j|^{p_j}\right)^{\frac{2}{p_i}},$$

thus

$$\sum_{i=1}^n \left(1 + \sum_{j=1}^n |z_j|^{p_j}\right)^{\frac{p_i-2}{p_i}} |z_i|^2 \le n \left(1 + \sum_{j=1}^n |z_j|^{p_j}\right).$$

This means, up to a constant n, the right hand side of (1.5) is smaller than or equals to the right hand side of (1.2).

Consider a special case, when

$$p_i \ge 2, \quad \text{for all } i = 1, 2, \cdots, n, \tag{1.10}$$

we get

$$|z_i|^{p_i-2} = (|z_i|^{p_i})^{\frac{p_i-2}{p_i}} \le \left(\sum_{j=1}^n |z_j|^{p_j}\right)^{\frac{p_i-2}{p_i}},$$

thus

$$\sum_{i=1}^n |z_i|^{p_i} \le \sum_{i=1}^n \left(\sum_{j=1}^n |z_j|^{p_j}\right)^{\frac{p_i-2}{p_i}} |z_i|^2.$$

This means that (1.5) implies (1.2) in the case (1.10) holds true. So we are, in this special case, in the framework of [7]. Unfortunately, we are not able to improve the degree of integrability that is proved in [7].

REMARK 1.3. The main feature of this paper lies in the case when

$$1 < p_i < 2, \quad \text{for all } i = 1, 2, \cdots, n. \tag{1.11}$$

In this case,

$$|z_i|^{p_i-2} = (|z_i|^{p_i})^{\frac{p_i-2}{p_i}} \geq \left(\sum_{j=1}^n |z_j|^{p_j}\right)^{\frac{p_i-2}{p_i}} \geq \left(1 + \sum_{j=1}^n |z_j|^{p_j}\right)^{\frac{p_i-2}{p_i}},$$

thus

$$\sum_{i=1}^n |z_i|^{p_i} \geq \sum_{i=1}^n \left(1 + \sum_{j=1}^n |z_j|^{p_j}\right)^{\frac{p_i-2}{p_i}} |z_i|^2.$$

This means that the condition in the left hand side of (1.5) is weaker than the one in the left hand side of (1.2). Since

$$\frac{1}{Q_i} = \frac{p_i - 2}{p_i \min\limits_{1 \leq j \leq n} \left(\frac{q_j}{p_j}\right)} + \frac{2}{q_i} \geq \frac{p_i - 2}{q_i} + \frac{2}{q_i} = \frac{p_i}{q_i},$$

then the conditions (1.9) on b become [7,(2.7)]. That is, in this case, the result of this paper is the same as [7, Theorem 2.1].

REMARK 1.4. If the density function $f(x, z)$ satisfies (1.2) and if it is convex, then the existence of minimizer of functional (1.1) can be guaranteed by the direct methods of calculus of variations. In case we assume (1.5) in place of (1.2), dropping the coercivity assumption, the existence of such minimizer remains unclear. The result of Theorem 1.1 remains valid under the condition that the minimizer of (1.1) under (1.5) is a priori existent.

2. PRELIMINARY LEMMAS.

In order to prove Theorem 1.1, we need two preliminary lemmas. The first one is the anisotropic embedding theorem, which can be found, for example, in [7, Theorem 3.1].

LEMMA 2.1. *Let Ω be a bounded open subset of R^n; let $p_1, \cdots, p_n$ be in $[1, +\infty)$; let $v : \Omega \to R$ be in $W_0^{1,(p_i)}(\Omega)$; if $\bar{p} < n$, then $v \in L^{\bar{p}^*}(\Omega)$ with*

$$\|v\|_{L^{\bar{p}^*}(\Omega)} \leq c_* \left(\prod_{i=1}^n \|D_i v\|_{L^{p_i}(\Omega)}\right)^{\frac{1}{n}},$$

where

$$c_* = \max_{i=1,\cdots,n} \left\{1 + \bar{p}^* \frac{p_i - 1}{p_i}\right\}.$$

The next lemma comes from [9].

Lemma 2.2. *Let $\tilde{C}, \alpha, \beta, L_0$ be positive constants. Let $\phi : [L_0, +\infty) \to [0, +\infty)$ be non-increasing and*

$$L_0 \le L < \tilde{L} \Rightarrow \phi(\tilde{L}) \le \frac{\tilde{C}}{(\tilde{L} - L)^\alpha} [\phi(L)]^\beta.$$

If $\beta < 1$, then

$$\phi(L) \le \left[\tilde{C}^{\frac{1}{1-\beta}} + L_0^{\frac{\alpha}{1-\beta}} \phi(L_0) \right] 2^{\frac{\alpha(2-\beta)}{(1-\beta)^2}} \left(\frac{1}{L} \right)^{\frac{\alpha}{1-\beta}}$$

for every $L \in [L_0, +\infty)$.

3. Proof of Theorem 1.1.

For $L \in (0, +\infty)$ and a function w, we let $T_L(w)$ to be the truncation of w at level L; that is,

$$T_L(w) = \begin{cases} w, & |w| \le L, \\ \operatorname{sign}(w) L, & |w| > L. \end{cases}$$

Let us consider

$$v = u - u_* - T_L(u - u_*) = \begin{cases} u - u_* + L, & \text{if } u - u_* < -L, \\ 0, & \text{if } -L \le u - u_* \le L, \\ u - u_* - L, & \text{if } u - u_* > L. \end{cases} \tag{3.1}$$

It is obvious, by the assumptions on u and u_*, that $v \in W_0^{1,(p_i)}(\Omega)$, and

$$Dv = (Du - Du_*) 1_{\{|u - u_*| > L\}}, \tag{3.2}$$

where $1_E(x)$ is the characteristic function for the set E, that is, $1_E(x) = 1$ for $x \in E$ and $1_E(x) = 0$ otherwise.

The elementary inequality

$$(a + b)^t \le 2^{t-1}(a^t + b^t), \quad a, b > 0, t \ge 1$$

implies

$$\begin{aligned} &\int_{\{|u-u_*|>L\}} \sum_{i=1}^n |D_i u - D_i u_*|^{p_i} dx \\ &\le 2^{p_m - 1} \left(\int_{\{|u-u_*|>L\}} \sum_{i=1}^n |D_i u|^{p_i} dx + \int_{\{|u-u_*|>L\}} \sum_{i=1}^n |D_i u_*|^{p_i} dx \right). \end{aligned} \tag{3.3}$$

We fix $i \in \{1, \cdots, n\}$. In the case $p_i \geq 2$, one has

$$
\begin{aligned}
\int_{\{|u-u_*|>L\}} |D_i u|^{p_i} dx &\leq \int_{\{|u-u_*|>L\}} \left(\sum_{j=1}^{n} |D_j u|^{p_j} \right)^{\frac{p_i-2}{p_i}} |D_i u|^2 dx \\
&\leq \int_{\{|u-u_*|>L\}} \left(1 + \sum_{j=1}^{n} |D_j u|^{p_j} \right)^{\frac{p_i-2}{p_i}} |D_i u|^2 dx.
\end{aligned} \tag{3.4}
$$

In the case $1 < p_i < 2$, one can use Hölder and Young inequalities to derive

$$
\begin{aligned}
&\int_{\{|u-u_*|>L\}} |D_i u|^{p_i} dx \\
&= \int_{\{|u-u_*|>L\}} \left(1 + \sum_{j=1}^{n} |D_j u|^{p_j} \right)^{\frac{p_i-2}{2}} |D_i u|^{p_i} \left(1 + \sum_{j=1}^{n} |D_j u|^{p_j} \right)^{\frac{2-p_i}{2}} dx \\
&\leq \left(\int_{\{|u-u_*|>L\}} \left(1 + \sum_{j=1}^{n} |D_j u|^{p_j} \right)^{\frac{p_i-2}{p_i}} |D_i u|^2 dx \right)^{\frac{p_i}{2}} \\
&\quad \left(\int_{\{|u-u_*|>L\}} \left(1 + \sum_{j=1}^{n} |D_j u|^{p_j} \right) dx \right)^{\frac{2-p_i}{2}} \\
&\leq C(\varepsilon) \int_{\{|u-u_*|>L\}} \left(1 + \sum_{j=1}^{n} |D_j u|^{p_j} \right)^{\frac{p_i-2}{p_i}} |D_i u|^2 dx \\
&\quad + \varepsilon \int_{\{|u-u_*|>L\}} \left(1 + \sum_{j=1}^{n} |D_j u|^{p_j} \right) dx \\
&\leq C(\varepsilon) \int_{\{|u-u_*|>L\}} \left(1 + \sum_{j=1}^{n} |D_j u|^{p_j} \right)^{\frac{p_i-2}{p_i}} |D_i u|^2 dx + \varepsilon |\{|u-u_*|>L\}| \\
&\quad + 2^{p_m-1} \varepsilon \left[\int_{\{|u-u_*|>L\}} \sum_{j=1}^{n} |D_j u - D_j u_*|^{p_j} dx \right. \\
&\quad \left. + \int_{\{|u-u_*|>L\}} \sum_{j=1}^{n} |D_j u_*|^{p_j} dx \right],
\end{aligned} \tag{3.5}
$$

where $0 < \varepsilon < 1$ is a constant to be determined later. In (3.5) and in the sequel, we use the notation $C(*, \cdots, *)$ to denote a constant that depends only on the quantities involved, and it may change at each appearance.

Taking into account $C(\varepsilon) > 1$, we derive that, in both cases $p_i \geq 2$ and $1 < p_i < 2$, (3.5) holds true, which together with (3.3) implies

$$
\begin{aligned}
&\int_{\{|u-u_*|>L\}} \sum_{i=1}^{n} |D_i u - D_i u_*|^{p_i} dx \\
&\quad \leq 2^{p_m-1} C(\varepsilon) \int_{\{|u-u_*|>L\}} \sum_{i=1}^{n} \left(1 + \sum_{j=1}^{n} |D_j u|^{p_j}\right)^{\frac{p_i-2}{p_i}} |D_i u|^2 dx \\
&\qquad + 2^{p_m-1} n\varepsilon \left|\{|u-u_*| > L\}\right| \\
&\qquad + 4^{p_m-1} n\varepsilon \int_{\{|u-u_*|>L\}} \sum_{j=1}^{n} |D_j u - D_j u_*|^{p_j} dx \\
&\qquad + 2^{p_m-1}(2^{p_m-1} n\varepsilon + 1) \int_{\{|u-u_*|>L\}} \sum_{i=1}^{n} |D_i u_*|^{p_i} dx.
\end{aligned} \tag{3.6}
$$

Take $\varepsilon = \frac{1}{n 4^{p_m - \frac{1}{2}}}$, then $4^{p_m-1} n\varepsilon = \frac{1}{2}$. Thus the third term in the right hand side of (3.6) is absorbed by the left hand side one. Therefore,

$$
\begin{aligned}
&\int_{\{|u-u_*|>L\}} \sum_{i=1}^{n} |D_i u - D_i u_*|^{p_i} dx \\
&\quad \leq C(p_m, n) \left[\int_{\{|u-u_*|>L\}} \sum_{i=1}^{n} \left(1 + \sum_{j=1}^{n} |D_j u|^{p_j}\right)^{\frac{p_i-2}{p_i}} |D_i u|^2 dx \right. \\
&\qquad \left. + \left|\{|u-u_*| > L\}\right| + \int_{\{|u-u_*|>L\}} \sum_{i=1}^{n} |D_i u_*|^{p_i} dx \right].
\end{aligned}
$$

The left hand side of assumption (1.5) implies

$$
\begin{aligned}
&\int_{\{|u-u_*|>L\}} \sum_{i=1}^{n} |D_i u - D_i u_*|^{p_i} dx \\
&\quad \leq C(p_m, n) \left[\int_{\{|u-u_*|>L\}} f(x, Du(x)) dx + \left|\{|u-u_*| > L\}\right| \right. \\
&\qquad \left. + \int_{\{|u-u_*|>L\}} \sum_{i=1}^{n} |D_i u_*|^{p_i} dx \right].
\end{aligned} \tag{3.7}
$$

Our next goal is to prove

$$\int_{\{|u-u_*|>L\}} f(x, Du(x))dx \leq \int_{\{|u-u_*|>L\}} f(x, Du_*(x))dx. \tag{3.8}$$

To this aim, we consider

$$w = u - v = u_* + T_L(u - u_*) = \begin{cases} u_* - L, & \text{if } u - u_* < -L, \\ u, & \text{if } -L \leq u - u_* \leq L, \\ u_* + L, & \text{if } u - u_* > L, \end{cases}$$

with v be as in (3.1). Then $w \in u_* + W_0^{1,(p_i)}(\Omega)$ and

$$Dw = (Du)1_{\{|u-u_*|\leq L\}} + (Du_*)1_{\{|u-u_*|>L\}}.$$

Then, minimality inequality (1.7) can be written as follows

$$\begin{aligned} &\int_{\{|u-u_*|\leq L\}} f(x, Du(x))dx + \int_{\{|u-u_*|>L\}} f(x, Du(x))dx \\ &\leq \int_{\{|u-u_*|\leq L\}} f(x, Dw(x))dx + \int_{\{|u-u_*|>L\}} f(x, Dw(x))dx \\ &= \int_{\{|u-u_*|\leq L\}} f(x, Du(x))dx + \int_{\{|u-u_*|>L\}} f(x, Du_*(x))dx. \end{aligned} \tag{3.9}$$

Since we assumed the anisotropic growth (1.5) and $D_i u, D_i u_* \in L^{p_i}(\Omega)$, then all the integrals above are finite and we can drop the integrals over $\{|u-u_*| \leq L\}$ from both sides in (3.9): this ends the proof of (3.8). Then (3.7) and (3.8) merge into

$$\begin{aligned} &\int_{\{|u-u_*|>L\}} \sum_{i=1}^n |D_i u - D_i u_*|^{p_i} dx \\ &\leq C(p_m, n) \left[\int_{\{|u-u_*|>L\}} f(x, Du_*(x))dx + |\{|u-u_*| > L\}| \right. \\ &\quad \left. + \int_{\{|u-u_*|>L\}} \sum_{i=1}^n |D_i u_*|^{p_i} dx \right]. \end{aligned}$$

Now we use the right hand side of (1.5), and we get

$$\begin{aligned} &\int_{\{|u-u_*|>L\}} \sum_{i=1}^n |D_i u - D_i u_*|^{p_i} dx \\ &\leq C(p_m, n, c) \left[\int_{\{|u-u_*|>L\}} \sum_{i=1}^n \left(1 + \sum_{j=1}^n |D_j u_*|^{p_j} \right)^{\frac{p_i - 2}{p_i}} |D_i u_*|^2 dx \right. \\ &\quad \left. + |\{|u-u_*| > L\}| + \int_{\{|u-u_*|>L\}} \sum_{i=1}^n |D_i u_*|^{p_i} dx \right]. \end{aligned} \tag{3.10}$$

By (1.6), one has

$$\left(1+\sum_{j=1}^{n}|D_ju_*|^{p_j}\right)^{\frac{p_i-2}{p_i}}|D_iu_*|^2\in L^{Q_i}(\Omega),\qquad i=1,\cdots,n, \tag{3.11}$$

where

$$Q_i:\frac{1}{Q_i}=\frac{p_i-2}{p_i\min\limits_{1\le j\le n}\left(\frac{q_j}{p_j}\right)}+\frac{2}{q_i},\qquad i=1,\cdots,n. \tag{3.12}$$

Let t_i be such that

$$p_i<t_i\le q_i \ \text{ and } \ \frac{t_i}{p_i}<Q_i,\quad i=1,\cdots,n. \tag{3.13}$$

Then by applying Hölder inequality with $p'=\frac{t_i}{p_i}$ and $q'=\frac{t_i}{t_i-p_i}$ on the first and third integrals in the right hand side of (3.10) we obtain

$$\begin{aligned}
&\int_{\{|u-u_*|>L\}}\sum_{i=1}^{n}\left(1+\sum_{j=1}^{n}|D_ju_*|^{p_j}\right)^{\frac{p_i-2}{p_i}}|D_iu_*|^2dx\\
&\le\sum_{i=1}^{n}\left\{\int_{\{|u-u_*|>L\}}\left[\left(1+\sum_{j=1}^{n}|D_ju_*|^{p_j}\right)^{\frac{p_i-2}{p_i}}|D_iu_*|^2\right]^{\frac{t_i}{p_i}}dx\right\}^{\frac{p_i}{t_i}}\\
&\qquad|\{|u-u_*|>L\}|^{\frac{t_i-p_i}{t_i}}\\
&=\sum_{i=1}^{n}\left[\int_{\{|u-u_*|>L\}}\left(1+\sum_{j=1}^{n}|D_ju_*|^{p_j}\right)^{\frac{(p_i-2)t_i}{p_i^2}}|D_iu_*|^{\frac{2t_i}{p_i}}dx\right]^{\frac{p_i}{ti}}\\
&\qquad|\{|u-u_*|>L\}|^{\frac{t_i-p_i}{t_i}}
\end{aligned}\tag{3.14}$$

and

$$\begin{aligned}
&\int_{\{|u-u_*|>L\}}\sum_{i=1}^{n}|D_iu_*|^{p_i}dx\\
&\le\sum_{i=1}^{n}\left(\int_{|u-u_*|>L}|D_iu_*|^{t_i}dx\right)^{\frac{p_i}{t_i}}|\{|u-u_*|>L\}|^{\frac{t_i-p_i}{t_i}}.
\end{aligned}\tag{3.15}$$

Combining (3.14) and (3.15) with (3.10) we arrive at

(3.16)

$$\begin{aligned}
&\int_{\{|u-u_*|>L\}} \sum_{i=1}^{n} |D_i u - D_i u_*|^{p_i} dx \\
&\le C(p_m, n, c) \left[\sum_{i=1}^{n} \left[\int_{\{|u-u_*|>L\}} \left(1 + \sum_{j=1}^{n} |D_j u_*|^{p_j} \right)^{\frac{(p_i-2)t_i}{p_i^2}} |D_i u_*|^{\frac{2t_i}{p_i}} dx \right]^{\frac{p_i}{ti}} \right. \\
&\qquad |\{|u-u_*|>L\}|^{\frac{t_i-p_i}{t_i}} \\
&\qquad + |\{|u-u_*|>L\}| \\
&\qquad \left. + \sum_{i=1}^{n} \left(\int_{|u-u_*|>L} |D_i u_*|^{t_i} dx \right)^{\frac{p_i}{t_i}} |\{|u-u_*|>L\}|^{\frac{t_i-p_i}{t_i}} \right].
\end{aligned}$$

We need that, for a suitable choice of $t_i \in (p_i, q_i]$, $i = 1, \cdots, n$, the exponent

$$b = \frac{t_i - p_i}{t_i} \tag{3.17}$$

does not depend on i. So, if we solve (3.17) with respect to t_i we get

$$t_i = \frac{p_i}{1-b}. \tag{3.18}$$

We keep in mind that all the t_i's should satisfy (3.13). The condition $t_i > p_i$ leads to $b > 0$, while for $t_i \le q_i$ we obtain that

$$b \le \min_{1\le i\le n} \left(1 - \frac{p_i}{q_i} \right), \tag{3.19}$$

and for $\frac{t_i}{p_i} < Q_i$ we can get

$$b < \min_{1\le i\le n} \left(1 - \frac{1}{Q_i} \right). \tag{3.20}$$

Thus, for every b satisfying

$$0 < b \le \min_{1\le i\le n} \left(1 - \frac{p_i}{q_i} \right) \quad \text{and} \quad b < \min_{1\le i\le n} \left(1 - \frac{1}{Q_i} \right),$$

it is enough to define t_i as in (3.18) to obtain that $p_i < t_i \le q_i$ and (3.17) holds true. Under these assumptions, by setting

$$M = \sum_{i=1}^{n} \left[\int_{\Omega} \left(1 + \sum_{j=1}^{n} |D_j u_*|^{p_j} \right)^{\frac{(p_i-2)t_i}{p_i^2}} |D_i u_*|^{\frac{2t_i}{p_i}} dx \right]^{\frac{p_i}{ti}} + \sum_{i=1}^{n} \left(\int_{\Omega} |D_i u_*|^{t_i} dx \right)^{\frac{p_i}{t_i}},$$

we obtain from (3.16) that

$$
\begin{aligned}
&\int_{\{|u-u_*|>L\}} \sum_{i=1}^{n} |D_i u - D_i u_*|^{p_i}\, dx \\
&\leq C(p_m, n, c, M) \left(|\{|u-u_*| > L\}|^b + |\{|u-u_*| > L\}| \right) \\
&= C(p_m, n, c, M) \big(|\{|u-u_*| > L\}|^b \\
&\quad + |\{|u-u_*| > L\}|^{1-b} |\{|u-u_*| > L\}|^b \big) \\
&\leq C(p_m, n, c, M) \left(|\{|u-u_*| > L\}|^b + |\Omega|^{1-b} |\{|u-u_*| > L\}|^b \right) \\
&= C(p_m, n, c, M, |\Omega|) |\{|u-u_*| > L\}|^b.
\end{aligned}
\tag{3.21}
$$

Now we estimate the left hand side of (3.21) from below by considering just one summand. Then we take both sides to the power $\frac{1}{p_i}$ and take the product with respect to i to obtain that

$$
\begin{aligned}
&\prod_{i=1}^{n} \left(\int_{\{|u-u_*|>L\}} |D_i u - D_i u_*|^{p_i}\, dx \right)^{\frac{1}{p_i}} \\
&\qquad \leq \left(C(p_m, n, c, M, |\Omega|) |\{|u-u_*| > L\}|^b \right)^{\sum_{i=1}^{n} \frac{1}{p_i}}.
\end{aligned}
\tag{3.22}
$$

Let us consider the test function (3.1). By Lemma 2.1, (3.2) and (3.22), we have

$$
\begin{aligned}
\left(\int_\Omega |v|^{\bar{p}^*}\, dx \right)^{\frac{1}{\bar{p}^*}} &\leq c_* \left[\prod_{i=1}^{n} \left(\int_\Omega |D_i v|^{p_i}\, dx \right)^{\frac{1}{p_i}} \right]^{\frac{1}{n}} \\
&= c_* \left[\prod_{i=1}^{n} \left(\int_{\{|u-u_*|>L\}} |D_i u - D_i u_*|^{p_i}\, dx \right)^{\frac{1}{p_i}} \right]^{\frac{1}{n}} \\
&\leq c_* \left(C(p_m, n, c, M, |\Omega|) \right)^{\frac{1}{\bar{p}}} |\{|u-u_*| > L\}|^{\frac{b}{\bar{p}}}.
\end{aligned}
\tag{3.23}
$$

Now, since $|v| = (|u-u_*| - L) 1_{\{|u-u_*|>L\}}$, for $\tilde{L} > L$ we have

$$
\begin{aligned}
&(\tilde{L} - L)^{\bar{p}^*} |\{|u-u_*| > \tilde{L}\}| \\
&\quad = \int_{\{|u-u_*|>\tilde{L}\}} (\tilde{L} - L)^{\bar{p}^*}\, dx \leq \int_{\{|u-u_*|>\tilde{L}\}} (|u-u_*| - L)^{\bar{p}^*}\, dx \\
&\quad \leq \int_{\{|u-u_*|>L\}} (|u-u_*| - L)^{\bar{p}^*}\, dx = \int_\Omega |v|^{\bar{p}^*}\, dx.
\end{aligned}
\tag{3.24}
$$

Finally, by (3.23) and (3.24), we obtain

$$
|\{|u-u_*| > \tilde{L}\}| \leq \frac{c_*^{\bar{p}^*} \left(C(p_m, n, c, M, |\Omega|) \right)^{\frac{\bar{p}^*}{\bar{p}}}}{(\tilde{L} - L)^{\bar{p}^*}} |\{|u-u_*| > L\}|^{\frac{b\bar{p}^*}{\bar{p}}},
\tag{3.25}
$$

which holds for every $\tilde{L}, L$ such that $\tilde{L} > L > 0$.

Setting $\phi(s) = |\{|u - u_*| > s\}|$, $\alpha = \bar{p}^*$, $\tilde{C} = c_*^{\bar{p}^*}(C(p_m, n, c, M, |\Omega|))^{\frac{\bar{p}^*}{\bar{p}}}$, $L_0 = 1$ and $\beta = \frac{b\bar{p}^*}{\bar{p}}$. (3.25) becomes

$$\phi(\tilde{L}) \le \frac{\tilde{C}}{(\tilde{L} - L)^\alpha}\phi(L)^\beta, \tag{3.26}$$

for $\tilde{L} > L \ge 1$.

Since we assume $b < \frac{\bar{p}}{\bar{p}^*}$ in (1.9), then $\beta < 1$. We get from Lemma 2.2 that

(3.27)

$$|\{|u - u_*| > L\}| \le \left\{\left[c_* C(p_m, n, c, M, |\Omega|)^{\frac{1}{\bar{p}}}\right]^{\frac{\bar{p}\bar{p}^*}{\bar{p} - b\bar{p}^*}} + |\Omega|\right\} 2^{\frac{\bar{p}\bar{p}^*(2\bar{p} - b\bar{p}^*)}{(\bar{p} - b\bar{p}^*)^2}} \left(\frac{1}{L}\right)^t,$$

where $t = \frac{\alpha}{1-\beta} = \frac{\bar{p}\bar{p}^*}{\bar{p} - b\bar{p}^*} > \bar{p}^*$ be as in (1.8). (3.27) means

$$u \in u_* + L^t_{weak}(\Omega), \quad t = \frac{\bar{p}\bar{p}^*}{\bar{p} - b\bar{p}^*}.$$

This ends the proof of Theorem 1.1.

4. Obstacle Problems.

In this section, we consider obstacle problem for the functional (1.1). Let

$$\mathcal{K}^{(p_i)}_{\psi, u_*}(\Omega) = \left\{v \in W^{1,(p_i)}(\Omega) : v \ge \psi, \text{ a.e. } \Omega, \text{ and } v - u_* \in W_0^{1,(p_i)}(\Omega)\right\},$$

where for the boundary datum u_* and the obstacle function ψ, we assume that

$$u_*, \psi \in W^{1,(q_i)}(\Omega), \text{ with } q_i > p_i, \text{ for every } i = 1, \cdots, n. \tag{4.1}$$

For a recent development related to anisotropic obstacle problem, we refer the reader to [2].

The next theorem shows that higher integrability of $\theta = \max\{\psi, u_*\}$ forces solutions $u \in \mathcal{K}^{(p_i)}_{\psi, u_*}(\Omega)$ to be more integrable.

Theorem 4.1. *Under the assumptions (1.5) and (4.1), let $u \in \mathcal{K}^{(p_i)}_{\psi, u_*}(\Omega)$ be a solution to the obstacle problem for the functional (1.1), that is,*

$$\int_\Omega f(x, Du(x))dx \le \int_\Omega f(x, Dw(x))dx, \quad \forall w \in \mathcal{K}^{(p_i)}_{\psi, u_*}(\Omega). \tag{4.2}$$

Then

$$u \in \theta + L^t_{weak}(\Omega),$$

where t satisfies (1.8), and b is any number such that (1.9) holds true.

PROOF. Let $u \in \mathcal{K}^{(p_i)}_{\psi,u_*}(\Omega)$ be a solution to the obstacle problem for the functional (1.1). For $L \in (0,+\infty)$ we define

$$v = u - \theta - T_L(u-\theta) = \begin{cases} u-\theta+L, & \text{if } u-\theta < -L, \\ 0, & \text{if } -L \le u-\theta \le L, \\ u-\theta-L, & \text{if } u-\theta > L. \end{cases} \tag{4.3}$$

We now show that $w = u - v \in \mathcal{K}^{(p_i)}_{\psi,u_*}(\Omega)$. Indeed, it is obvious that $w \in W^{1,(p_i)}(\Omega)$; for the first case $u-\theta < -L$, we obviously have $w = u - v = \theta - L > u \ge \psi$; for the second case $-L \le u-\theta \le L$, one has $w = u \ge \psi$; for the third case $u-\theta > L$, we have $w = \theta + L \ge \theta \ge \psi$; since $u \in u_* + W^{1,(p_i)}_0(\Omega)$ and $u \ge \psi$ a.e. Ω, then $\theta = \max\{\psi, u_*\} = u_* = u$ on $\partial\Omega$, thus $v = 0$ on $\partial\Omega$. This implies $w = u$ on $\partial\Omega$, and therefore $w \in u_* + W^{1,(p_i)}_0(\Omega)$.

Since $Dw = D\theta 1_{\{|u-\theta|>L\}} + Du 1_{\{|u-\theta|\le L\}}$, then inequality (4.2) implies

$$\int_{\{|u-\theta|>L\}} f(x, Du(x))dx \le \int_{\{|u-\theta|>L\}} f(x, D\theta(x))dx. \tag{4.4}$$

The next proof is similar to the proof of Theorem 1.1 with θ in place of u_* and (4.4) in place of (3.8). We omit the details. □

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