RAD HAZU. MATEMATIČKE ZNANOSTI
Vol. 20 = 528 (2016): 37-50

3-JENSEN-CONVEXITY AT A POINT AND 3-WRIGHT-CONVEXITY AT A POINT AND RELATED RESULTS

Sadia Khalid, Josip Pečarić and Marjan Praljak

Abstract. Two new classes of convex functions at a point are introduced and some interesting related results are deduced.

1. Introduction and Preliminaries

The notion of convex function is one of the most important concepts in the theory of inequalities (see [4, p.1]). Throughout this paper I is an interval in $\mathbb{R}$.

Definition 1.1. *A function $f : I \to \mathbb{R}$ is said to be convex if for all $x, y \in I$ and for all $\lambda \in [0, 1]$, the inequality*

$$f(\lambda x + (1-\lambda) y) \leq \lambda f(x) + (1-\lambda) f(y)$$

holds.

The following definition of Jensen-convex (J-convex) function is given in [4, p.5].

Definition 1.2. *A function $f : I \to \mathbb{R}$ is said to be convex in the Jensen sense or J-convex if for all $x, y \in I$, the inequality*

$$f\left(\frac{x+y}{2}\right) \leq \frac{f(x) + f(y)}{2}$$

holds.

Wright-convex (W-convex) function is defined as follows (see [4, p.7]):

Definition 1.3. *A function $f : I \to \mathbb{R}$ is said to be W-convex if for all $x, y + h \in I$ such that $x \leq y$, $h > 0$, the inequality*

$$f(x+h) - f(x) \leq f(y+h) - f(y) \tag{1.1}$$

2010 *Mathematics Subject Classification.* 26A51, 26D15.
Key words and phrases. Jensen-convexity, Wright-convexity, 3-Jensen-convex function at a point, 3-Wright-convex function at a point, n-Wright convexity.

holds. The function f is said to be W-concave if the reversed inequality holds in (1.1).

Let Δ_h stands for the difference operator defined by $(\Delta_h f)(x) = f(x+h) - f(x)$, where $h > 0$. Then (1.1) takes the form $\Delta_h f(x) \leq \Delta_h f(y)$ such that $x \leq y$.

DEFINITION 1.4. *Let $n \in \mathbb{N}$ and $h_1, \ldots, h_n > 0$. A function $f : I \to \mathbb{R}$ is said to be n-Wright-convex if $(\Delta_{h_1} \ldots \Delta_{h_n} f)(x) \geq 0$ holds whenever $x, x + h_1 \ldots + h_n \in I$.*

REMARK 1.5. Note that the 2-Wright-convex functions are simply the Wright-convex functions.

The following theorem is given in [4, p.53].

THEOREM 1.6. *If f is a J-convex function defined on I, then for all points $x_1, \ldots, x_n \in I$ and for all rational non-negative numbers $p_1, \ldots, p_n$ such that $\sum_{i=1}^{n} p_i = 1$, the following inequality*

$$f\left(\sum_{i=1}^{n} p_i x_i\right) \leq \sum_{i=1}^{n} p_i f(x_i) \tag{1.2}$$

holds.

The following theorem is given in [4, p.161] (see also [6]).

THEOREM 1.7. *Let $x_1 \geq x_2 \geq \ldots \geq x_{2n+1}$ or $x_1 \leq x_2 \leq \ldots \leq x_{2n+1}$, $x_i \in I$ for $i = 1, \ldots, 2n+1$ and let f be a W-convex function defined on I. Then the following inequality is valid*

$$f\left(\sum_{i=1}^{2n+1} (-1)^{i-1} x_i\right) \leq \sum_{i=1}^{2n+1} (-1)^{i-1} f(x_i). \tag{1.3}$$

The following theorem is a generalization of an inequality of Z. Opial [3] given in [4, p.162].

THEOREM 1.8. *Let $x_k \in I$ for $k = 1, \ldots, 2n+1$ and let $\sum_{i=1}^{2k+1} (-1)^{i-1} x_i \in I$ for $k = 1, \ldots, n$.*

(*i*) *If*

$$x_{2k} \leq x_{2k+1}, \quad \sum_{i=1}^{2k} (-1)^{i-1} x_i \geq 0 \quad for \quad k = 1, \ldots, n, \tag{1.4}$$

then the reverse of (1.3) *holds for every W-convex function $f : I \to \mathbb{R}$. Further, if reverse of the inequalities in* (1.4) *hold, then reverse of* (1.3) *is also valid.*

(*ii*) *If instead of* (1.4)*, the following conditions hold*

$$x_{2k} \leq x_{2k+1}, \quad \sum_{i=1}^{2k} (-1)^{i-1} x_i \leq 0 \quad for \quad k = 1, \dots, n, \tag{1.5}$$

then (1.3) *is valid. If the reversed inequalities in* (1.5) *hold, then* (1.3) *is also valid.*

The following theorem is given in [4, p.322] and its proof can be obtained easily from the proof of Theorem 1.8.

THEOREM 1.9. *Let* $x_i, y_i \in I$ $(i = 1, \dots, n)$*,* $c_k = \sum_{i=1}^{k-1} (x_i - y_i)$ *for* $k = 2, \dots, n$ *and let* $x_k + c_k \in I$ *for all* k*.*

(*i*) *If*

$$x_{k+1} \leq y_k \quad for \quad k = 1, \dots, n-1, \tag{1.6}$$

$$\sum_{i=1}^{k} x_i \leq \sum_{i=1}^{k} y_i \quad for \quad k = 1, \dots, n-1, \tag{1.7}$$

and

$$\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i, \tag{1.8}$$

then

$$\sum_{i=1}^{n} f(x_i) \leq \sum_{i=1}^{n} f(y_i) \tag{1.9}$$

holds for every W-convex function $f : I \to \mathbb{R}$*. Furthermore,* (1.9) *holds for every W-convex function* f *if reverse of the inequalities in* (1.6) *and* (1.7) *hold.*

(*ii*) *If* (1.6) *and* (1.8) *hold and reverse of* (1.7) *holds, then reverse of* (1.9) *holds for every W-convex function* $f : I \to \mathbb{R}$*. Furthermore, the same is true if* (1.7) *and* (1.8) *hold and reverse of* (1.6) *holds.*

A special case of Theorem 1.9 is given in [4, Remark 12.10].

REMARK 1.10. Let $a_1 \geq a_3 \geq \dots \geq a_{2n+1}$, $a_{2k} \geq 0$ $(k = 1, \dots, n)$, $a_1, a_{2n+1} \in I$, $a_k + a_{k+1} \in I$ $(k = 1, \dots, 2n)$. Then for all the W-convex functions $f : I \to \mathbb{R}$, we have

$$\begin{aligned} & f(a_1) + f(a_2 + a_3) + f(a_4 + a_5) + \dots + f(a_{2n} + a_{2n+1}) \\ & \qquad \leq f(a_1 + a_2) + f(a_3 + a_4) + \dots + f(a_{2n-1} + a_{2n}) + f(a_{2n+1}). \end{aligned}$$

In 1997, I. Perić (see [5, p.10]) proved the following theorem for W-convex functions.

Theorem 1.11. *Let* $f : [a, b] \to \mathbb{R}$ *be a W-convex function,* $0 < a \leq y_1 \leq \ldots \leq y_n$ *and let* $C_n \geq 0$*,* $n \in \mathbb{N}$*. Let* $C_k y_k, C_n y_n, C_{k+1} y_k, \sum_{k=1}^{n} C_k (y_k - y_{k-1}) \in [a, b]$ *for all* $k = 1, \ldots, n-1$ *with* $y_0 \equiv 0$*. If*

$$\sum_{k=1}^{n} C_k \left(y_k - y_{k-1}\right) \geq C_{n+1} y_n, \quad n \geq 1, \tag{1.10}$$

then

$$f\left(\sum_{k=1}^{n} C_k (y_k - y_{k-1})\right) + \sum_{k=1}^{n-1} f\left(C_{k+1} y_k\right) \geq \sum_{k=1}^{n} f\left(C_k y_k\right), \quad n \in \mathbb{N}. \tag{1.11}$$

If f *is W-concave, then the reversed inequality holds in* (1.11).

Definition 1.12. *Let* p_k $(k \in \mathbb{N})$ *be real numbers such that* $p_i > 0$ $(i = 1, \ldots, k)$ *with* $P_k = \sum_{i=1}^{k} p_i$ $(k \in \mathbb{N})$*. A sequence* $(x_k, k \in \mathbb{N}) \subset \mathbb{R}$ *is said to be non-increasing in* **p***-weighted mean, if the inequality*

$$\frac{1}{P_n} \sum_{k=1}^{n} p_k x_k \geq \frac{1}{P_{n+1}} \sum_{k=1}^{n+1} p_k x_k, \quad n \in \mathbb{N}, \tag{1.12}$$

holds. A sequence $(x_k, k \in \mathbb{N}) \subset \mathbb{R}$ *is said to be non-decreasing in* **p***-weighted mean, if the reversed inequality holds in* (1.12).

The following theorem is given in [2, Theorem 3].

Theorem 1.13. *Let* x_k *and* p_k $(k = 1, \ldots, n)$ *be real numbers such that* $x_k \geq 0$ *and* $p_k \geq 0$ *with* $P_k = \sum_{i=1}^{k} p_i$ $(k = 1, \ldots, n)$*. Let* $p_1 x_1, \sum_{k=1}^{n} p_k x_k$*,* $P_k x_k, P_{k-1} x_k \in [a, b]$ *for all* $k = 2, \ldots, n$ *and* $f : [a, b] \to \mathbb{R}$ *be a W-convex function.*

(*i*) *If the sequence* $(x_k, k = 1, \ldots, n)$ *is non-increasing in* **p***-weighted mean, then we have*

$$f\left(\sum_{k=1}^{n} p_k x_k\right) \geq f\left(p_1 x_1\right) + \sum_{k=2}^{n} \left(f\left(P_k x_k\right) - f\left(P_{k-1} x_k\right)\right). \tag{1.13}$$

(*ii*) *If the sequence* $(x_k, k = 1, \ldots, n)$ *is non-decreasing in* **p***-weighted mean, then we have*

$$f\left(\sum_{k=1}^{n} p_k x_k\right) \leq f\left(p_1 x_1\right) + \sum_{k=2}^{n} \left(f\left(P_k x_k\right) - f\left(P_{k-1} x_k\right)\right). \tag{1.14}$$

If the function f *is W-concave, then the reversed inequalities hold in* (1.13) *and* (1.14).

For a W-convex function f, Theorems 1.11 and 1.13 (i) are equivalent. By making the substitutions $C_k = x_k$ and $y_k - y_{k-1} = p_k$ $(k = 1, \ldots, n)$ condition (1.10) is equivalent to the condition that the sequence $(x_k, \ k = 1, \ldots, n)$

is non-increasing in **p**-weighted mean and inequality (1.11) is equivalent to (1.13).

2. Main Results

In [1], I. A. Baloch, J. Pečarić and M. Praljak introduced a new class of functions $K_1^c(a,b)$ that extends 3-convex functions and can be interpreted as functions that are 3-convex at point c. They also proved some of the properties of this new class. In particular, they proved that a function is 3-convex on an interval if and only if it is 3-convex at every point of the interval.

In this paper we define a class of 3-J-convex functions at a point $c \in I$ denoted by $\Gamma_1^c(I)$ (a class of 3-J-concave functions at a point $c \in I$ denoted by $\Gamma_2^c(I)$) and a class of 3-W-convex functions at a point $c \in I$ denoted by $\Xi_1^c(I)$ (a class of 3-W-concave functions at a point $c \in I$ denoted by $\Xi_2^c(I)$).

Definition 2.1. *Let I be an interval in $\mathbb{R}$ and $c \in I$. A function $f : I \to \mathbb{R}$ is said to be 3-J-convex at a point c (3-J-concave at a point c) if there exists a constant $\tilde{A}$ such that the function $F(x) = f(x) - \frac{\tilde{A}}{2}x^2$ is J-concave (J-convex) on $I \cap (-\infty, c]$ and J-convex (J-concave) on $I \cap [c, \infty)$.*

Definition 2.2. *Let I be an interval in $\mathbb{R}$ and $c \in I$. A function $f : I \to \mathbb{R}$ is said to be 3-W-convex at a point c (3-W-concave at a point c) if there exists a constant A such that the function $G(x) = f(x) - \frac{A}{2}x^2$ is W-concave (W-convex) on $I \cap (-\infty, c]$ and W-convex (W-concave) on $I \cap [c, \infty)$.*

The following theorem is our first main result.

Theorem 2.3. *Let $x_i \in I \cap (-\infty, c]$ $(i = 1, \ldots, n)$ and $y_j \in I \cap [c, \infty)$ $(j = 1, \ldots, m)$ and let $p_1, \ldots, p_n$ and $w_1, \ldots, w_m$ be rational non-negative numbers such that $\sum_{i=1}^n p_i = \sum_{i=1}^m w_i = 1$. If $f \in \Gamma_1^c(I)$ and if*

$$\sum_{i=1}^{n} p_i x_i^2 - \left(\sum_{i=1}^{n} p_i x_i\right)^2 = \sum_{j=1}^{m} w_j y_j^2 - \left(\sum_{j=1}^{m} w_j y_j\right)^2, \tag{2.1}$$

then

$$\sum_{i=1}^{n} p_i f(x_i) - f\left(\sum_{i=1}^{n} p_i x_i\right) \leq \sum_{j=1}^{m} w_j f(y_j) - f\left(\sum_{j=1}^{m} w_j y_j\right), \tag{2.2}$$

while for $f \in \Gamma_2^c(I)$, the reverse of (2.2) holds.

Proof. Since $f \in \Gamma_1^c(I)$, there exists a constant $\tilde{A}$ such that $F(x) = f(x) - \frac{\tilde{A}}{2}x^2$ is J-concave on $I \cap (-\infty, c]$ and J-convex on $I \cap [c, \infty)$. By applying inequality (1.2) with f and I replaced by $-F$ and $I \cap (-\infty, c]$ respectively, we have

$$\sum_{i=1}^{n} p_i F(x_i) \leq F\left(\sum_{i=1}^{n} p_i x_i\right),$$

equivalent to

$$\sum_{i=1}^{n} p_i f(x_i) - f\left(\sum_{i=1}^{n} p_i x_i\right) \leq \frac{\tilde{A}}{2}\left(\sum_{i=1}^{n} p_i x_i^2 - \left(\sum_{i=1}^{n} p_i x_i\right)^2\right). \tag{2.3}$$

By applying inequality (1.2) with f and I replaced by F and $I \cap [c, \infty)$ respectively, we have

$$\sum_{j=1}^{m} w_j f(y_j) - f\left(\sum_{j=1}^{m} w_j y_j\right) \geq \frac{\tilde{A}}{2}\left(\sum_{j=1}^{m} w_j y_j^2 - \left(\sum_{j=1}^{m} w_j y_j\right)^2\right). \tag{2.4}$$

From (2.3) and (2.4), we have

$$\sum_{i=1}^{n} p_i f(x_i) - f\left(\sum_{i=1}^{n} p_i x_i\right) - \frac{\tilde{A}}{2}\left(\sum_{i=1}^{n} p_i x_i^2 - \left(\sum_{i=1}^{n} p_i x_i\right)^2\right) \leq 0 \leq$$

$$\sum_{j=1}^{m} w_j f(y_j) - f\left(\sum_{j=1}^{m} w_j y_j\right) - \frac{\tilde{A}}{2}\left(\sum_{j=1}^{m} w_j y_j^2 - \left(\sum_{j=1}^{m} w_j y_j\right)^2\right),$$

which, together with (2.1), yields inequality (2.2). If $f \in \Gamma_2^c(I)$, the inequalities above are reversed and the second inequality of the theorem follows. ∎

REMARK 2.4. From the proof of Theorem 2.3 it is clear that for $f \in \Gamma_1^c(I)$, the following refinement of inequality (2.2) holds

$$\begin{aligned} &\sum_{i=1}^{n} p_i f(x_i) - f\left(\sum_{i=1}^{n} p_i x_i\right) \\ &\leq \frac{\tilde{A}}{2}\left(\sum_{i=1}^{n} p_i x_i^2 - \left(\sum_{i=1}^{n} p_i x_i\right)^2\right) = \frac{\tilde{A}}{2}\left(\sum_{j=1}^{m} w_j y_j^2 - \left(\sum_{j=1}^{m} w_j y_j\right)^2\right) \\ &\leq \sum_{j=1}^{m} w_j f(y_j) - f\left(\sum_{j=1}^{m} w_j y_j\right), \end{aligned} \tag{2.5}$$

while for $f \in \Gamma_2^c(I)$, the reversed inequalities hold in (2.5) and we obtain the refinement of the reverse of (2.2).

THEOREM 2.5. *Let* $x_i \in I \cap (-\infty, c]$ $(i = 1, \dots, 2n+1)$ *be such that* $x_1 \geq x_2 \geq \dots \geq x_{2n+1}$ *or* $x_1 \leq x_2 \leq \dots \leq x_{2n+1}$ *and* $y_j \in I \cap [c, \infty)$ $(j = 1, \dots, 2m+1)$ *be such that* $y_1 \geq y_2 \geq \dots \geq y_{2m+1}$ *or* $y_1 \leq y_2 \leq \dots \leq y_{2m+1}$ *with the condition*

$$\sum_{i=1}^{2n+1} (-1)^{i-1} (x_i - \bar{x})^2 = \sum_{j=1}^{2m+1} (-1)^{j-1} (y_j - \bar{y})^2 , \tag{2.6}$$

where $\bar{x} = \sum_{i=1}^{2n+1} (-1)^{i-1} x_i$ *and* $\bar{y} = \sum_{j=1}^{2m+1} (-1)^{j-1} y_j$. *If* $f \in \Xi_1^c(I)$, *then*

$$\begin{aligned} &\sum_{i=1}^{2n+1} (-1)^{i-1} f(x_i) - f\left(\sum_{i=1}^{2n+1} (-1)^{i-1} x_i\right) \\ &\leq \sum_{j=1}^{2m+1} (-1)^{j-1} f(y_j) - f\left(\sum_{j=1}^{2m+1} (-1)^{j-1} y_j\right), \end{aligned} \tag{2.7}$$

while for $f \in \Xi_2^c(I)$ *the reverse of* (2.7) *holds.*

PROOF. Since $f \in \Xi_1^c(I)$, there exists a constant A such that $G(x) = f(x) - \frac{A}{2}x^2$ is W-concave on $I \cap (-\infty, c]$ and W-convex on $I \cap [c, \infty)$. By applying inequality (1.3) with f and I replaced by $-G$ and $I \cap (-\infty, c]$ respectively, we have

$$\begin{aligned} 0 \geq &\sum_{i=1}^{2n+1} (-1)^{i-1} G(x_i) - G\left(\sum_{i=1}^{2n+1} (-1)^{i-1} x_i\right) \\ = &\sum_{i=1}^{2n+1} (-1)^{i-1} f(x_i) - \frac{A}{2} \sum_{i=1}^{2n+1} (-1)^{i-1} x_i^2 \\ &- \left[f\left(\sum_{i=1}^{2n+1} (-1)^{i-1} x_i\right) - \frac{A}{2} \left(\sum_{i=1}^{2n+1} (-1)^{i-1} x_i\right)^2 \right] \\ = &\sum_{i=1}^{2n+1} (-1)^{i-1} f(x_i) - f\left(\sum_{i=1}^{2n+1} (-1)^{i-1} x_i\right) \\ &- \frac{A}{2} \sum_{i=1}^{2n+1} (-1)^{i-1} (x_i - \bar{x})^2 . \end{aligned} \tag{2.8}$$

By applying inequality (1.3) with f and I replaced by G and $I \cap [c, \infty)$ respectively, we have

$$
\begin{aligned}
0 &\leq \sum_{j=1}^{2m+1} (-1)^{j-1} G(y_j) - G\left(\sum_{j=1}^{2m+1} (-1)^{j-1} y_j\right) \\
&= \sum_{j=1}^{2m+1} (-1)^{j-1} f(y_j) - \frac{A}{2} \sum_{j=1}^{2m+1} (-1)^{j-1} y_j^2 \\
&\quad - \left[f\left(\sum_{j=1}^{2m+1} (-1)^{j-1} y_j\right) - \frac{A}{2}\left(\sum_{j=1}^{2m+1} (-1)^{j-1} y_j\right)^2 \right] \\
&= \sum_{j=1}^{2m+1} (-1)^{j-1} f(y_j) - f\left(\sum_{j=1}^{2m+1} (-1)^{j-1} y_j\right) \\
&\quad - \frac{A}{2} \sum_{j=1}^{2m+1} (-1)^{j-1} (y_j - \bar{y})^2 .
\end{aligned} \tag{2.9}
$$

From (2.8) and (2.9), we have

$$
\begin{aligned}
&\sum_{i=1}^{2n+1} (-1)^{i-1} f(x_i) - f\left(\sum_{i=1}^{2n+1} (-1)^{i-1} x_i\right) - \frac{A}{2} \sum_{i=1}^{2n+1} (-1)^{i-1} (x_i - \bar{x})^2 \leq 0 \leq \\
&\sum_{j=1}^{2m+1} (-1)^{j-1} f(y_j) - f\left(\sum_{j=1}^{2m+1} (-1)^{j-1} y_j\right) - \frac{A}{2} \sum_{j=1}^{2m+1} (-1)^{j-1} (y_j - \bar{y})^2 ,
\end{aligned}
$$

which, together with (2.6), yields inequality (2.7). If $f \in \Xi_2^c(I)$ the inequalities above are reversed and the second inequality of the theorem follows. ∎

Corollary 2.6. *Let $n \in \mathbb{N}$ and let $\boldsymbol{x} = (x_1, \ldots, x_{2n+1}) \in [0, c]^{2n+1}$ and $\boldsymbol{y} = (y_1, \ldots, y_{2n+1}) \in [c, 2c]^{2n+1}$ be monotonic and satisfy*

$$x_1 + y_1 = \ldots = x_{2n+1} + y_{2n+1} = 2c. \tag{2.10}$$

If $f \in \Xi_1^c(I)$, then (2.7) holds with $n = m$.

Proof. One can easily see that (2.10) implies $\bar{y} = 2c - \bar{x}$ and (2.6) with $m = n$ and $I = [0, 2c]$. ∎

REMARK 2.7. From the proof of Theorem 2.5, it is clear that for $f \in \Xi_1^c(I)$ the following refinement of inequality (2.7) holds

$$\begin{aligned} &\sum_{i=1}^{2n+1} (-1)^{i-1} f(x_i) - f\left(\sum_{i=1}^{2n+1} (-1)^{i-1} x_i\right) \\ &\le \frac{A}{2} \sum_{i=1}^{2n+1} (-1)^{i-1} (x_i - \bar{x})^2 = \frac{A}{2} \sum_{j=1}^{2m+1} (-1)^{j-1} (y_j - \bar{y})^2 \\ &\le \sum_{j=1}^{2m+1} (-1)^{j-1} f(y_j) - f\left(\sum_{j=1}^{2m+1} (-1)^{j-1} y_j\right), \end{aligned} \tag{2.11}$$

while for $f \in \Xi_2^c(I)$, the reversed inequalities in (2.11) hold.

The results in the next remark weakens the assumption (2.6) of Theorem 2.5.

REMARK 2.8. Let $n, m \in \mathbb{N}$ and let $\mathbf{x} = (x_1, \ldots, x_{2n+1}) \in [a,c]^{2n+1}$ and $\mathbf{y} = (y_1, \ldots, y_{2m+1}) \in [c,b]^{2m+1}$ be monotonic. If $f \in \Xi_1^c([a,b])$ and if A is such that $G(x) = f(x) - \frac{A}{2}x^2$ is W-concave on $[a,c]$ and W-convex on $[c,b]$, then from the proof of Theorem 2.5 we conclude that inequalities (2.8) and (2.9) hold and, combined together, they can be rewritten as

$$\begin{aligned} &\frac{A}{2}\left[\sum_{j=1}^{2m+1} (-1)^{j-1} (y_i - \bar{y})^2 - \sum_{i=1}^{2n+1} (-1)^{i-1} (x_i - \bar{x})^2\right] \\ &\le \sum_{j=1}^{2m+1} (-1)^{j-1} f(y_j) - f\left(\sum_{j=1}^{2m+1} (-1)^{j-1} y_j\right) \\ &- \left[\sum_{i=1}^{2n+1} (-1)^{i-1} f(x_i) - f\left(\sum_{i=1}^{2n+1} (-1)^{i-1} x_i\right)\right]. \end{aligned} \tag{2.12}$$

Therefore, for inequality (2.7) to hold it is enough to assume that

$$0 \le A\left[\sum_{j=1}^{2m+1} (-1)^{j-1} (y_i - \bar{y})^2 - \sum_{i=1}^{2n+1} (-1)^{i-1} (x_i - \bar{x})^2\right].$$

For example, this condition is satisfied in the following situation. Since G is W-concave on $[a,c]$ for $x + h_1 + h_2 \le c$ and W-convex on $[c,b]$ for $y + h_1 + h_2 \le b$, $h_1, h_2 > 0$, we have

$$0 \ge \Delta_{h_1} \Delta_{h_2} G(x) = \Delta_{h_1} \Delta_{h_2} f(x) - A h_1 h_2 \tag{2.13}$$

and

$$0 \le \Delta_{h_1} \Delta_{h_2} G(y) = \Delta_{h_1} \Delta_{h_2} f(y) - A h_1 h_2 \tag{2.14}$$

respectively.

(i) If $\Delta_{h_1}\Delta_{h_2} f(x) \geq 0$, then (2.13) implies $A \geq 0$ and in addition if

$$\sum_{i=1}^{2n+1} (-1)^{i-1} (x_i - \bar{x})^2 \leq \sum_{j=1}^{2m+1} (-1)^{j-1} (y_j - \bar{y})^2 ,$$

then the left hand side of (2.12) is nonnegative, which yields inequality (2.7).

or

(ii) If $\Delta_{h_1}\Delta_{h_2} f(y) \leq 0$, then from (2.14) we conclude that $A \leq 0$ and in addition if

$$\sum_{i=1}^{2n+1} (-1)^{i-1} (x_i - \bar{x})^2 \geq \sum_{j=1}^{2m+1} (-1)^{j-1} (y_j - \bar{y})^2 ,$$

then (2.7) follows again from the nonnegativity of the left hand side of (2.12).

If $f \in \Xi_2^c([a,b])$, then we have reversed inequality in (2.12) and by the same arguing as above, the reverse of the inequality (2.7) holds.

Remark 2.9. In fact, we have shown that under the assumptions of Remark 2.8, the following refinement of inequality (2.7) for $f \in \Xi_1^c([a,b])$ holds

$$\sum_{i=1}^{2n+1} (-1)^{i-1} f(x_i) - f\left(\sum_{i=1}^{2n+1} (-1)^{i-1} x_i\right) \leq \frac{A}{2} \sum_{i=1}^{2n+1} (-1)^{i-1} (x_i - \bar{x})^2$$

$$\leq \frac{A}{2} \sum_{j=1}^{2m+1} (-1)^{j-1} (y_i - \bar{y})^2 \leq \sum_{j=1}^{2m+1} (-1)^{j-1} f(y_j) - f\left(\sum_{j=1}^{2m+1} (-1)^{j-1} y_j\right),$$

while the reversed inequalities hold for $f \in \Xi_2^c([a,b])$.

Remark 2.10. We have stated Remarks 2.8 and 2.9 that extend the results given in Theorem 2.5. Analogous extension hold for all other results of this section, namely Theorems 2.3, 2.11, 2.12 and 2.14, but we will not state them explicitly.

The next theorem is a generalization of Theorem 1.8 and its proof, which is omitted, is analogous to the proof of Theorem 2.5.

Theorem 2.11. *Let* $x_k \in I \cap (-\infty, c]$ $(k = 1, \ldots, 2n+1)$ *and* $y_l \in I \cap [c, -\infty)$ $(l = 1, \ldots, 2m+1)$ *satisfy* (2.6) *and let* $\sum_{i=1}^{2k+1} (-1)^{i-1} x_i \in I \cap (-\infty, c]$ *for* $k = 1, \ldots, n$ *and* $\sum_{j=1}^{2l+1} (-1)^{j-1} y_j \in I \cap [c, \infty)$ *for* $l = 1, \ldots, m$.

(*i*) *If*

$$x_{2k} \le x_{2k+1}, \quad \sum_{i=1}^{2k} (-1)^{i-1} x_i \ge 0 \quad \textit{for} \quad k = 1, \dots, n, \tag{2.15}$$

and

$$y_{2l} \le y_{2l+1}, \quad \sum_{j=1}^{2l} (-1)^{j-1} y_j \ge 0 \quad \textit{for} \quad l = 1, \dots, m, \tag{2.16}$$

hold, then reverse of (2.7) *is valid for every* $f \in \Xi_1^c(I)$. *Further, if reverse of the inequalities in* (2.15) *and* (2.16) *are valid, then reverse of* (2.7) *is also valid for every* $f \in \Xi_1^c(I)$.

(*ii*) *If instead of* (2.15) *and* (2.16),

$$x_{2k} \le x_{2k+1}, \quad \sum_{i=1}^{2k} (-1)^{i-1} x_i \le 0 \quad \textit{for} \quad k = 1, \dots, n, \tag{2.17}$$

and

$$y_{2l} \le y_{2l+1}, \quad \sum_{i=1}^{2l} (-1)^{i-1} y_j \le 0, \quad \textit{for} \quad l = 1, \dots, m, \tag{2.18}$$

hold, then (2.7) *is valid for every* $f \in \Xi_1^c(I)$. *Further, if reverse of the inequalities in* (2.17) *and* (2.18) *are valid, then* (2.7) *is also valid for every* $f \in \Xi_1^c(I)$.

The next theorem is a generalization of Theorem 1.9 and its proof, which is omitted, is analogous to the proof of Theorem 2.5.

Theorem 2.12. *Let* $x_i, y_i \in I \cap (-\infty, c]$ $(i = 1, \dots, n)$ *and* $u_j, v_j \in I \cap [c, \infty)$ $(j = 1, \dots, m)$ *and let* $c_k = \sum_{i=1}^{k-1} (x_i - y_i)$ *for* $k = 2, \dots, n$ *and* $d_l = \sum_{j=1}^{l-1} (u_j - v_j)$ *for* $l = 2, \dots, m$. *Also assume that* $x_k + c_k \in I \cap (-\infty, c]$ *for all* k *and* $u_l + d_l \in I \cap [c, \infty)$ *for all* l.

(*i*) *If*

$$x_{k+1} \le y_k \ \textit{for} \ k = 1, \dots, n-1; \ u_{l+1} \le v_l \ \textit{for} \ l = 1, \dots, m-1, \tag{2.19}$$

$$\sum_{i=1}^{k} x_i \le \sum_{i=1}^{k} y_i \ \textit{for} \ k = 1, \dots, n-1; \ \sum_{j=1}^{l} u_j \le \sum_{j=1}^{l} v_j \ \textit{for} \ l = 1, \dots, m-1, \tag{2.20}$$

$$\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i, \ \sum_{j=1}^{m} u_j = \sum_{j=1}^{m} v_j \tag{2.21}$$

and

$$\sum_{i=1}^{n}\left(x_i^2-y_i^2\right)=\sum_{j=1}^{m}\left(u_j^2-v_j^2\right), \tag{2.22}$$

then for every $f\in\Xi_1^c(I)$*, the following inequality holds*

$$\sum_{i=1}^{n}\left(f(x_i)-f(y_i)\right)\geq\sum_{j=1}^{m}\left(f(u_j)-f(v_j)\right). \tag{2.23}$$

Furthermore, (2.23) *holds for every* $f\in\Xi_1^c(I)$ *if the reversed inequalities in* (2.19) *and* (2.20) *hold.*

(*ii*) *If* (2.19), (2.21) *and* (2.22) *hold and the reverse of the inequalities in* (2.20) *hold, then the reversed inequality in* (2.23) *holds for every* $f\in\Xi_1^c(I)$*. Furthermore, the same is true if* (2.20), (2.21) *and* (2.22) *hold and the reversed inequalities in* (2.19) *hold.*

A special case of Theorem 2.12 is given in the following remark.

Remark 2.13. Let $c\geq a_1\geq a_3\geq\ldots\geq a_{2n+1}\geq a$, $a_{2k}\geq 0$ $(k=1,\ldots,n)$, $a_k+a_{k+1}\in[a,c]$ $(k=1,\ldots,2n)$ and let $b\geq b_1\geq b_3\geq\ldots\geq b_{2m+1}\geq c$, $b_{2l}\geq 0$ $(l=1,\ldots,m)$, $b_l+b_{l+1}\in[c,b]$ $(l=1,\ldots,2m)$. Then the sequences

$$\begin{aligned}
&x_1=a_1,\quad x_k=a_{2k-2}+a_{2k-1}\text{ for }k=2,\ldots,n+1,\\
&y_k=a_{2k-1}+a_{2k}\text{ for }k=1,\ldots,n,\quad y_{n+1}=a_{2n+1},\\
&u_1=b_1,\quad u_k=b_{2k-2}+b_{2k-1}\text{ for }k=2,\ldots,m+1,\\
&v_k=b_{2k-1}+b_{2k}\text{ for }k=1,\ldots,m,\quad v_{m+1}=b_{2m+1}
\end{aligned}$$

satisfy conditions (2.19) and (2.20) for $k=1,\ldots,n$ and $l=1,\ldots,m$ and (2.21) for $i=1,\ldots,n+1$ and $j=1,\ldots,m+1$. Furthermore, for $i=1,\ldots,n+1$ and $j=1,\ldots,m+1$ condition (2.22) is equivalent to

$$\sum_{i=1}^{2n}(-1)^i a_i a_{i+1}=\sum_{j=1}^{2m}(-1)^j b_j b_{j+1}.$$

If all of these assumptions are satisfied, then for every $f\in\Xi_1^c([a,b])$, the following inequality

$$\begin{aligned}
&f(a_1+a_2)+f(a_3+a_4)+\ldots+f(a_{2n-1}+a_{2n})+f(a_{2n+1})\\
&\qquad-\left(f(a_1)+f(a_2+a_3)+\ldots+f(a_{2n}+a_{2n+1})\right)\\
&\quad\leq f(b_1+b_2)+f(b_3+b_4)+\ldots+f(b_{2m-1}+b_{2m})+f(b_{2m+1})\\
&\qquad\qquad-\left(f(b_1)+f(b_2+b_3)+\ldots+f(b_{2m}+b_{2m+1})\right)
\end{aligned}$$

holds.

The next theorem is a generalization of Theorem 1.13 and its proof, which is omitted, is analogous to the proof of Theorem 2.5.

THEOREM 2.14. *Let* $x_k \in I \cap (-\infty, c]$ *and* p_k $(k = 1, \ldots, n)$ *be real numbers such that* $x_k \geq 0$ *and* $p_k \geq 0$ *with* $P_k = \sum_{i=1}^{k} p_i$ $(k = 1, \ldots, n)$ *and let* $p_1 x_1, \sum_{k=1}^{n} p_k x_k, P_k x_k, P_{k-1} x_k \in I \cap (-\infty, c]$ *for all* $k = 2, \ldots, n$. *Let* $y_l \in I \cap [c, \infty)$ *and* q_l $(l = 1, \ldots, m)$ *be real numbers such that* $y_l \geq 0$ *and* $q_l \geq 0$ *with* $Q_l = \sum_{j=1}^{l} q_j$ $(l = 1, \ldots, m)$ *and let* $q_1 y_1, \sum_{l=1}^{m} q_l y_l, Q_l y_l, Q_{l-1} y_l \in I \cap [c, \infty)$ *for all* $l = 2, \ldots, m$. *Let* $f \in \Xi_1^c(I)$ *and*

$$\left(\sum_{k=1}^{n} p_k x_k\right)^2 - (p_1 x_1)^2 - \sum_{k=2}^{n} x_k^2 \left(P_k^2 - P_{k-1}^2\right)$$
$$= \left(\sum_{l=1}^{m} q_l y_l\right)^2 - (q_1 y_l)^2 - \sum_{l=2}^{m} y_l^2 \left(Q_l^2 - Q_{l-1}^2\right).$$

(*i*) *If the sequences* $(x_k,\, k = 1, \ldots, n)$ *and* $(y_l,\, l = 1, \ldots, m)$ *are non-increasing in* **p**-*weighted and* **q**-*weighted mean, respectively, then we have*

$$f\left(\sum_{k=1}^{n} p_k x_k\right) - f(p_1 x_1) - \sum_{k=2}^{n} \left(f(P_k x_k) - f(P_{k-1} x_k)\right)$$
$$\leq f\left(\sum_{l=1}^{m} q_l y_l\right) - f(q_1 y_1) - \sum_{l=2}^{m} \left(f(Q_l y_l) - f(Q_{l-1} y_l)\right). \tag{2.24}$$

(*ii*) *If the sequences* $(x_k,\, k = 1, \ldots, n)$ *and* $(y_l,\, l = 1, \ldots, m)$ *are non-decreasing in* **p**-*weighted and* **q**-*weighted mean, respectively, then we have*

$$f\left(\sum_{l=1}^{m} q_l y_l\right) - f(q_1 y_1) - \sum_{l=2}^{m} \left(f(Q_l y_l) - f(Q_{l-1} y_l)\right)$$
$$\leq f\left(\sum_{k=1}^{n} p_k x_k\right) - f(p_1 x_1) - \sum_{k=2}^{n} \left(f(P_k x_k) - f(P_{k-1} x_k)\right). \tag{2.25}$$

If $f \in \Xi_2^c(I)$, *then the reversed inequalities hold in* (2.24) *and* (2.25).

REFERENCES

[1] I. A. Baloch, J. Pečarić and M. Praljak, *Generalization of Levinson's inequality*, J. Math. Inequal. **9** (2) (2015), 571–586.

[2] S. Khalid, J. Pečarić and M. Praljak, *On an inequality of I. Perić*, Math. Commun. **19** (2014), 201–222.

[3] Z. Opial, *Sur une inégalité*, Ann. Polon. Math. **8** (1960), 29–32.

[4] J. Pečarić, F. Proschan and Y. L. Tong, Convex Functions, Partial Orderings, and Statistical Applications, Academic Press, New York, 1992.

[5] I. Perić, Converses of Hölder type inequalities, PhD dissertation, University of Zagreb, 1997 (in Croatian).
[6] G. Szegö, *Über eine Verallgemeinergung des Dirichletschen Integrals*, Math. Z. **52** (1950), 676–685.

3-Jensen-konveksnost u točki i 3-Wright-konveksnost u točki i povezani rezultati

Sadia Khalid, Josip Pečarić i Marjan Praljak

Sažetak. Uvedene su dvije nove klase konveksnih funkcija u točki i izvedeni su neki zanimljivi povezani rezultati.

Sadia Khalid, Josip Pečarić
Abdus Salam School of Mathematical Sciences
GC University
68-B, New Muslim Town, Lahore 54600, Pakistan
and
Faculty of Textile Technology
University of Zagreb
Prilaz baruna Filipovića 28a, 10000 Zagreb, Croatia
E-mail: saadiakhalid176@gmail.com, skhalid@ttf.hr
E-mail: pecaric@element.hr

Marjan Praljak
Faculty of Food Technology and Biotechnology
University of Zagreb
Pierottijeva 6, 10000 Zagreb, Croatia
E-mail: mpraljak@pbf.hr

Received: 6.3.2016.
Revised: 19.7.2016.