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SQUARE EULERIAN QUADRUPLES

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Abstract. We consider the problem of finding four different rational squares, such that the product of any two plus the sum of the same two always gives a square. We give some historical background to this problem and exhibit the small number of solutions found so far.

1. Introduction

The genesis of this work is the following section taken from Chapter 19 in Volume 2 of Dickson's "History of the Theory of Numbers" [1].

Fermat treated the problem to find four numbers such that the product of any two increased by the sum of those two gives a square. He made use of three squares such that the product of any two increased by the sum of the same two gives a square. Stating that there is an infinitude of such sets of three squares, he cited $4, 3504384/d, 2019241/d$*, where* $d = 203401$*. However, he actually used the squares* $25/9, 64/9, 196/9$*, of Diophantus V,* 5*, which have the additional property that the product of any two increased by the third gives a square. Taking these three squares as three of our numbers and* x *as the fourth, we are to satisfy*

$$\frac{34}{9}x + \frac{25}{9} = \Box, \qquad \frac{73}{9}x + \frac{64}{9} = \Box, \qquad \frac{205}{9}x + \frac{196}{9} = \Box.$$

This "triple equation" with squares as constant terms is readily solved. T.L. Heath found x *to be the ratio of two numbers each of* 21 *digits.*

This section generated several questions:
1. Is there "an infinitude of such sets of three squares"?
2. Where did the example $4, 3504384/d, 2019241/d$ come from?
3. What is the x found by Heath and can we find a smaller value - smaller meaning fewer digits in its rational form?

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4. What would the solution have been with the original example of Fermat?
5. Can we use the set of 3 squares to find a fourth square, so that the 4 squares are a solution to Fermat's problem?

Given a set $\{a_1, a_2, \ldots, a_m\}$ such that $a_i a_j + a_i + a_j = \Box$ for $1 \leq i < j \leq m$, Dujella [2, 3] called such a set an *Eulerian m-tuple*, in honour of Euler who found the first example of a quadruple of numbers.

To be historically accurate, however, we should note that what Euler was considering was sets $\{b_1, b_2, b_3, b_4\}$ with $b_i b_j - 1 = \Box$. This is related to the present problem since

$$a_i a_j + a_i + a_j = (a_i + 1)(a_j + 1) - 1$$

2. Infinitude of Eulerian Triples of Squares

We wish to find x, y, z such that

$$x^2y^2 + x^2 + y^2 = a^2, \tag{2.1}$$

$$x^2z^2 + x^2 + z^2 = b^2, \tag{2.2}$$

$$y^2z^2 + y^2 + z^2 = c^2. \tag{2.3}$$

Rather than have a complicated analysis, we show that there are an infinite number of solutions with $x = 2$.

Thus we look for y, z with

$$5y^2 + 4 = a^2, \qquad 5z^2 + 4 = b^2, \qquad y^2z^2 + y^2 + z^2 = c^2. \tag{2.4}$$

Rational solutions of $5t^2 + 4 = \Box$ can be parameterised as $t = 4f/(5 - f^2)$, with f rational. Thus $y = 4m/(5 - m^2)$ and $z = 4n/(5 - n^2)$, which can be substituted into the third part of (1).

We find that m and n must satisfy

$$d^2 = m^2n^4 + (m^4 - 4m^2 + 25)n^2 + 25m^2$$

for d rational.

Define $d = Y/m$ and $n = X/m$, giving the quartic relation

$$Y^2 = X^4 + (m^4 - 4m^2 + 25)X^2 + 25m^4$$

This quartic has an obvious rational point $X = 0, Y = 5m^2$, and so is birationally equivalent to an elliptic curve if $m \neq 0$. Using the standard method described by Mordell [4], we find the elliptic curve (with $m = p/q$)

$$J^2 = K^3 - 2(p^4 - 4p^2q^2 + 25q^4)K^2 + (p^8 - 8p^6q^2 - 34p^4q^4 - 200p^2q^6 + 625q^8)K \tag{2.5}$$

with the relation $n = J/(2pqK)$.

The curve has 3 points of order 2 with $J = 0$, which lead to $z = 0$, so we need other rational points for non-trivial solutions. Evidence suggests that the torsion subgroup is just isomorphic to $\mathbb{Z}2 \times \mathbb{Z}2$, but this would be difficult to prove. If true, we need the curves to have rank greater than 0 for solutions.

If we experiment, we quickly find that $m = 1/2$ gives a curve $J^2 = K^3 - 770K^2 + 146625K$ which has rank 1 with generator $P = (245, 2100)$. $m = 1/2$ gives $y = 8/19$ and the point P gives $n = 15/7$ and $z = 21$, and it easily checked that $\{4, 64/361, 441\}$ is a square Eulerian triple.

Since the curve has rank 1 we have that integral multiples of P are also rational points on the curve. For example, doubling the point P leads to the values $K = 187489/441, J = 651232/9261$ and so $n = 376/9093$ and an alternative z of $13675872/413271869$. There are thus an infinite number of Eulerian triples with $x = 2, y = 8/19$.

3. Numerical Values

It is totally unclear from Fermat's original work where his numerical example involving 4 comes from, as he just states this result with no supporting algebra or computation. As we saw in this last section, there are much simpler sets which include 4.

We now consider the system

$$\frac{34}{9}x + \frac{25}{9} = \square, \qquad \frac{73}{9}x + \frac{64}{9} = \square, \qquad \frac{205}{9}x + \frac{196}{9} = \square. \tag{3.1}$$

We first write

$$\frac{34}{9}x + \frac{25}{9} = \left(\frac{5}{3} + fx\right)^2$$

which gives $x = 2(17 - 15f)/(9f^2)$.

Substituting this into the second and third equations, we find the following two equations must have rational solutions:

$$\begin{aligned} 2(288f^2 - 1095f + 1241) &= e^2, \\ 2(882f^2 - 3075f + 3485) &= h^2. \end{aligned}$$

Now, in the first equation, $f = 17/15$ gives $e = 136/5$, so, if we substitute $e = 136/5 + g(f - 17/15)$, we can solve to find

$$f = \frac{17g^2 - 816g - 23058}{15(g^2 - 576)}$$

and if we substitute this into the other quadratic relation we find, clearing denominators, that we must have a rational solution to

$$s^2 = 14161g^4 + 731544g^3 + 28206441g^2 + 639486144g + 6471280836 = G(g) \tag{3.2}$$

Since $14161 = 119^2$ we can attempt to complete the square, by forming $G(g) - (\alpha g^2 + \beta g + \gamma)^2$. Simple arithmetic shows that if $\alpha = 119, \beta =$

$21516/7, \gamma = 919177353/11662$ then

$$G(g) - (\alpha g^2 + \beta g + \gamma)^2 = \frac{1581221503125(13328g + 22275)}{136002244}$$

Thus, we have a square solution when $g = -22275/13328$, which gives $f = 142415972261/56567733755$ and finally

$$x = \frac{-459818598496844787200}{631629004828419699201}$$

where both numerator and denominator have 21 digits.

Because equation (3.2) has leading coefficient 119^2 it can be transformed into an elliptic curve. Mordell's method leads to the curve $j^2 = k^3 + 20478k^2 + 99801585k$ with

$$g = \frac{3(105j - 44(163k + 2348685))}{34(49k + 560880)}.$$

This curve has 3 points of order 2 with $k = 0, -7999, -12483$ all with $j = 0$. The first leads to $x = 0$ whilst the other two give undefined values for f. Thus we need points of infinite order. Cremona's `mwrank` program shows the rank to be 2 with generators $(-9984, -222768)$ and $(-8379, -114912)$. The first gives $g = -543/8$, $f = 269/147$ and $x = -50176/72361$, significantly simpler than Heath's value.

With regard to the $\{4, 3504384/203401, 2019241/203401\}$ triple of squares we can perform an identical analysis. The corresponding value of x from completing the square is

$$x = \frac{-28448417598272924003671204878289354665765410185967616}{36828906078832095599985737816846193226885934523284161}$$

where both numerator and denominator have 53 digits.

Attempts, as before, to find a smaller value of x lead to the elliptic curve

$$v^2 = u^3 + 104508834248050u^2 + 2673491566832365510467420 0u.$$

The completing the square values lead to a point of infinite order on this curve with $u = -9390695817653070336/2019241$. Investigations with `APECS`, `mwrank`, and `SAGE` were unable to find other generators. Both `APECS` and `mwrank` give 3 as an upper bound for the rank. The root number is -1, so the parity conjecture suggests the rank is 1 or 3, but we are unable to be exact.

4. Square Quadruples

We now consider the problem of finding $\{x_1, x_2, x_3, x_4\}$ with all 6 combinations $x_i^2 x_j^2 + x_i^2 + x_j^2 = \square$ for $i \neq j$.

Consider first finding a triple (t, y, z) satisfying the square condition for each possible pairing.

The rational solutions to $(t^2+1)s^2+t^2=\square$ can be parameterized by $s=2\,t\,m/(t^2+1-m^2)$ with $m\in\mathbb{Q}$, so let

$$y=\frac{2\,t\,m}{t^2-m^2+1},\qquad z=\frac{2\,t\,n}{t^2-n^2+1},\tag{4.1}$$

where $m,n\in\mathbb{Q}$.

We find that $y^2z^2+y^2+z^2=\square$ requires

$$m^2n^4+(m^4-4m^2+t^4+2t^2+1)n^2+m^2(t^4+2t^2+1)=\square.\tag{4.2}$$

Considering m and t as parameters, this quartic in n has an obvious solution when $n=0$, so is birationally equivalent to an elliptic curve. After some standard algebra we find the elliptic curve to be

$$v^2=u(u-T_1)(u-T_2)\tag{4.3}$$

where

$$T_1=(m^2+2m+t^2+1)(m^2-2m+t^2+1),$$
$$T_2=(m^2+2m-t^2-1)(m^2-2m-t^2-1)$$

with the reverse transformation

$$n=\frac{v}{2mu}.\tag{4.4}$$

This elliptic curve often has rank zero, but we can find enough curves with strictly positive rank to generate as many triples as we need. For example, $m=3,t=1$ gives a curve with rank 1 and generator $(4,54)$. This gives $n=9/4$ and the triple (1, 6/7, 72/49). It seems (numerically) that this curve has rank 0 roughly half the time, as expected.

To expand to a fourth value, we require to find w with $(k^2+1)w^2+k^2=\square$ assuming that k is a known fixed value from (t,y,z). From before, $w=2rt/(t^2+1-r^2)$, which we substitute into $(y^2+1)w^2+w^2=\square$, to find that r must satisfy

$$y^2r^4+(2t^2(y^2+2)-2y^2)r^2+t^4y^2+2t^2y^2+y^2=\square.\tag{4.5}$$

This quartic has an obvious solution when $r=0$, so is birationally equivalent to an elliptic curve. We find the curve

$$V^2=U^3+(t^2(2y^2+1)+y^2)U^2+(t^4y^2(y^2+1)+t^2y^2(y^2+1))U\tag{4.6}$$

with $r=V/(y(t^2(y^2+1)+U))$.

This elliptic curve can be easily transformed to have integer coefficients. We first restricted our programs to searching for integer points on the curve, some of which lead to values of w different from t,y,z. The final test is to compute $w^2z^2+w^2+z^2$ and to test whether this is square.

This methodology was coded using the simple multiple precision `UBASIC` system, and run for several hours on a PC. The code finds hundreds of examples with 5 of the 6 identities equal to a square, but only one example with all 6 square. This solution is

$$\left\{ \left(\frac{3}{5}\right)^2, \left(\frac{224}{107}\right)^2, \left(\frac{8}{5}\right)^2, 18^2 \right\}.$$

We then coded the method using `Pari-gp` and found rational generators of the elliptic curve. This, however, did not discover a single extra solution despite extensive computing. To find further solutions, if possible, we had to specialize the search procedure as follows.

Numerical investigation showed that $u = m^2(m+2)^2 - (t^2+1)^2$ would give a point on the curve (4.3) if

$$4m^2 - (t^2+1)^2 = \square.$$

This quadric has a solution when $m = 5(t^2+1)/6$ and the standard quadric parameterisation method gives

$$m = \frac{(t^2+1)(5k^2-16k+20)}{6(k^2-4)}. \tag{4.7}$$

Doing all the various substitutions eventually gives

$$n = \frac{8(k-1)(k-4)}{5k^2-16k+20} \tag{4.8}$$

and all of these can be substituted into (4.1) to give a parametric triple (t, y, z) with

$$y = \frac{12t(4-k^2)(5k^2-16k+20)}{(5k^2-16k+20)^2t^2 - (k^2+16k-44)(11k^2-16k-4)},$$

$$z = \frac{16t(k-1)(k-4)(5k^2-16k+20)}{(5k^2-16k+20)^2t^2 - 3(k^2-8k+4)(13k^2-56k+52)}.$$

All this can be plugged into (4.6) but the resulting equation is horrible and there is no point printing it out. This approach gives only one elliptic curve per set of parameters to search for solutions and is thus faster. With this methodology we found the second solution

$$\left\{ \left(\frac{352}{3419}\right)^2, \left(\frac{3}{13}\right)^2, \left(\frac{72}{199}\right)^2, \left(\frac{10}{13}\right)^2 \right\}.$$

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References

[1] L. E. Dickson, History of the Theory of Numbers, Vol. II: Diophantine Analysis, AMS, Chelsea, New York, 1992.
[2] A. Dujella, *An extension of an old problem of Diophantus and Euler*, Fibonacci Quart. **37** (1999), 312–314.
[3] A. Dujella, *An extension of an old problem of Diophantus and Euler. II*, Fibonacci Quart. **40** (2002), 118–123.
[4] L. J. Mordell, Diophantine Equations, Academic Press, New York, 1968.

Kvadratne Eulerove četvorke

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Sažetak. Razmatramo problem nalaženja četiri različita racionalna kvadrata sa svojstvom da produkt svaka dva među njima uvećan za njihovu sumu daje potpun kvadrat. Dajemo povijesni pregled ovog problema i prikazujemo nekoliko rješenja.

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