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BOUNDARY PERTURBATION FOR THE DIRICHLET BOUNDARY VALUE PROBLEM

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Abstract. We study the effects of small boundary perturbations on the solutions of the boundary value problems posed in such domains. We start from the domain Ω and then perturb its boundary by the product of a small parameter and some smooth function. The zeroth order approximation is simply the same boundary value problem posed in domain Ω, but the first order corrector is also found, containing some interesting effects.

1. Introduction

Perturbation theory [8, 13] has an important role in many branches of mathematical physics and especially in fluid dynamics. Because of the common non-linearity issues, for the simplification of the mathematical model one usually uses idealized geometries for the flow domains. These geometries can be close to or far from the shapes in practice. It is of a practical interest to keep the approximation and the approximate solution close to the exact one. With the introduction of a small parameter as the perturbation quantity, e.g. a coordinate perturbation for the domain change [5, 7, 9, 11], the approximation becomes more accurate as the parameter tends to zero. The book [7] discusses how perturbation of the boundary stays rather neglected topic in the study of PDEs, due to the trivial change of variables followed by long and tedious calculations. In [11], the Stokes resolvent system is considered in a domain with a perturbed boundary and the asymptotic expansion is justified by the layer potential techniques. Authors also proved the continuity of the solution with respect to the small parameter. The application of the layer potential theory is also found in [1], dealing with the asymptotic expansions of boundary perturbations of steady-state voltage potentials resulting from small perturbations of the shape of a conductivity inclusion.

More general setting may be found in the papers [5, 6] on the topic of parabolic and elliptic boundary value problems on varying domains. Significant

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part deals with the discussion and formulation of the domain convergence

$$\Omega_n \longrightarrow \Omega,$$

after which the author shows how solutions behave as a sequence of the observed domains approaches an open set.

A useful application of the asymptotic methods may also be found in the elasticity theory. The approach within some of the papers is different from ours, so caution is needed in the use of term boundary perturbation. For example, in [2] boundary perturbations are referred to in a sense of effects on the boundary data of the solution in the presence of internal small defects (elastic linear cracks). Similar case happens in [4], with the asymptotic expansion of the boundary displacement in the presence of elastic inhomogeneities, and in [14] with perturbations in the electromagnetic fields. These are the cases when the perturbations are observed as a direct consequence of the underlying dynamic systems. On the other hand, from our point of view, the external perturbation of boundary affects the solutions of boundary value problems.

If we assume that the domain is almost ideal in shape, the question we are posing is about the influence of small irregularities and imperfections of the real domain. In the present paper, we study the effects of small boundary perturbations on the solutions of the boundary value problems posed in such domains. We start from the domain Ω and then first perturb the geometry of its boundary for the value of $\varepsilon h(x)$ where ε is a small parameter that controls the level of perturbation and $h \in C^\infty(\mathbf{R})$ is some smooth function, given in advance. We consider the problem both in rectangular and polar coordinates, but the perturbation appears only in the direction normal to the boundary. The zeroth order approximation is simply the same boundary value problem posed in the domain Ω, which is an obvious choice. However, a corrector of order ε can be found, containing some interesting effects. It turns out that the perturbation is not local, i.e. it does not affect the solution only in the vicinity of the boundary, but in the whole domain.

2. Perturbation method in rectangular coordinates

2.1. *Description of the problem.* We consider the domain with a perturbed boundary in rectangular coordinates. For example, if we start with the unit square $]0,1[\times]0,1[$ in $\mathbf{R}^2$, the perturbation of the boundary can be obtained depending on the small parameter ε and the given function $h(x)$ as in

$$\Omega_\varepsilon = \left\{ (x,y) \in \mathbf{R}^2 \,;\, 0 < x < 1 \,,\, 0 < y < 1 - \varepsilon\, h(x) \right\} \ .$$

In this example, the perturbation is present only on the top side of the original square. Let us consider first the Poisson equation in Ω_ε with the given function f and the homogeneous Dirichlet boundary condition

$$-\Delta u^\varepsilon = f \ \text{ in } \Omega_\varepsilon \ , \ u^\varepsilon = 0 \ \text{ on } \ \partial\Omega_\varepsilon \ . \tag{2.1}$$

The numerical solution to this type of equation (using FreeFem++ solver) for values $\varepsilon = 0.01$, $h(x) = \sin(9\pi x)$ and $f(x, y) = y \cos x$ is presented in Figure 1.

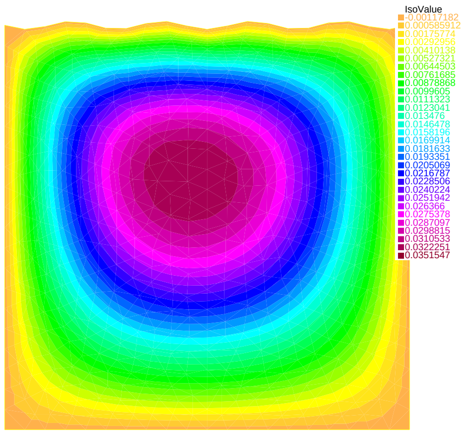

FIGURE 1. Solution to the Poisson equation in a rectangular Ω_ε

2.2. *Change of variable.* Let us look closer to the equation (2.1) and the domain Ω_ε. We want to eliminate the dependence of boundary and by that the domain itself on the small parameter ε. That is why we introduce a new variable

$$z = \frac{y}{1 - \varepsilon\, h(x)} \,. \tag{2.2}$$

We should calculate and simultaneously express $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$ in the new variables:

$$\frac{\partial z}{\partial x} = \frac{\varepsilon y\, h'}{(1 - \varepsilon\, h)^2} = \frac{\varepsilon z\, h'}{1 - \varepsilon\, h} \,, \qquad \frac{\partial z}{\partial y} = \frac{1}{1 - \varepsilon\, h} \,.$$

Now, in the new variables (x, z) the domain Ω_ε becomes a square $\Omega =]0, 1[^2$. Along with the intended domain change, the change of variables affects functions and partial derivatives, as well. We change the notation of the solution to (2.1) from $u^\varepsilon(x,y)$ to $U^\varepsilon(x,z)$, but keep the same notation for f, for simplicity. The derivatives in the new variables are changing as follows:

$$
\begin{aligned}
\frac{\partial}{\partial x} \quad &\rightsquigarrow \frac{\partial}{\partial x} + \frac{\partial z}{\partial x}\,\frac{\partial}{\partial z} = \frac{\partial}{\partial x} + \varepsilon\,\frac{y\,h'}{(1-\varepsilon\,h)^2}\,\frac{\partial}{\partial z} = \frac{\partial}{\partial x} + \varepsilon\,\frac{z\,h'}{1-\varepsilon\,h}\,\frac{\partial}{\partial z} = \\
&= \frac{\partial}{\partial x} + \varepsilon\,z\,h' \sum_{k=0}^{\infty} \varepsilon^k\;h^k\,\frac{\partial}{\partial z}\\
\frac{\partial}{\partial y} \quad &\rightsquigarrow \frac{\partial z}{\partial y}\,\frac{\partial}{\partial z} = \frac{1}{1-\varepsilon\,h}\,\frac{\partial}{\partial z} = \sum_{k=0}^{\infty} \varepsilon^k\;h^k\,\frac{\partial}{\partial z}
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial^2}{\partial x^2} \quad &\rightsquigarrow \left(\frac{\partial}{\partial x} + \frac{\partial z}{\partial x}\,\frac{\partial}{\partial z}\right)\left(\frac{\partial}{\partial x} + \frac{\partial z}{\partial x}\,\frac{\partial}{\partial z}\right)\\
&= \left(\frac{\partial}{\partial x} + \frac{\varepsilon\,z\,h'}{1-\varepsilon\,h}\,\frac{\partial}{\partial z}\right)\left(\frac{\partial}{\partial x} + \frac{\varepsilon\,z\,h'}{1-\varepsilon\,h}\,\frac{\partial}{\partial z}\right)\\
&= \frac{\partial^2}{\partial x^2} + \left(\frac{\varepsilon\,z\,h''}{1-\varepsilon\,h} + \frac{\varepsilon^2\,z\,(h')^2}{(1-\varepsilon\,h)^2}\right)\frac{\partial}{\partial z} + 2\,\frac{\varepsilon z\,h'}{1-\varepsilon\,h}\,\frac{\partial^2}{\partial z \partial x}\\
&\quad + \frac{\varepsilon^2 z\,(h')^2}{(1-\varepsilon\,h)^2}\,\frac{\partial}{\partial z} + \frac{\varepsilon^2 z^2\,(h')^2}{(1-\varepsilon\,h)^2}\,\frac{\partial^2}{\partial z^2}\\
&= \frac{\partial^2}{\partial x^2} + \varepsilon\,\frac{z\,h''}{1-\varepsilon\,h}\,\frac{\partial}{\partial z} + 2\,\varepsilon^2\frac{z\,(h')^2}{(1-\varepsilon\,h)^2}\,\frac{\partial}{\partial z} + \varepsilon\,\frac{2z\,h'}{1-\varepsilon\,h}\,\frac{\partial^2}{\partial z \partial x}\\
&\quad + \varepsilon^2\frac{z^2\,(h')^2}{(1-\varepsilon\,h)^2}\,\frac{\partial^2}{\partial z^2} =\\
&= \frac{\partial^2}{\partial x^2} + \varepsilon\left(z\,h''\,\frac{\partial}{\partial z} + 2\,z\,h'\,\frac{\partial^2}{\partial z\,\partial x}\right)\\
&\quad + \sum_{k=2}^{\infty} \varepsilon^k h^{k-2}\Bigg[\left(zh''h + 2(k-1)z(h')^2\right)\frac{\partial}{\partial z} + 2zh'h\frac{\partial^2}{\partial z \partial x}\\
&\quad + (zh')^2(k-1)\,\frac{\partial^2}{\partial z^2}\Bigg]
\end{aligned}
$$

$$
\frac{\partial^2}{\partial y^2} \rightsquigarrow \frac{1}{(1-\varepsilon\,h)^2}\,\frac{\partial^2}{\partial z^2} = \sum_{k=0}^{\infty} \varepsilon^k\,(k+1)\,h^k\,\frac{\partial^2}{\partial z^2}\;.
$$

To summarize, we write down the first few orders of approximations of the derivatives:

$$\begin{aligned}
\frac{\partial}{\partial x} &\rightsquigarrow \frac{\partial}{\partial x} + \varepsilon \cdot z\, h' \,\frac{\partial}{\partial z} + \varepsilon^2 \cdot z\, h\, h' \,\frac{\partial}{\partial z} + O(\varepsilon^3)\\
\frac{\partial}{\partial y} &\rightsquigarrow \frac{\partial}{\partial z} + \varepsilon \cdot h \,\frac{\partial}{\partial z} + \varepsilon^2 \cdot h^2 \,\frac{\partial}{\partial z} + O(\varepsilon^3)\\
\frac{\partial^2}{\partial x^2} &\rightsquigarrow \frac{\partial^2}{\partial x^2} + \varepsilon \cdot \left(z\, h'' \,\frac{\partial}{\partial z} + 2z\, h' \,\frac{\partial^2}{\partial z \partial x}\right) + O(\varepsilon^2)\\
\frac{\partial^2}{\partial y^2} &\rightsquigarrow \frac{\partial^2}{\partial z^2} + \varepsilon \cdot 2\, h \,\frac{\partial^2}{\partial z^2} + \varepsilon^2 \cdot 3\, h^2 \,\frac{\partial^2}{\partial z^2} + O(\varepsilon^3)\ .
\end{aligned}$$

2.3. *Asymptotic expansion.* We would like to construct an approximation to the solution of the boundary value problem, the approximation that is asymptotically equal to the original one with respect to the parameter ε. We are going to use the definition from [8].

DEFINITION 2.1. *Given $f(\varepsilon)$ and $\varphi(\varepsilon)$, we say that $\varphi(\varepsilon)$ is an asymptotic approximation to $f(\varepsilon)$ as $\varepsilon \searrow 0$ whenever $f(\varepsilon) = \varphi(\varepsilon) + o(\varphi)$ as $\varepsilon \searrow 0$. In that case we write*

$$f \sim \varphi \quad \textit{as} \quad \varepsilon \searrow 0.$$

We look for the solution $u^\varepsilon(x,y) = U^\varepsilon(x,z)$ to (2.1) in the form of an asymptotic expansion, a formal power series in the small parameter known as a perturbation series,

$$U^\varepsilon(x,z) = U_0 + \varepsilon\, U_1 + \varepsilon^2\, U_2 + \varepsilon^3\, U_3 + \cdots \tag{2.3}$$

for certain functions $U_k(x,z)$, yet to be determined. For simplicity, after the change of variables, we kept the same notation f for the function on the right hand side of (2.1). Depending on the given formula for f, it needs to be expanded in the same manner

$$f(x,z) = f_0(x,z) + \varepsilon\, f_1(x,z) + \varepsilon^2\, f_2(x,z) + \cdots \tag{2.4}$$

By introducing the expansions (2.3) and (2.4) into the Poisson equation (2.1), changing the derivatives in the new variables and collecting equal powers of ε, we get

$$\begin{aligned}
-\Bigg\{ &\left(\frac{\partial^2 U_0}{\partial x^2} + \frac{\partial^2 U_0}{\partial z^2} \right) + \varepsilon \cdot \left(\frac{\partial^2 U_1}{\partial x^2} + \frac{\partial^2 U_1}{\partial z^2} + z\, h'' \,\frac{\partial U_0}{\partial z} + 2\, z\, h' \,\frac{\partial^2 U_0}{\partial z\, \partial x} + 2\, h \,\frac{\partial^2 U_0}{\partial z^2} \right)\\
&+ \sum_{k=2}^{\infty} \varepsilon^k \cdot \Bigg[\frac{\partial^2 U_k}{\partial x^2} + \frac{\partial^2 U_k}{\partial z^2} + \sum_{\ell=2}^{k} h^{\ell-2} \Bigg[\Big(z h'' h + 2(\ell-1) z (h')^2 \Big) \frac{\partial U_{k-\ell}}{\partial z}\\
&+ 2 z h' h \frac{\partial^2 U_{k-\ell}}{\partial z \partial x} + (z h')^2 (\ell - 1)\, \frac{\partial^2 U_{k-\ell}}{\partial z^2} \Bigg]\Bigg]\Bigg\} = f_0 + \varepsilon\, f_1 + \varepsilon^2\, f_2 + \cdots \ \text{ in } \Omega\ .
\end{aligned}$$

Denoting

$$\Delta_{xz} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} \ ,$$

we end up with a recurrent system of problems for U_k, $k = 0, 1, \dots$:

$$-\Delta_{xz} U_0 = f_0 \ \text{ in } \Omega \ , \ \ U_0 = 0 \ \text{ on } \partial\Omega \tag{2.5}$$

$$\begin{aligned} -\Delta_{xz} U_1 = f_1 + z\, h'' \frac{\partial U_0}{\partial z} + 2\, z\, h' \ \frac{\partial^2 U_0}{\partial z\, \partial x} + 2h \frac{\partial^2 U_0}{\partial z^2} \ \text{ in } \Omega, \\ U_1 = 0 \ \text{ on } \partial\Omega \end{aligned} \tag{2.6}$$

$$\begin{aligned} -\Delta_{xz} U_k = f_k + \sum_{\ell=2}^{k} h^{\ell-2} \Bigg[\Big(z\, h''\, h + 2(\ell-1)\, z\, (h')^2 \Big) \frac{\partial U_{k-\ell}}{\partial z} + \\ + 2z\, h'\, h \frac{\partial^2 U_{k-\ell}}{\partial z \partial x} + (z\, h')^2\, (\ell-1) \frac{\partial^2 U_{k-\ell}}{\partial z^2} \Bigg] \ \text{ in } \Omega, \\ U_k = 0 \ \text{ on } \partial\Omega, \ k = 2, 3, \dots \end{aligned} \tag{2.7}$$

THEOREM 2.2. *Let $u^\varepsilon(x,y)$ be the solution to the boundary value problem (2.1) and $U_k(x,z)$, $(k = 0, 1, \ldots)$ obtained from (2.5), (2.6) and (2.7). Then for $m \in \mathbf{N}$ we have*

$$u^\varepsilon(x,y) \ \sim \ \sum_{k=0}^{m} \varepsilon^k\, U_k \left(x, \frac{y}{1 - \varepsilon\, h} \right)$$

as $\varepsilon \searrow 0$.

PROOF. Let U^ε be the expansion (2.3) with U_k obtained from (2.5), (2.6) and (2.7). Since

$$-\Delta_{xz} \left(U^\varepsilon - \sum_{k=0}^{m} \varepsilon^k\, U_k \right) = O(\varepsilon^{m+1}) \ \text{ in } \Omega \ , \quad U^\varepsilon - \sum_{k=0}^{m} \varepsilon^k\, U_k = 0 \text{ on } \partial\Omega \ ,$$

due to the maximum principle we have

$$\left| U^\varepsilon - \sum_{k=0}^{m} \varepsilon^k\, U_k \right| = O(\varepsilon^{m+1}) = o(\varepsilon^m) \ .$$

According to Definition 2.1, this means that $U^\varepsilon \sim \sum_{k=0}^{m} \varepsilon^k\, U_k$. The result now follows from the fact that

$$u^\varepsilon(x,y) = U^\varepsilon \left(x, \frac{y}{1 - \varepsilon\, h} \right) \ .$$

□

3. Perturbation method in polar coordinates

3.1. *Star-shaped domain.* We can generalize the idea on a star-shaped domain Ω_ε that can be described in the polar coordinates (r,φ) as

$$\Omega_\varepsilon = \left\{\, (r,\varphi) \; ; \; 0 \le \varphi < 2\pi \; , \; 0 \le r < g(\varphi) - \varepsilon\, h(\varphi) \,\right\}$$

which can be seen as a small perturbation of the domain

$$\Omega = \left\{\, (r,\varphi) \; ; \; 0 \le \varphi < 2\pi \; , \; 0 \le r < g(\varphi) \,\right\} \; .$$

We can assume that $g(\varphi), h(\varphi) \in C^\infty(\mathbf{R})$, that both are periodic with period 2π, and the solution to (2.1) is now $u^\varepsilon(r,\varphi)$. One example and the visualization of the numerical solution to the Poisson equation in this type of Ω_ε for $f(r,\varphi) = 1$ is shown in Figure 2. To keep this example simple, the unperturbed domain is taken to be a unit circle, so $g(\varphi) = 1$.

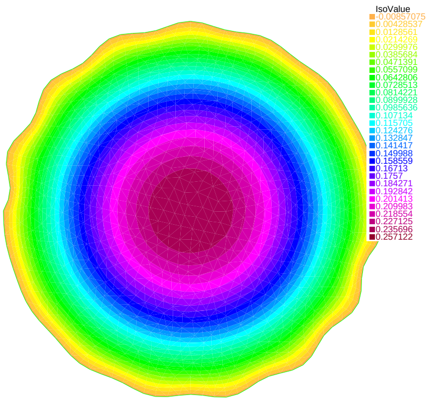

Figure 2. Solution to the Poisson equation in a star-shaped domain Ω_ε

In this type of problem we introduce a new variable

$$\rho = \frac{g(\varphi)\; r}{g(\varphi) - \varepsilon h(\varphi)} \; . \tag{3.1}$$

This transformation maps Ω_ε on Ω, but the Laplace operator in the new variables changes again and depends on the small parameter ε explicitly. First of all,

$$\frac{\partial\rho}{\partial r} = \frac{g(\varphi)}{g(\varphi)-\varepsilon h(\varphi)} = \frac{1}{1-\varepsilon\frac{h}{g}} = \sum_{k=0}^{\infty}\varepsilon^k \left(\frac{h}{g}\right)^k$$

$$\begin{aligned}\frac{\partial\rho}{\partial\varphi} &= \frac{g'(g-\varepsilon h)-g\,(g'-\varepsilon h')}{(g-\varepsilon h)^2}\,r = \varepsilon\,\frac{gh'-hg'}{(g-\varepsilon h)^2}\,r \\ &= \varepsilon\,\frac{gh'-hg'}{g^2}\;\frac{1}{(1-\varepsilon\frac{h}{g})^2}\,r = \varepsilon\,\left(\frac{h}{g}\right)'\frac{1}{1-\varepsilon\frac{h}{g}}\;\rho \\ &= \rho\,\left(\frac{h}{g}\right)'\sum_{k=0}^{\infty}\varepsilon^{k+1}\,\left(\frac{h}{g}\right)^k \quad .\end{aligned}$$

The Laplace operator in polar coordinates reads

$$\Delta = \frac{1}{r}\frac{\partial}{\partial r}\left(r\,\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial\varphi^2} \quad . \tag{3.2}$$

Denoting $\mathcal{U}^\varepsilon(\rho,\varphi) = u^\varepsilon(r,\varphi)$, in the new variables we get the expression

$$\begin{aligned}\Delta\mathcal{U}^\varepsilon = &\left(1-\varepsilon\frac{h}{g}\right)^{-2}\left\{\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\,\frac{\partial\mathcal{U}^\varepsilon}{\partial\rho}\right)+\frac{1}{\rho^2}\,\frac{\partial^2\mathcal{U}^\varepsilon}{\partial\varphi^2}+\frac{2\varepsilon}{\rho\left(1-\varepsilon\frac{h}{g}\right)}\,\left(\frac{h}{g}\right)'\frac{\partial^2\mathcal{U}^\varepsilon}{\partial\rho\partial\varphi}\right.\\ &\left.+\frac{\varepsilon}{\rho\left(1-\varepsilon\frac{h}{g}\right)}\left(\varepsilon\frac{2}{\left(1-\varepsilon\frac{h}{g}\right)}\left[\left(\frac{h}{g}\right)'\right]^2+\left(\frac{h}{g}\right)''\right)\frac{\partial\mathcal{U}^\varepsilon}{\partial\rho}+\frac{\varepsilon^2}{\left(1-\varepsilon\frac{h}{g}\right)^2}\left[\left(\frac{h}{g}\right)'\right]^2\frac{\partial^2\mathcal{U}^\varepsilon}{\partial\rho^2}\right\}\\ = &\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\,\frac{\partial\mathcal{U}^\varepsilon}{\partial\rho}\right)+\frac{1}{\rho^2}\,\frac{\partial^2\mathcal{U}^\varepsilon}{\partial\varphi^2}\\ &+\varepsilon\,\left\{2\,\frac{h}{g}\,\left[\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\,\frac{\partial\mathcal{U}^\varepsilon}{\partial\rho}\right)+\frac{1}{\rho^2}\,\frac{\partial^2\mathcal{U}^\varepsilon}{\partial\varphi^2}\right]+\frac{2}{\rho}\left(\frac{h}{g}\right)'\frac{\partial^2\mathcal{U}^\varepsilon}{\partial\rho\partial\varphi}+\frac{1}{\rho}\left(\frac{h}{g}\right)''\frac{\partial\mathcal{U}^\varepsilon}{\partial\rho}\right\}+O(\varepsilon^2)\end{aligned}$$

Now we turn to the equation (2.1) set in the polar coordinates. We look for the solution $u^\varepsilon(r,\varphi)$ in the form of an asymptotic expansion

$$\mathcal{U}^\varepsilon(\rho,\varphi) = U_0(\rho,\varphi)+\varepsilon\,U_1(\rho,\varphi)+\cdots$$

and for the right hand side term expanded as

$$f(\rho,\varphi) = f_0(\rho,\varphi)+\varepsilon\,f_1(\rho,\varphi)+\cdots$$

We obtain following problems for the first two terms

$$-\left(\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\,\frac{\partial U_0}{\partial\rho}\right)+\frac{1}{\rho^2}\,\frac{\partial^2 U_0}{\partial\varphi^2}\right) = f_0 \text{ in } \Omega\ ,\ \ U_0 = 0 \text{ for } \rho = g(\varphi) \tag{3.3}$$

$$\begin{aligned}-\left(\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\,\frac{\partial U_1}{\partial\rho}\right)+\frac{1}{\rho^2}\,\frac{\partial^2 U_1}{\partial\varphi^2}\right) &= f_1-2\,\frac{h}{g}\,f_0+\frac{2}{\rho}\left(\frac{h}{g}\right)'\frac{\partial^2 U_0}{\partial\rho\partial\varphi}+\\ &+\frac{1}{\rho}\left(\frac{h}{g}\right)''\frac{\partial U_0}{\partial\rho} \text{ in } \Omega\ ,\ \ U_1 = 0 \text{ for } \rho = g(\varphi)\ .\end{aligned} \tag{3.4}$$

3.2. *Application of the boundary perturbation method.* Steady viscous fluid flow through a long straight pipe of uniform circular cross-section and induced by a constant pressure difference or pressure drop between the two ends is generally known as the Poiseuille flow (see e.g. [3] or some more recent and related results [10], [12]). Because of the nature of geometry involved, polar coordinates are used in the governing equations.

EXAMPLE. *Consider the Poiseuille flow through a pipe which has a cross-section that can be described as*

$$\omega_\varepsilon = \{\, (r,\varphi)\, ;\, r \le R - \varepsilon\, h(\varphi)\,\} \ .$$

The pipe is then

$$\Omega_\varepsilon =]0,1[\, \times\, \omega_\varepsilon \ .$$

Governing equations. The Navier-Stokes system that governs the flow is

$$\begin{cases} -\Delta \mathbf{v}_\varepsilon + Re\,(\mathbf{v}_\varepsilon\, \cdot\, \nabla\,)\mathbf{v}_\varepsilon + \nabla p_\varepsilon = 0 \ \text{ in } \Omega_\varepsilon \\ \operatorname{div} \mathbf{v}_\varepsilon = 0 \\ \mathbf{v}_\varepsilon = 0 \text{ on }]0,1[\, \times \partial\omega_\varepsilon \ , \ p_\varepsilon = q_x \ , \ \mathbf{v}_\varepsilon\, \times\, \mathbf{n} = 0 \text{ for } x = 0,1 \end{cases}$$

Solution. The solution is given by

$$\begin{cases} \mathbf{v}_\varepsilon = u_\varepsilon(r,\varphi)\cdot(q_1 - q_0) \\ p_\varepsilon = q_0 + \, x\cdot(q_1 - q_0) \end{cases} ,$$

where u_ε is the solution to the boundary value problem

$$-\Delta\, u_\varepsilon = 1 \ \text{ in } \omega_\varepsilon \ , \ u_\varepsilon = 0 \ \text{ on } \partial\omega_\varepsilon \ . \tag{3.5}$$

The problem (3.5) is a simplified version of the problem considered in the previous section with $g(\varphi) = R$ and $f = f_0 = 1$. The expansion for $\mathcal{U}^\varepsilon(\rho,\varphi) = u_\varepsilon(r,\varphi)$ has a form

$$\mathcal{U}^\varepsilon(\rho,\varphi) = U_0(\rho,\varphi) + \varepsilon\, U_1(\rho,\varphi) + \cdots \ .$$

Now (3.3) becomes

$$-\left(\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\,\frac{\partial U_0}{\partial\rho}\right) + \frac{1}{\rho^2}\,\frac{\partial^2 U_0}{\partial\varphi^2}\right) = 1 \text{ in } \Omega \ , \ U_0 = 0 \text{ for } \rho = R \ ,$$

so that

$$U_0 = \frac{1}{4}(R^2 - \rho^2) \ .$$

The second term U_1, according to (3.4), satisfies

$$-\left(\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\,\frac{\partial U_1}{\partial\rho}\right) + \frac{1}{\rho^2}\,\frac{\partial^2 U_1}{\partial\varphi^2}\right) = -2\,\frac{h}{R} + \frac{1}{\rho}\frac{h''}{R}\frac{\partial U_0}{\partial\rho} = -2\,\frac{h}{R} - \frac{1}{2}\frac{h''}{R} \ .$$

Function

$$W_1 = \frac{\rho^2}{2R}\,h(\varphi)$$

satisfies that equation, but not the boundary condition since

$$W_1(R,\varphi) = \frac{R}{2}\, h(\varphi) \neq 0 \ .$$

To overcome this, we look for U_1 in the form $U_1 = W_1 + V_1$ where

$$-\left(\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\,\frac{\partial V_1}{\partial\rho}\right) + \frac{1}{\rho^2}\,\frac{\partial^2 V_1}{\partial\varphi^2}\right) = 0 \ \text{ for } \rho < R \ , \quad V_1(R,\varphi) = -\frac{R}{2}\, h(\varphi) \ .$$

That problem is easily solved using the Fourier method and we get

$$V_1 = \sum_{k=0}^{\infty} \rho^k \left(A_k \, \cos\, k\varphi + B_k \, \sin\, k\varphi\right) \ ,$$

with (for $k = 1, 2, \ldots$)

$$\begin{aligned} A_k &= -\frac{R^{1-k}}{2\pi}\int_0^{2\pi} h(t)\, \cos\, kt\, dt \\ B_k &= -\frac{R^{1-k}}{2\pi}\int_0^{2\pi} h(t)\, \sin\, kt\, dt \\ A_0 &= -\frac{R}{4\pi}\int_0^{2\pi} h(t)\, dt \ . \end{aligned} \tag{3.6}$$

Thus, we have computed the first two terms in the asymptotic expansion of the solution

$$u_\varepsilon(r,\varphi) = \mathcal{U}^\varepsilon(\rho,\varphi) \sim U_0(\rho,\varphi) + \varepsilon U_1(\rho,\varphi) \ .$$

Taking into account that

$$\rho = \frac{R\,r}{R - \varepsilon\, h} = r + \varepsilon\, r\, \frac{h}{R} + O(\varepsilon^2) \ \Rightarrow \ \rho^2 = r^2 + 2\,\varepsilon\, r^2\, \frac{h}{R} + O(\varepsilon^2) \ ,$$

we get

$$\frac{1}{4}\left(R^2 - \rho^2\right) = \frac{1}{4}\left(R^2 - r^2 - 2\,\varepsilon\, r^2\,\frac{h}{R}\right) + O(\varepsilon^2) \ .$$

Therefore,

$$\begin{aligned} u_\varepsilon(r,\varphi) &= \frac{1}{4}(R^2 - r^2) + \varepsilon\, \left(\sum_{k=0}^{\infty} r^k \left(A_k\, \cos\, k\varphi + B_k\, \sin\, k\varphi\right)\right) + O(\varepsilon^2) \\ &= \frac{1}{4}(R^2 - r^2) + \varepsilon\; H(r,\varphi) + O(\varepsilon^2) \ , \end{aligned}$$

where

$$H(r,\varphi) = \sum_{k=0}^{\infty} r^k \left(A_k\, \cos\, k\varphi + B_k\, \sin\, k\varphi\right) \ ,$$

with Fourier coefficients A_k and B_k given by (3.6). We see that the zeroth order approximation is simply the solution to the boundary value problem posed in domain Ω, which is the usual Poiseuille quadratic velocity profile

$$\mathbf{v}_\varepsilon^{(0)} = \frac{1}{4}(R^2 - r^2) \cdot (q_1 - q_0) \ .$$

In addition, we found a corrector of order ε for the Poiseuille velocity

$$\mathbf{v}_\varepsilon^{(1)} = H(r, \varphi) \cdot (q_1 - q_0) \ ,$$

depending only on the choice of perturbation function $h(\varphi)$, through the calculation of Fourier coefficients (3.6). To illustrate the example, let us take $h(\varphi) = \cos 5\varphi$ and $R = 1$. Calculations show that the only non-zero coefficient in the Fourier series is

$$A_5 = -\frac{1}{2} \ ,$$

so the first order approximation for the velocity is

$$\mathbf{v}_\varepsilon = \left(\frac{1}{4}(1 - r^2) - \frac{1}{2}\, \varepsilon\, r^5 \cdot \cos 5\varphi \right) \cdot (q_1 - q_0) + O(\varepsilon^2) \ .$$

The cross-section and the numerical solution for $\varepsilon = \frac{1}{5}$ are presented in Figure 3.

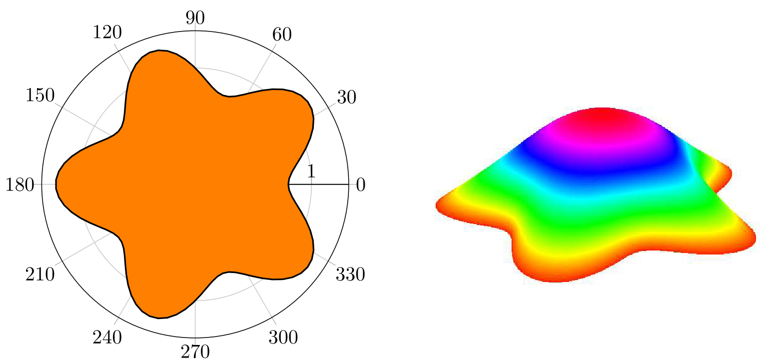

FIGURE 3. Star-shaped cross-section ω_ε and the FreeFem++ solution (velocity profile)

To visualize our result, the asymptotic approximations of zeroth and first order for the velocity profile are given in Figure 4, followed by the calculated corrector of order ε in Figure 5.

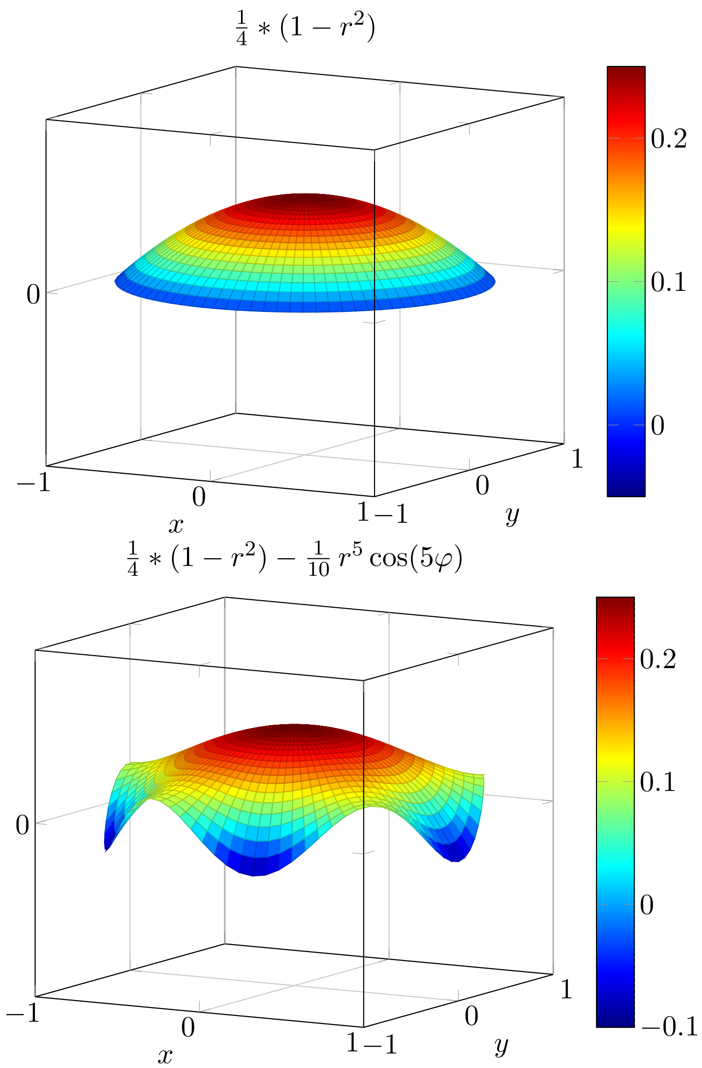

Figure 4. Velocity profile. Zeroth and first order approximation.

4. Conclusion

Methods of the asymptotic analysis used in this paper can easily be applied to the similar boundary value problems and similar domains with the perturbed boundary. Results, however, depend on the possibility to do the

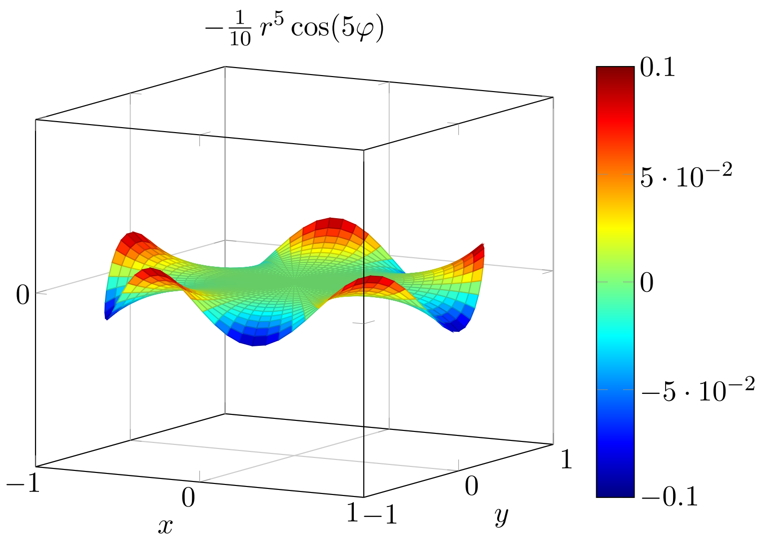

FIGURE 5. Corrector for the velocity profile.

calculations and obtain the exact solution. We considered two cases, boundary value problems set in the rectangular coordinates and in the polar coordinates. Utilizing the same idea of the appropriate change of variable and the asymptotic expansion, we found the approximations of solutions to the Dirichlet problems. To justify our approach, we formalized the ideas of Section 2 in Theorem 2.2. By choosing the Poiseuille flow through a perturbed pipe in Section 3.2 and finding explicit approximation of the velocity profile, we showed that in some cases even the first order approximation gives effective and satisfying results. Comparison with the zeroth order approximation (unperturbed solution) is the informative way to study the effects of the boundary perturbation. Their difference is quantified by the corrector, showing the magnitude of deviation from the initial problem.

We expect to obtain further results by this method, considering the Dirichlet and the Neumann spectrum of the Laplace operator acting in the star-shaped domain with the perturbed boundary. This problem is connected to the question "Can one hear the shape of the drum?" posed by M. Kac in the middle of the last century.

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Perturbacija ruba za Dirichletov rubni problem

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SAŽETAK. Proučavamo utjecaj malih perturbacija ruba domene na rješenja rubnih problema zadanih na takvom području. Polazna točka u promatranju je domena Ω, čijem se rubu dodaje pomak u obliku umnoška malog parametra i neke glatke funkcije. Nulta aproksimacija je upravo originalni rubni problem u području Ω, dok aproksimacija prvog reda sadrži korektor koji ima zanimljiv učinak na rješenja Dirichletovog problema.

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