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ON PARABOLAS RELATED TO THE CYCLIC QUADRANGLE IN ISOTROPIC PLANE

Marija Šimić Horvath, Vladimir Volenec and Jelena Beban-Brkić

Abstract. The geometry of the cyclic quadrangle in the isotropic plane has been discussed in [11]. Therein, its diagonal triangle and diagonal points were introduced. Hereby, we turn our attention to parabolas inscribed to non tangential quadrilaterals of the cyclic quadrangle. Non tangential quadrilaterals of the cyclic quadrangle are formed by taking its four sides out of six. Several properties of these parabolas related to diagonal points of the cyclic quadrangle are studied.

1. Introduction

The isotropic plane is a real projective metric plane whose absolute figure is a pair (Ω, ω), Ω being an *absolute point* and ω an *absolute line* incident to it. If $T = (x : y : z)$ denotes any point in the plane presented in homogeneous coordinates then usually a projective coordinate system where $\Omega = (0 : 1 : 0)$ and the line ω with the equation $z = 0$ is chosen. For the isotropic plane the notation $I_2(\mathbb{R})$ will be used. Metric quantities and all the notions related to the geometry of the isotropic plane can be found in [9] and [8]. As the principle of duality is valid in the projective plane it is preserved in the isotropic plane as well.

The geometry of a non tangential quadrilateral and the geometry of a non cyclic quadrangle in the isotropic plane are discussed in [10] and [2], respectively. Further, the cyclic quadrangle in the isotropic plane is introduced in [11]. The diagonal triangle and the diagonal points of the cyclic quadrangle are presented there. In the chapters that follow we give several properties on parabolas related to the cyclic quadrangle, specially, to its diagonal triangle.

2. On a parabola in the isotropic plane

A complete classification of conics in $I_2(\mathbb{R})$ can be found in [8] and [1]. Some facts on conics in $I_2(\mathbb{R})$ and an interesting approach in analyzing them

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are given in [3]. In this section, we will introduce a parabola and its elements by use of a similar approach.

A *conic section* or shortly *conic* is the locus of all points $(x : y : z)$ in a real projective plane that are solutions of an equation

$$\varphi(x,y,z) = lx^2 + my^2 + nz^2 + 2oyz + 2pzx + 2qxy = 0, \quad l,m,n,o,p,q \in \mathbb{R}, \tag{2.1}$$

and

$$\delta = \det \begin{pmatrix} l & q & p \\ q & m & o \\ p & o & n \end{pmatrix} = lmn + 2opq - lo^2 - mp^2 - nq^2. \tag{2.2}$$

We say that the conic (2.1) is a *regular* or a *singular* one whether $\delta \neq 0$ or $\delta = 0$, respectively.

All tangent lines $\mathcal{T} = (X : Y : Z)$ to the regular conic $\Phi(X,Y,Z)$ satisfy

$$\Phi(X,Y,Z) = LX^2 + MY^2 + NZ^2 + 2OYZ + 2PZX + 2QXY = 0, \tag{2.3}$$

where

$$L = mn-o^2, M = nl-p^2, N = lm-q^2, O = pq-lo, P = qo-mp, Q = op-nq, \tag{2.4}$$

i.e. it is the conic equation expressed in line coordinates. These tangents form the second class curve.

Proposition 2.1. *The regular conic (2.1) has the tangential equation of the form (2.3) with the coefficients (2.4), and the regular conic with the tangential equation (2.3) has the point equation (2.1) with the coefficients given by*

$$\begin{array}{lll} MN - O^2 = \delta l, & NL - P^2 = \delta m, & LM - Q^2 = \delta n, \\ PQ - LO = \delta o, & QO - MP = \delta p, & OP - NQ = \delta q. \end{array} \tag{2.5}$$

Now, we transfer our attention on the isotropic plane. Any point (x, y) in $I_2(\mathbb{R})$ in homogeneous coordinates is represented by $(x : y : 1)$.

Theorem 2.2. *Any conic in the isotropic plane has the equation of the form*

$$lx^2 + my^2 + n + 2oy + 2px + 2qxy = 0. \tag{2.6}$$

The conic is a regular or a singular one depending on whether within the equality (2.2) $\delta \neq 0$ or $\delta = 0$ is valid. If the conic (2.6) has a pair of directrices their equations are of the form

$$Nx^2 - 2Px + L = 0, \tag{2.7}$$

while its other tangents $y = kx + t$ satisfy the equation

$$Lk^2 + 2Pkt + Nt^2 - 2Qk - 2Ot + M = 0 \tag{2.8}$$

with (2.4) being valid.

The conic (2.1) and the absolute line ω have common points satisfying

$$lx^2 + 2qxy + my^2 = 0. \tag{2.9}$$

(2.9) is a quadratic equation in the ratio $x : y$ with a discriminant that, owing to (2.4), equals

$$q^2 - lm = -N.$$

Therefore, the regular conic (2.1) is an ellipse if $N > 0$, a hyperbola if $N < 0$ and a parabola or a circle if $N = 0$.

Let us recall the theorem stated and proved in ([3] p.259).

THEOREM 2.3. *The conic with the equation (2.6) has the axis*

$$qx + my + o = 0, \tag{2.10}$$

and abscissae of its foci are solutions on x of

$$(q^2 - lm)x^2 + 2(oq - mp)x + o^2 - mn = 0. \tag{2.11}$$

According to Theorem 2.2, a pair of directrices of the conic (2.6) have the equation $Nx^2 - 2Px + L = 0$ that because of (2.4) turns into (2.11).

COROLLARY 2.4. *A parabola with the equation (2.6) (where $lm = q^2$) has the directrix*

$$2(oq - mp)x + o^2 - mn = 0. \tag{2.12}$$

Since in the case of parabola we find that $N = 0$, its equation in line coordinates equals

$$Lk^2 + 2Pkt - 2Qk - 2Ot + M = 0. \tag{2.13}$$

Out of (2.12) and because of (2.4) the equation of the directrix of a parabola can be written as

$$x = \frac{L}{2P}.$$

Applying (2.5) in (2.6) we get

$$-O^2x^2 - P^2y^2 + LM - Q^2 + 2(PQ - LO)y + 2(QO - MP)x + 2OPxy = 0. \tag{2.14}$$

The focus of the parabola has abscissa $x = \frac{L}{2P}$, and putting it into (2.14) one gets its ordinate as well. Whereby, we obtain

$$4P^4y^2 - 4P^2(2PQ - LO)y + O^2L^2 + 4Q^2P^2 - 4PQOL = 0,$$

that is,

$$(2P^2y - 2PQ + LO)^2 = 0.$$

This yields the coordinates of the focus of the parabola:

$$x = \frac{L}{2P}, \quad y = \frac{2PQ - LO}{2P^2}. \tag{2.15}$$

The conic equation in line coordinates as well as the form of the focus of a parabola are notions that will be needed in the study of cyclic quadrangles that follow in the sequel.

3. On a cyclic quadrangle in the isotropic plane

The cyclic quadrangle in the isotropic plane together with its diagonal triangle and diagonal points are introduced in [11]. In this section we first recall basic facts of the cyclic quadrangle described in [11]. Several properties of the cyclic quadrangle related to its inscribed parabolas will follow.

Lemma 3.1. *([11], p. 267) For any cyclic quadrangle $ABCD$ there exist four distinct real numbers a, b, c, d such that, in the defined canonical affine coordinate system, the vertices have the form*

$$A = (a, a^2), B = (b, b^2), C = (c, c^2), D = (d, d^2), \tag{3.1}$$

the circumscribed circle has the equation

$$y = x^2 \tag{3.2}$$

and the sides are given by

$$\begin{array}{llll} AB & \ldots\ y = (a+b)x - ab, & DA & \ldots\ y = (a+d)x - ad, \\ BC & \ldots\ y = (b+c)x - bc, & AC & \ldots\ y = (a+c)x - ac, \\ CD & \ldots\ y = (c+d)x - cd, & BD & \ldots\ y = (b+d)x - bd. \end{array} \tag{3.3}$$

Tangent lines $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$ to the circle (3.2) at the points (3.1) are of the form

$$\begin{array}{llll} \mathcal{A} & \ldots\ y = 2ax - a^2, & \mathcal{B} & \ldots\ y = 2bx - b^2, \\ \mathcal{C} & \ldots\ y = 2cx - c^2, & \mathcal{D} & \ldots\ y = 2dx - d^2. \end{array} \tag{3.4}$$

For the cyclic quadrangle $ABCD$ from Lemma 3.1 it is said to be in a *standard position* or it is a *standard quadrangle.*
Without loss of generality, choosing

$$s = a + b + c + d = 0, \tag{3.5}$$

the centroid takes the form

$$G = (0, -\frac{ab + ac + ad + bc + bd + cd}{2}). \tag{3.6}$$

Besides, the y-axis coincides with the diameter of the circle (3.2) and passes through G, while the x-axis is its tangent at a point parallel to G. The direction of the x-axis is called the *main direction of a cyclic quadrangle* ([11]).

A diagonal triangle of the cyclic quadrangle ([11], p. 268.) has the vertices $U = AC \cap BD$, $V = AB \cap CD$ and $W = AD \cap BC$ given by

$$\begin{aligned} U &= \Big(\frac{ac-bd}{a+c-b-d}, \frac{ac(b+d)-bd(a+c)}{a+c-b-d}\Big), \\ V &= \Big(\frac{ab-cd}{a+b-c-d}, \frac{ab(c+d)-cd(a+b)}{a+b-c-d}\Big), \\ W &= \Big(\frac{ad-bc}{a+d-b-c}, \frac{ad(b+c)-bc(a+d)}{a+d-b-c}\Big), \end{aligned} \tag{3.7}$$

while its sides are

$$\begin{aligned} UV &\ \ldots\ y = \frac{2(ad-bc)}{a+d-b-c}x - \frac{ad(b+c)-bc(a+d)}{a+d-b-c}, \\ UW &\ \ldots\ y = \frac{2(ab-cd)}{a+b-c-d}x - \frac{ab(c+d)-cd(a+b)}{a+b-c-d}, \\ VW &\ \ldots\ y = \frac{2(ac-bd)}{a+c-b-d}x - \frac{ac(b+d)-bd(a+c)}{a+c-b-d}. \end{aligned} \tag{3.8}$$

Before continuing, let us recall the next two notions, that of an allowable triangle and a quadrangle (see [2], [6], [10], [11]). A triangle is called *allowable* if none of its sides is isotropic. A cyclic quadrangle that has the allowable diagonal triangle is called *an allowable cyclic quadrangle*. Now on we will discuss new properties of the cyclic quadrangle related to its inscribed parabolas.

THEOREM 3.2. *Let $ABCD$ be an allowable cyclic quadrangle. Its four sides taken out of six sides, form three non tangential quadrilaterals. The foci of the parabolas inscribed to these non tangential quadrilaterals are the vertices of the orthic triangle of the diagonal triangle UVW.*

(For the Euclidean case see [5].)

PROOF. According to (2.13), the line equation of the parabola inscribed into a non tangential quadrilateral formed by the sides AB, BC, CD, and DA of the cyclic quadrangle $ABCD$ is

$$\begin{aligned} &(ac-bd)k^2 + (a-b+c-d)kt - [ac(a+c)-bd(b+d)]k \\ &\qquad - (a^2+c^2-b^2-d^2)t + a^2c^2 - b^2d^2 = 0. \end{aligned} \tag{3.9}$$

Out of the equality

$$\begin{aligned} &(ac-bd)(a+b)^2 - (a-b+c-d)(a+b)ab - (a+b)[ac(a+c)-bd(b+d)] \\ &\qquad + ab(a^2+c^2-b^2-d^2) + a^2c^2 - b^2d^2 = 0 \end{aligned}$$

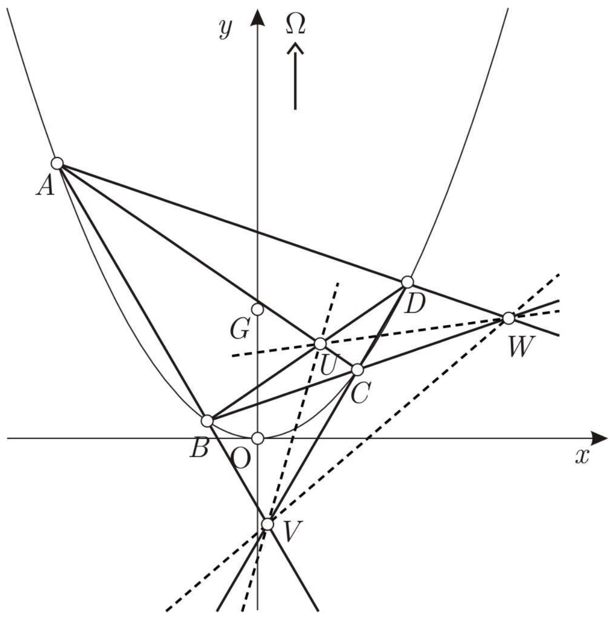

FIGURE 1. The standard cyclic quadrangle

it follows that AB is tangent to the parabola (3.9). The same feature can be proved for the sides BC, CD, and AD as well.

Referring (2.15) the parabola (3.9) has the focus

$$F_{AC,BD} = \left(\tfrac{ac-bd}{a-b+c-d}, \tfrac{(a-b+c-d)[ac(a+c)-bd(b+d)]-(ac-bd)(a^2+c^2-b^2-d^2)}{(a-b+c-d)^2}\right).$$

Due to the symmetry on a, b, c, d, the four lines AD, BC, AC, BD as well as the four lines AB, CD, AC, BD form another two non tangential quadrilaterals with foci

$$F_{AB,CD} = \left(\tfrac{ab-cd}{a+b-c-d}, \tfrac{(a+b-c-d)[ab(a+b)-cd(c+d)]-(ab-cd)(a^2+b^2-c^2-d^2)}{(a+b-c-d)^2}\right),$$
$$F_{AD,BC} = \left(\tfrac{ad-bc}{a+d-b-c}, \tfrac{(a+d-b-c)[ad(a+d)-bc(b+c)]-(ad-bc)(a^2+d^2-b^2-c^2)}{(a+d-b-c)^2}\right),$$

respectively.

The vertices of the orthic triangle we are searching for are the feet of the altitudes of the diagonal triangle UVW. We obtain exactly the foci $F_{AC,BD}, F_{AB,CD}$, $F_{AD,BC}$ standing for these vertices. Indeed, since

$$\frac{(bc-ad)\left(a^2-b^2-c^2+d^2\right)+(a-b-c+d)(ad(a+d)-bc(b+c))}{(a-b-c+d)^2}$$
$$= \frac{2(ad-bc)^2}{(a-b-c+d)^2} - \frac{-abc+abd+acd-bcd}{a-b-c+d}$$

the point $F_{AD,BC}$ is incident with the side UV. On the other hand, because of the form of its abscissa it is incident with the height $x = \dfrac{ad-bc}{a+d-b-c}$ dragged from the point W. □

The next two theorems are dealing with foci and parabolas described in the proof of the previous theorem. Their Euclidean cases can be found in [4].

Theorem 3.3. *Let $ABCD$ be an allowable cyclic quadrangle. Its diagonal points and the foci of the corresponding inscribed parabolas of its non tangential quadrilaterals are inverse points, i.e. their midpoint is the point of intersection of the directrix of the non tangential quadrilateral and the circumscribed circle of the cyclic quadrangle.*

Proof. Let us study the non tangential quadrilateral formed by AB, BC, CD, DA. By direct calculation, the midpoint of the straight line segment $UF_{AC,BD}$ has coordinates

$$\begin{aligned} x &= \frac{ad-bc}{a+d-b-c}, \\ y &= \frac{1}{2(a+c-b-d)^2}\{(a+c-b-d)[ac(b+d)-bd(a+c)] \\ &\quad + (a+c-b-d)[ac(a+c)-bd(b+d)] - (ac-bd)(a^2+c^2-b^2-d^2)\}. \end{aligned}$$

On the other hand, the directrix of the non-tangential quadrilateral is the directrix of its inscribed parabola as well ([10], p. 120). Therefore, the point of intersection of the directrix $x = \dfrac{ac-bd}{a+c-b-d}$ of the parabola (3.9) and the circle $y = x^2$ is the point

$$P = \left(\frac{ac-bd}{a+c-b-d}, \frac{(ac-bd)^2}{(a+c-b-d)^2}\right).$$

Because of

$$\begin{aligned} \tfrac{(ac-bd)^2}{(a-b+c-d)^2} = {} & \tfrac{(a-b+c-d)(ac(a+c)-bd(b+d))-(ac-bd)\left(a^2-b^2+c^2-d^2\right)}{2(a-b+c-d)^2} \\ & + \tfrac{ac(b+d)-bd(a+c)}{2(a-b+c-d)} \end{aligned}$$

the ordinate of the point P coincides with the ordinate in (3.10). □

Theorem 3.4. *Each bisector of the pairs of opposite sides AC, BD; AD, BC, and AB, CD of the allowable cyclic quadrangle $ABCD$ is the tangent of the inscribed parabola to corresponding non tangential quadrilateral, respectively.*

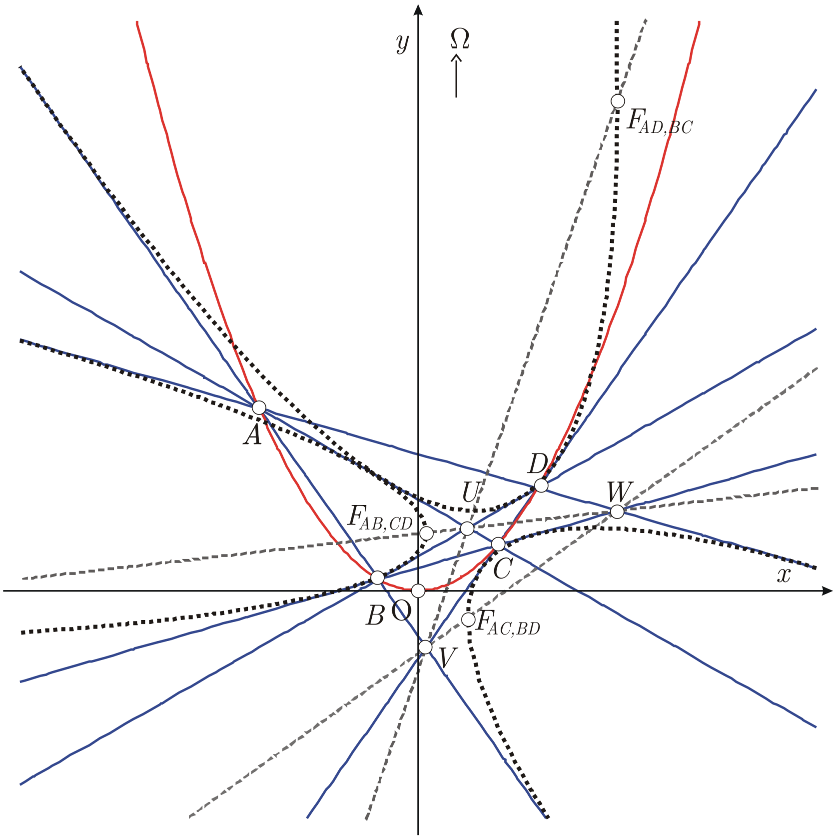

FIGURE 2. Parabolas inscribed to corresponding non tangential quadrilateral

PROOF. According to (3.3), AC and BD are given by $y = (a + c)x - ac$, $y = (b + d)x - bd$, respectively. Hence, their bisector is of the form

$$y = \frac{a + b + c + d}{2}x - \frac{ac + bd}{2}$$

i.e., referring to (3.5),

$$y = -\frac{ac + bd}{2}.$$

Owing to

$$(a^2 - b^2 + c^2 - d^2)(ac + bd) + 2(a^2c^2 - b^2d^2) = 0,$$

it follows that the bisector of the diagonals AC and BD is tangent to its inscribed parabola (3.9). ∎

The foci of the parabolas discussed in Theorem 3.2 have two more interesting properties. Taking for example $F_{AC,BD}$, the following is valid:

1. Referring to (3.7), W stands for the diagonal point, i.e. the point of intersection of the sides AD and BC.

   By direct calculation we obtain the circle circumscribed to the triangle ABW that is

   $$(a-b)y = (a+d-b-c)x^2 + (a+b)(c-d)x - ab(c-d).$$

   It can be easily verified that W is incident with this circle. On the other hand, owing to

   $$\begin{aligned}&\tfrac{(a-b)\left((bd-ac)\left(a^2-b^2+c^2-d^2\right)+(a-b+c-d)(ac(a+c)-bd(b+d))\right)}{(a-b+c-d)^2}\\ &= \tfrac{(a-b-c+d)(ac-bd)^2}{(a-b+c-d)^2} + \tfrac{(a+b)(c-d)(ac-bd)}{a-b+c-d} - ab(c-d)\end{aligned}$$

   the focus $F_{AC,BD}$ is incident with it as well.

2. Using the notation given above where $F_{AC,BD}$ stands for the focus of the inscribed parabola to the non tangential quadrilateral formed by AB, BC, CD, and DA and U, V, W are the diagonal points (3.7) of the cyclic quadrangle $ABCD$,

   $$\begin{aligned}d(F_{AC,BD}, A) \cdot d(F_{AC,BD}, C) &= d(F_{AC,BD}, B) \cdot d(F_{AC,BD}, D)\\ &= d(F_{AC,BD}, V) \cdot d(F_{AC,BD}, W).\end{aligned}$$

   Precisely,

   $$\begin{aligned}&\left(a - \frac{ac-bd}{a-b+c-d}\right)\left(c - \frac{ac-bd}{a-b+c-d}\right)\\ &= \left(b - \frac{ac-bd}{a-b+c-d}\right)\left(d - \frac{ac-bd}{a-b+c-d}\right),\end{aligned}$$

   and

   $$\begin{aligned}&\left(\frac{ad-bc}{a-b-c+d} - \frac{ac-bd}{a-b+c-d}\right)\left(\frac{ab-cd}{a+b-c-d} - \frac{ac-bd}{a-b+c-d}\right)\\ &= \left(a - \frac{ac-bd}{a-b+c-d}\right)\left(c - \frac{ac-bd}{a-b+c-d}\right).\end{aligned}$$

In a similar manner, the same considerations can be carried out for the foci $F_{AB,CD}$ and $F_{AD,BC}$. The analogous result in the Euclidean case can be found in [5].

At the end, motivated again by the Euclidean case, we present a theorem dealing with the parabolas inscribed to the triangles of the cyclic quadrangle.

For the Euclidean case see [7]. Before stating it we need the following definition: two lines are said to be *antiparallel* with respect to the third line if they form two opposite angles with this line.

THEOREM 3.5. *Let $ABCD$ be a cyclic quadrangle, and let the points A, B, C, and D be the foci of the parabolas inscribed to the triangles BCD, CDA, DAB, and ABC, respectively. The axes of these parabolas $\mathcal{O}_A, \mathcal{O}_B, \mathcal{O}_C$, and $\mathcal{O}_D$ and the corresponding tangents $\mathcal{A}, \mathcal{B}, \mathcal{C}$, and $\mathcal{D}$ of the circle (3.2) at the points A, B, C, and D are antiparallel with respect to the main direction of the quadrangle.*

PROOF. Let us consider, for example, the parabola inscribed to the triangle BCD with A as its focus. According to (2.10) its equation is

$$ak^2+kt-(ab+ac+ad)k-(b+c+d-a)t+(abc+abd+acd-bcd)=0. \tag{3.10}$$

Since,

$$a(c+d)^2-(ab+ac+ad+cd)(c+d)+cd(b+c+d-a)+(abc+abd+acd-bcd)=0$$

the line CD is a tangent of (3.10).

Now from (2.5), the coefficients l, m, n, o, p, q and the axis of the parabola can be obtained:

$$\begin{aligned} &qx+my+o=0, \quad \text{i.e.} \\ &\mathcal{O}_A \ \ldots \ y=(b+c+d-a)x+a(2a-b-c-d). \end{aligned}$$

Obviously, $\mathcal{O}_A$ and the tangent $\mathcal{A}$ from (3.4) are antiparallel with respect to the x-axis. As a matter of fact,

$$b+c+d-a+2a=0,$$

which together with (3.5) results in $0=0$.

The same properties of the other three parabolas inscribed to the triangles CDA, DAB, and ABC, can be proved in an analogous way. ∎

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O parabolama tetivnog četverovrha u izotropnoj ravnini

Marija Šimić Horvath, Vladimir Volenec i Jelena Beban-Brkić

Sažetak. Geometrija tetivnog četverovrha u izotropnoj ravnini proučavana je u [11] gdje se uvodi dijagonalni trokut i dijagonalne točke. U ovom članku stavlja se naglasak na parabole upisane netangencijalnim četverostranima tetivnog četverovrha. Četiri stranice od postojećih šest stranica tetivnog četverovrha formiraju netangencijalni četverostran. Proučavaju se svojstva spomenutih parabola vezana za dijagonalne točke tetivnog četverovrha.

M. Šimić Horvath
Faculty of Architecture
University of Zagreb
10 000 Zagreb, Croatia
E-mail: `marija.simic@arhitekt.hr`

V. Volenec
Department of Mathematics
University of Zagreb
10 000 Zagreb, Croatia
E-mail: `volenec@math.hr`

J. Beban-Brkić
Faculty of Geodesy
University of Zagreb
10 000 Zagreb, Croatia
E-mail: `jbeban@geof.hr`

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