RAD HAZU. MATEMATIČKE ZNANOSTI
Vol. 20 = 528 (2016): 83-95

STEINER POINT OF A TRIANGLE IN AN ISOTROPIC PLANE

Ružica Kolar–Šuper, Zdenka Kolar–Begović and Vladimir Volenec

Abstract. The concept of the Steiner point of a triangle in an isotropic plane is defined in this paper. Some different concepts connected with the introduced concepts such as the harmonic polar line, Ceva's triangle, the complementary point of the Steiner point of an allowable triangle are studied. Some other statements about the Steiner point and the connection with the concept of the complementary triangle, the anticomplementary triangle, the tangential triangle of an allowable triangle as well as the Brocard diameter and the Euler circle are also proved.

1. Introduction

The isotropic (or Galilean) plane is a projective–metric plane, where the absolute consists of one line, the absolute line ω, and one point on that line, the absolute point Ω. The lines through the point Ω are *isotropic lines*, and the points on the line ω are isotropic points (the points at infinity). In an isotropic plane, the *distance* between the two points $P_i = (x_i, y_i)$ $(i = 1, 2)$ is defined by $P_1P_2 = x_2 - x_1$ and two lines with the equations $y = k_ix + l_i$ $(i = 1, 2)$ form the *angle* k_2-k_1. Two points P_1, P_2 with $x_1 = x_2$ are said to be *parallel*, and we shall say they are on the same isotropic line. Any isotropic line is *perpendicular* to any nonisotropic line. Two lines with $k_1 = k_2$ are *parallel*. For two parallel points P_1, P_2 their span is defined by $s(P_1, P_2) = y_2 - y_1$. The required facts about the isotropic plane can be found in [9] and [10].

A triangle is said to be *allowable* if none of its sides is isotropic. Each allowable triangle ABC can be set by a suitable choice of the coordinate system in the *standard position*, in which its circumscribed circle has the equation $y = x^2$, its vertices are the points

$$A = (a, a^2), \quad B = (b, b^2), \quad C = (c, c^2), \tag{1.1}$$

2010 *Mathematics Subject Classification.* 51N25.
Key words and phrases. Isotropic plane, Steiner point, Steiner ellipse, Ceva's triangle, orthic triangle.

and its sides BC, CA, AB have the equations

$$y = -ax - bc, \quad y = -bx - ca, \quad y = -cx - ab, \tag{1.2}$$

where

$$a + b + c = 0. \tag{1.3}$$

We shall say then that ABC is the *standard triangle* (Figure 1). To prove geometric facts for each allowable triangle it is sufficient to give a proof for the standard triangle (see [7]).

With the labels

$$p = abc, \quad q = bc + ca + ab \tag{1.4}$$

a number of useful equalities are proved in [7], as e.g. $a^2 = bc - q$, $q + 3bc = -(b-c)^2$, $2q - 3bc = (c-a)(a-b)$, $b^2c^2 + c^2a^2 + a^2b^2 = q^2$.

In an isotropic plane, the concept of Steiner ellipses of a triangle has been considered in [12]. In this paper we investigate the concept of the Steiner point of a triangle in an isotropic plane.

2. Steiner point of a triangle in an isotropic plane

In [12], the *Steiner point* of the allowable triangle ABC is defined as the fourth (in addition to A, B, C) common point S of the circumscribed circle and the circumscribed Steiner ellipse of that triangle (Figure 1). In the case of the standard triangle ABC this point is of the form

$$S = \left(-\frac{3p}{q}, \frac{9p^2}{q^2}\right). \tag{2.1}$$

For each point P, let $U = BC \cap AP$, $V = CA \cap BP$, $W = AB \cap CP$, and for each line $\mathcal{P}$, let $U' = BC \cap \mathcal{P}$, $V' = CA \cap \mathcal{P}$, $W' = AB \cap \mathcal{P}$. We shall say that the line $\mathcal{P}$ is the harmonic polar line of the point P with respect to the triangle ABC if the pairs of points: B, C; U, U' and C, A; V, V' and A, B; W, W' are in harmonicity.

Theorem 2.1. *The joint line of the centroid and the symmedian center of an allowable triangle is a harmonic polar line of its Steiner point (Figure 1).*

(In [2], Cesaro gives the statement in the Euclidean case.)

Proof. The line with the equation

$$y = \left(a - \frac{3p}{q}\right)x + \frac{3ap}{q} \tag{2.2}$$

obviously passes through the points $A = (a, a^2)$ and S from (2.1), so it is the line AS. From equation (2.2) and the equation $y = -ax - bc$ of the line BC

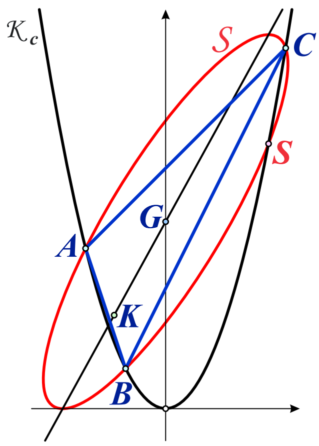

FIGURE 1

for the abscissa of the point $AS \cap BC$ we get the equation

$$\left(2a - \frac{3p}{q}\right) x = -\frac{3ap}{q} - bc,$$

i.e., $a(2q - 3bc)x = -bc(q + 3a^2)$, which due to

$$q + 3a^2 = q + 3(bc - q) = -(2q - 3bc)$$

has the solution

$$x = \frac{bc}{a}. \tag{2.3}$$

The line with the equation

$$y = \frac{2q^2}{9p}x - \frac{2}{3}q \tag{2.4}$$

passes through the points

$$G = \left(0, -\frac{2}{3}q\right), \qquad K = \left(\frac{3p}{2q}, -\frac{q}{3}\right),$$

owing to [7] and [6], the centroid and the symmedian center of the triangle ABC. From equation (2.4) and the equation $y = -ax - bc$ of the line BC for the abscissa of the point $GK \cap BC$ we get the following equation

$$\left(\frac{2q^2}{9p} + a\right) x = \frac{2}{3}q - bc.$$

Since

$$2q^2 + 9ap = 2q^2 + 9bc(bc - q) = 2q^2 - 9bcq + 9b^2c^2 = (2q - 3bc)(q - 3bc),$$

this equation has the solution $x = x'$, where

$$x' = \frac{3p}{q - 3bc}. \tag{2.5}$$

The points of the line BC with the abscissas b, c and x, x' are in harmonicity if and only if

$$(x-b)(x'-c) + (x-c)(x'-b) = 0,$$

i.e.,

$$2xx' + 2bc = (x+x')(b+c).$$

Owing to (2.3) and (2.5) we get

$$\begin{aligned} 2xx' + 2bc + a(x+x') &= \frac{1}{q-3bc}[6b^2c^2 + 2bc(q-3bc) + bc(q-3bc) + 3a^2bc] \\ &= \frac{3bc}{q-3bc}(q - bc + a^2) = 0. \end{aligned}$$

□

THEOREM 2.2. *The joint lines of the corresponding vertices of the anticomplementary and the tangential triangle of the allowable triangle ABC are the sides of Ceva's triangle of its Steiner point (Figure 2).*

PROOF. If $A_nB_nC_n$ and $A_tB_tC_t$ are the anticomplementary and the tangential triangle of the triangle ABC, respectively, then according to [7] and [1], we have e.g.

$$A_n = (-2a, -2bc), \qquad A_t = \left(-\frac{a}{2}, bc\right).$$

The line with the equation

$$y = \frac{2bc}{a}x + 2bc$$

obviously passes through these two points, so it is the line A_nA_t. From this equation and the equation $y = -bx - ca$ of the line CA we get the equation

$$\left(\frac{2bc}{a} + b\right)x = -2bc - ca,$$

i.e., $b(2c+a)x = -ac(a+2b)$ or $b(c-b)x = -ac(b-c)$ with the solution $x = \frac{ca}{b}$ for the abscissa of the point $CA \cap A_nA_t$. Analogously, the abscissa of the point $AB \cap A_nA_t$ is $x = \frac{ab}{c}$. The obtained abscissas are hence the abscissas of the points $BS \cap CA$ and $CS \cap AB$ because they are analogous to the abscissa (2.3) of the point $AS \cap BC$. □

THEOREM 2.3. *If D, E, F are the intersections of the corresponding sides of the complementary triangle $A_mB_mC_m$ and the orthic triangle $A_hB_hC_h$ of the allowable triangle ABC, then the points D, E, F lie on the polar lines of the points A, B, C with regard to the Euler circle of the triangle ABC, and the*

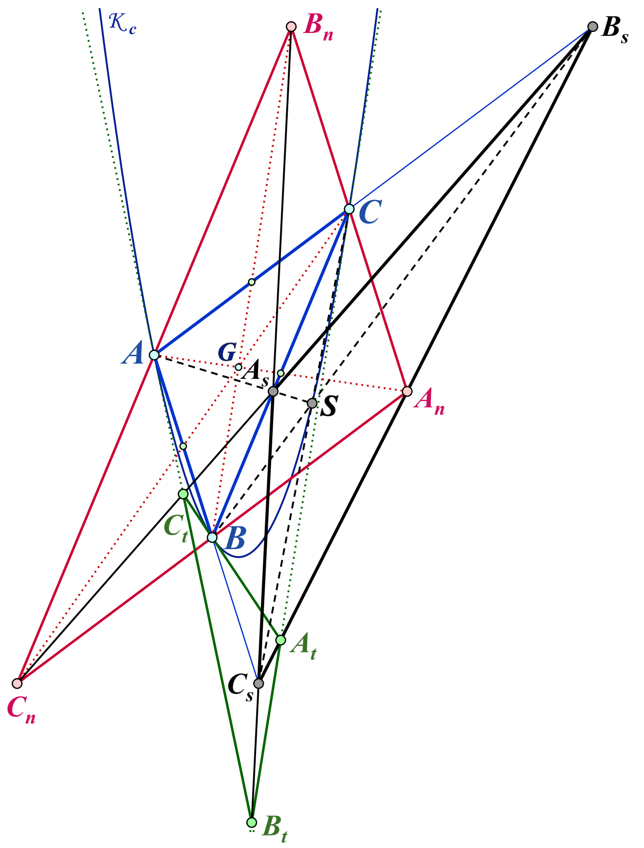

FIGURE 2

lines A_mD, B_mE, C_mF pass through the point S' which is a complementary point to the Steiner point of that triangle (Figure 3).

(In the Euclidean case, Godt gives this statement in [4] and [5].)

PROOF. According to [7], the lines B_mC_m and B_hC_h have the equations $y = -ax + \frac{bc}{2} - q$ and $y = 2ax + 2bc - q$. The point $D = \left(-\frac{bc}{2a}, a^2\right)$ lies on these lines because of $bc - q = a^2$. According to [1], the Euler circle of the triangle ABC has the equation $y = -2x^2 - q$, and with regard to that circle the polar of the point (x_o, y_o) has the equation $y + y_o = -4x_ox - 2q$. With $x_o = a$, $y_o = a^2$, there follows $y + a^2 = -4ax - 2q$, i.e., $y = -4ax - bc - q$, the equation of the polar of the point A with regard to the Euler circle. The

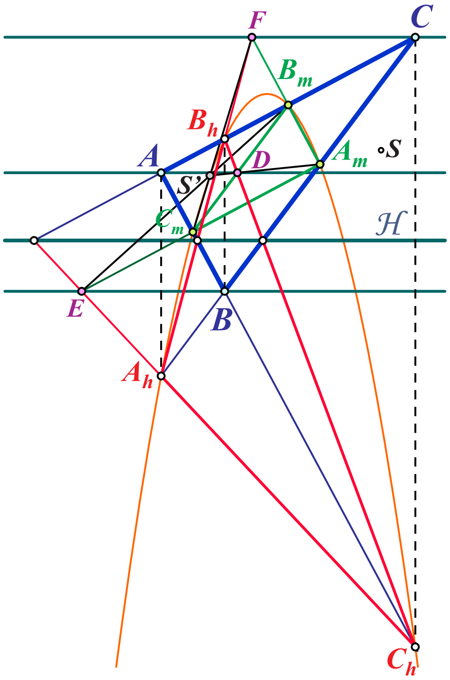

FIGURE 3

point D lies on that polar because of

$$-4a\left(-\frac{bc}{2a}\right)-bc-q=bc-q=a^2.$$

According to [7], we have the point

$$A_m=\left(-\frac{a}{2},-\frac{bc}{2}-\frac{q}{2}\right).$$

That point and the point D lie on the line with the equation

$$y=\left(a-\frac{3p}{q}\right)x-q+\frac{3}{2}bc-\frac{3}{2q}b^2c^2 \tag{2.6}$$

because of

$$\begin{aligned}\left(a-\frac{3p}{q}\right)\left(-\frac{a}{2}\right)-q+\frac{3}{2}bc-\frac{3}{2q}b^2c^2&=-\frac{a^2}{2}+\frac{3bc}{2q}(a^2-bc)-q+\frac{3}{2}bc\\&=-\frac{1}{2}(bc-q)-\frac{3bc}{2}-q+\frac{3}{2}bc\\&=-\frac{bc}{2}-\frac{q}{2},\end{aligned}$$

$$\left(a-\frac{3p}{q}\right)\left(-\frac{bc}{2a}\right)-q+\frac{3}{2}bc-\frac{3}{2q}b^2c^2=bc-q+\frac{3bc}{2q}\left(\frac{p}{a}-bc\right)=a^2.$$

Due to [7], the point

$$S'=\left(\frac{3p}{2q},-\frac{9p^2}{2q^2}-q\right)$$

is a complementary point to the point S. It also lies on the line (2.6) because of

$$\left(a-\frac{3p}{q}\right)\frac{3p}{2q}-q+\frac{3}{2}bc-\frac{3}{2q}b^2c^2=\frac{3bc}{2q}(a^2+q-bc)-\frac{9p^2}{2q^2}-q=-\frac{9p^2}{2q^2}-q.$$

□

The corresponding sides of the triangle ABC and its orthic triangle $A_hB_hC_h$ intersect at three points which lie on the same line. By analogy with the Euclidean case, this line is called an orthic axis of the observed triangle. In [7], it is shown that the orthic axis $\mathcal{H}$ of the standard triangle ABC has the equation $y=-\frac{q}{3}$.

The points A and D lie on the line with the equation $y=a^2$ and we get $AD||\mathcal{H}$ and similarly $BE||\mathcal{H}$, $CF||\mathcal{H}$ i.e.

Corollary 2.4. *With the labels from Theorem 2.3, the lines AD, BE, CF are parallel to the orthic axis of the triangle ABC (Figure 3).*

Theorem 2.5. *Lines parallel to the sides of the allowable triangle ABC through its Steiner point S meet its circumscribed circle again in the points S_a, S_b, S_c such that the Brocard diameter of the triangle ABC is a perpendicular bisector of the segments AS_a, BS_b, CS_c (Figure 4).*

(Thébault [11] gives this statement in the Euclidean case.)

Proof. The line with the eqaution

$$y=-ax+\frac{9p^2}{q^2}-\frac{3ap}{q}$$

is parallel to the line BC and it passes through the point

$$S=\left(-\frac{3p}{q},\frac{9p^2}{q^2}\right).$$

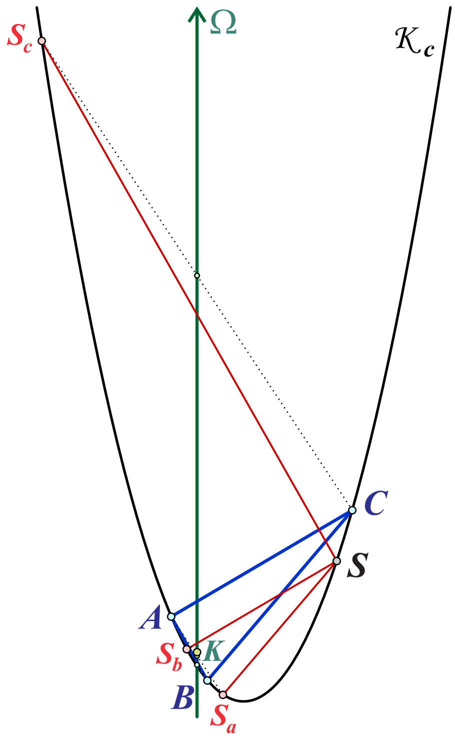

FIGURE 4

Besides that, it also passes through the point

$$S_a = \left(\frac{3p}{q} - a, \left(\frac{3p}{q} - a\right)^2\right)$$

because of

$$\left(\frac{3p}{q} - a\right)^2 + a\left(\frac{3p}{q} - a\right) - \frac{9p^2}{q^2} + \frac{3ap}{q} = 0.$$

The point S_a obviously lies on the circumscribed circle with the equation $y = x^2$. A perpendicular bisector of AS_a has the equation $x = \frac{3p}{2q}$, so due to [6], it is the Brocard diameter of the triangle ABC. □

THEOREM 2.6. *If the lines parallel to the lines BC, CA, AB through the points A, B, C meet the circumscribed circle of the allowable triangle ABC at the points A_{nh}, B_{nh}, C_{nh} again and if A_s, B_s, C_s are the intersections $BC \cap B_{nh}C_{nh}$, $CA \cap C_{nh}A_{nh}$, $AB \cap A_{nh}B_{nh}$, then the lines AA_s, BB_s, CC_s pass through the Steiner point S of the triangle ABC (Figure 5).*

(In [3], [8] and [13], this statement is given in the Euclidean case.)

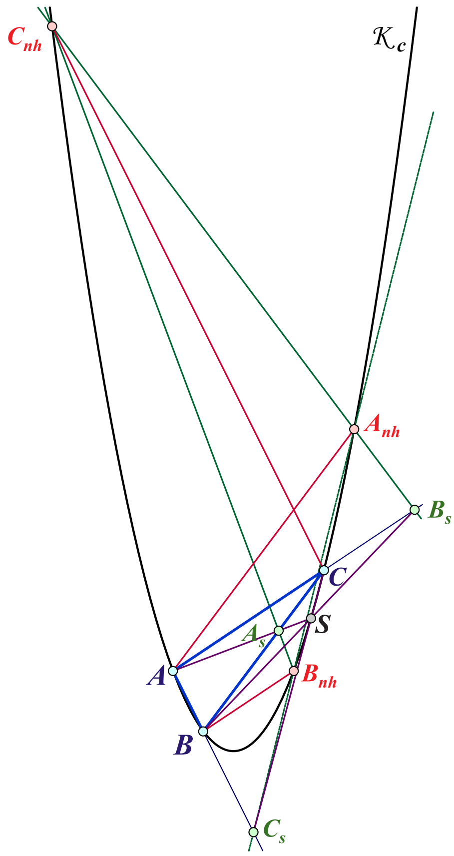

FIGURE 5

PROOF. Let us consider the points

$$A_{nh} = (-2a, 4a^2), \quad B_{nh} = (-2b, 4b^2), \quad C_{nh} = (-2c, 4c^2) \tag{2.7}$$

on the circumscribed circle of the triangle ABC. The line AA_{nh} has the slope $a - 2a = -a$, so it is parallel to the line BC. The line with the equation

$$y = 2ax - 4bc$$

passes through the points B_{nh} and C_{nh} since e.g. for the point B_{nh} we have $-4ab - 4bc = 4b^2$, so it is the line $B_{nh}C_{nh}$. The point

$$A_s = \left(\frac{bc}{a}, -2bc\right),$$

lies on that line and it also lies on the line BC with the equation $y = -ax - bc$, thus we have $A_s = BC \cap B_{nh}C_{nh}$. This point has $\frac{bc}{a}$ as its abscissa, and then, owing to the proof of Theorem 2.1, we get $A_s = AS \cap BC$. □

From the previous proof and the proof of Theorem 2.2 there follows:

Corollary 2.7. *Ceva's triangle $A_sB_sC_s$ of the Steiner point S of the standard triangle ABC has the vertices*

$$A_s = \left(\frac{bc}{a}, -2bc\right), \quad B_s = \left(\frac{ca}{b}, -2ca\right), \quad C_s = \left(\frac{ab}{c}, -2ab\right),$$

and its sides B_sC_s, C_sA_s, A_sB_s have the equations

$$y = 2\frac{bc}{a}x + 2bc, \quad y = 2\frac{ca}{b}x + 2ca, \quad y = 2\frac{ab}{c}x + 2ab.$$

In [7], it is shown that the orthic triangle $A_hB_hC_h$ of the triangle ABC has e.g. the vertex $A_h = (a, q - 2bc)$. According to (2.7), because of

$$2(q - 2bc) + 4a^2 = 2q - 4q = 3\left(-\frac{2}{3}q\right)$$

we have the equality

$$2A_h + A_{nh} = 3G.$$

Therefore, the point A_{nh} is anticomplementary to the point A_h, i.e., $A_{nh}B_{nh}C_{nh}$ is the orthic triangle of the anticomplementary triangle $A_nB_nC_n$ of the triangle ABC. For this reason, the statement of Theorem 2.6 is in fact the statement of Theorem 2.3 for the anticomplementary triangle $A_nB_nC_n$ of the triangle ABC.

From the proof of Theorem 2.6 there follows:

Corollary 2.8. *The orthic triangle $A_{nh}B_{nh}C_{nh}$ of the anticomplementary triangle of the standard triangle ABC has the vertices given by formulae (2.7) and sides given by equations*

$$y = 2ax - 4bc, \quad y = 2bx - 4ca, \quad y = 2cx - 4ab. \tag{2.8}$$

Theorem 2.9. *The equality*

$$\frac{AS^2}{BC^2} + \frac{BS^2}{CA^2} + \frac{CS^2}{AB^2} = 2$$

holds for the Steiner point S of the triangle ABC.

(Thébault [11] gives this statement in the Euclidean case.)

Proof. Due to (1.1) and (2.5), we get

$$AS = -\frac{3p}{q} - a = -\frac{a}{q}(q + 3bc) = \frac{a}{q}(b - c)^2,$$

and analogous equalities follow for BS and CS. Hence we obtain

$$\begin{aligned}\frac{AS^2}{BC^2}+\frac{BS^2}{CA^2}+\frac{CS^2}{AB^2}&=\frac{1}{q^2}[a^2(b-c)^2+b^2(c-a)^2+c^2(a-b)^2]\\&=\frac{1}{q^2}[2(b^2c^2+c^2a^2+a^2b^2)-2abc(a+b+c)]\\&=\frac{1}{q^2}\cdot 2q^2=2.\end{aligned}$$

□

COROLLARY 2.10. *With the same labels, the equality*

$$\frac{AS}{BC}+\frac{BS}{CA}+\frac{CS}{AB}=0$$

also holds.

THEOREM 2.11. *The circles, whose tangents at the vertices A, B, C of the allowable triangle ABC are parallel to the lines BC, CA, AB and which pass through the Steiner point S of that triangle, have radii (Figure 6)*

$$\frac{BC^2}{2\cdot CA\cdot AB},\quad \frac{CA^2}{2\cdot AB\cdot BC},\quad \frac{AB^2}{2\cdot BC\cdot CA}.$$

(Gallatly ([3]) gives incorrect expressions for these radii in the Euclidean case.)

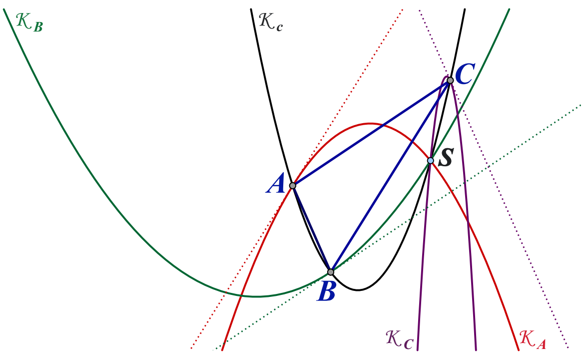

FIGURE 6

PROOF. The circle with the equation

$$(aq+3p)y=(3p-2aq)x^2+3a(aq-3p)x+9a^2p \tag{2.9}$$

passes through the point S since we get

$$(3p-2aq)\frac{9p^2}{q^2}-3a(aq-3p)\frac{3p}{q}+9a^2p=\frac{9p^2}{q^2}(aq+3p).$$

From the equation of this circle and the equation $y = -ax + 2a^2$ of the line parallel to the line BC which obviously passes through the point $A = (a, a^2)$, for the abscissa x of their intersection we obtain the following equation:

$$(aq + 3p)(-ax + 2a^2) = (3p - 2aq)x^2 + 3a(aq - 3p)x + 9a^2 p.$$

As we have

$$3a(aq - 3p) + a(aq + 3p) = -2a(3p - 2aq),$$
$$9a^2 p - 2a^2(aq + 3p) = a^2(3p - 2aq),$$

this equation has the form $(3p - 2aq)(x^2 - 2ax + a^2) = 0$ and it has a double solution $x = a$, thus the circle (2.9) and the considered line touch each other at the point A. The circle with the equation $y = ux^2 + vx + w$ has the radius $\frac{1}{2u}$. Because of that, the circle (2.9) has the radius

$$\frac{1}{2} \cdot \frac{aq + 3p}{3p - 2aq} = \frac{1}{2} \cdot \frac{q + 3bc}{3bc - 2q} = \frac{1}{2} \cdot \frac{(b - c)^2}{(c - a)(a - b)} = \frac{BC^2}{2 \cdot CA \cdot AB}.$$

□

Acknowledgements.
The authors are grateful to the referee for very useful suggestions.

References

[1] J. Beban–Brkić, R. Kolar–Šuper, Z. Kolar–Begović and V. Volenec, *On Feuerbach's theorem and a pencil of circles in the isotropic plane*, J. Geom. Graph. **10** (2006), 125–132.
[2] E. Cesàro, *Sur l'emploi des coordonnées barycentriques,* Mathesis **10** (1890), 177–190.
[3] W. Gallatly, *Note on the Steiner point*, Amer. Math. Monthly **15** (1908), 226–227.
[4] W. Godt, *Zur Figur des Feuerbach'schen Kreises,* Arch. Math. Phys. (2) **4** (1886), 436–441.
[5] W. Godt, *Ueber den Feuerbach'schen Kreis und die Steiner'sche Curve vierter Ordnung und dritter Classe*, Jahresber. Deutsch. Math. Verein. **4** (1894–95), 161–162.
[6] Z. Kolar–Begović, R. Kolar–Šuper, J. Beban–Brkić and V. Volenec, *Symmedians and the symmedian center of the triangle in an isotropic plane*, Math. Pannon. **17** (2006), 287–301.
[7] R. Kolar–Šuper, Z. Kolar–Begović, V. Volenec and J. Beban–Brkić, *Metrical relationships in a standard triangle in an isotropic plane*, Math. Commun. **10** (2005), 149–157.
[8] A. Mineur, *Sur le point de Steiner*, Mathesis **41** (1927), 414–415.
[9] H. Sachs, Ebene isotrope Geometrie, Vieweg–Verlag, Braunschweig–Wiesbaden, 1987.
[10] K. Strubecker, *Geometrie in einer isotropen Ebene*, Math. Naturwiss. Unterricht **15** (1962), 297–306, 343–351, 385–394.
[11] V. Thébault, *Sur les points de Steiner et de Tarry*, Ann. Soc. Sci. Bruxelles **64** (1950), 131–138; Mathesis **60** (1951), supplement, 8pp.
[12] V. Volenec, Z. Kolar–Begović and R. Kolar–Šuper, *Steiner's ellipses in an isotropic plane*, Math. Pannon. **21** (2010), 229–238.
[13] F. Weaver, *Sur deux points remarquables du triangle*, Mathesis **42** (1928), 293–303.

Steinerova točka trokuta u izotropnoj ravnini

Ružica Kolar–Šuper, Zdenka Kolar–Begović, Vladimir Volenec

Sažetak. Pojam Steinerove točke trokuta u izotropnoj ravnini je uveden u ovom članku. Proučavani su različiti pojmovi povezani s uvedenim pojmom, kao što su harmonička polara, Cevin trokut, komplementarna točka Steinerovoj točki dopustivog trokuta. Dokazane su tvrdnje o Steinerovoj točki i vezi s komplementarnim i antikomplementarnim trokutom, tangencijalnim trokutom dopustivog trokuta kao i Brocardovim dijametrom i Eulerovom kružnicom.

Ružica Kolar–Šuper
Faculty of Education
University of Osijek
31 000 Osijek
Croatia
E-mail: rkolar@foozos.hr

Zdenka Kolar-Begović
Department of Mathematics
University of Osijek
31 000 Osijek
Croatia
E-mail: zkolar@mathos.hr

Vladimir Volenec
Department of Mathematics
University of Zagreb
10 000 Zagreb
Croatia
E-mail: volenec@math.hr

Received: 18.9.2015.
Revised: 20.11.2015.