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Constructive Elaboration of the Pedal Surfaces of the (1,2) Congruence

This paper is dedicated to late colleague and friend mr. sc. JOSIP TADIĆ (1947.-1996).

Constructive Elaboration of the Pedal Surfaces of the (1,2) Congruence
ABSTRACT
In this paper the pedal surfaces of the 1st order and 2nd class congruences are deduced by quartic inversion in projective space, using the methods of synthetic geometry. It is shown that there are the 4th order surfaces with a double line, and the constructive elaboration method is given. For one class of these surfaces the parametric equations are derived, and it is shown how they could be drawn using Mathematica®. Some properties of the surfaces are proved and illustrated with the pictures drawn by Mathematica®.

Key Words
(1,2) congruence, pedal surfaces of congruences, quartic inversion, 4th order surfaces with one double line, 4th order surfaces with one double line which contain absolute conic, pinch-points

Konstruktivna obrada nožišnih ploha jedne (1,2) kongruencije
SAŽETAK
U radu su nožišne plohe kongruencija 1. reda i 2. razreda izvedene pomoću kvartne inverzije u projektivnom prostoru P^3. Dokazano je da su to plohe 4. reda s jednim dvostrukim pravcem koje sadrže apsolutnu koniku kao jednostruku krivulju. Pokazano je da se ove plohe, s obzirom na broj njihovih realnih jednostrukih pravaca, mogu razvrstati u tri skupine.
Za specijalnu (1,2) kongruenciju, kojoj su ravnalice pravac i kružnica koja leži u ravnini okomitoj na taj pravac, izvedene su parametarske jednadžbe nožišnih ploha. Istražena su pojedina svojstva nožišnih ploha (sjecišta kružnih presjeka s dvostrukim pravcem, kuspidalne točke, slučajevi raspada i degeneracije) od kojih su neka ilustrirana konstrukcijama nacrtanim u programu Mathematica®.

Ključne riječi
(1,2) kongruencija, nožišne plohe kongruencija, kvartna inverzija, plohe 4. reda s jednim dvostrukim pravcem, plohe 4. reda s jednim dvostrukim pravcemkoje sadrže apsolutu, kuspidalne točke

INTRODUCTION

There are two things that have motivated me to write this type of paper for this journal. The first one is the supposition that the investigation and *presentation* of numerous forms of high order surfaces in projective space fall into the discipline of constructive geometry. The second one is the fact that by using a computer and combining methods of synthetic and analytic geometry we can achieve good results more easily. In this paper I have used the above mentioned procedure for one class of quartics with a double line. The forms of these quartics could be elaborated in detail by using method of differential geometry, but it is beyond the concept of this paper.

1. THE PEDAL SURFACES OF 1ST ORDER AND 2ND CLASS CONGRUENCES

Let K_n^m be one nth order and mth class congruence and P any finite point in the projective space P^3. Between the rays of the K_n^m and the planes of the sheaf $\{P\}$ the (1,1) correspondence is defined. Corresponding rays and planes are *perpendicular*. (The perpendicularity in P^3 derives from *polarity* with respect to the *absolute conic* [7, p.357]). The locus of intersections of the corresponding rays and planes is a surface which is called the *pedal surface* of congrence K_n^m for the *pole P*. According to Kranjčević[2] it is $(2n+m)^{th}$ order surface which passes n times through the absolute conic.

The purpose of this paper is to elaborate some pedal surfaces of congruence K_1^2. It is shown by Sturm[9, p.37] that the rays of congruence K_1^2 can be determined as bisecants of a broken up space curve of 3rd order. Let c be a conic and d a line having a unique common point O. Let γ be the plane of the conic c, and ω be the plane determined by the line d and the tangent o to the conic c at the point O (Fig.1). If lines of the star $\{O\}$ which are not incident with planes γ and ω, and lines in the plane γ which are not incident with a point O, are excluded, the other bisecants of c and d form the congruence $K_1^2(c,d)$. (Fig.2 illustrates the rising of the unique ray through every real point, and Fig.3 the rising of two rays in arbitrary plane when they are real).

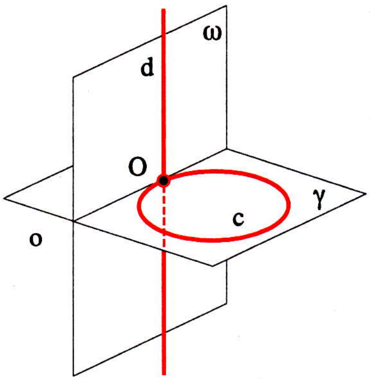

Fig.1

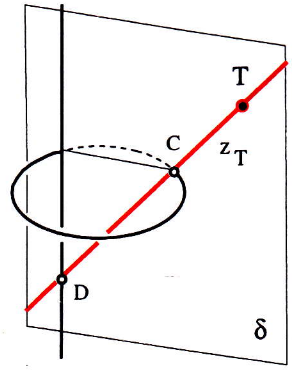

Fig.2

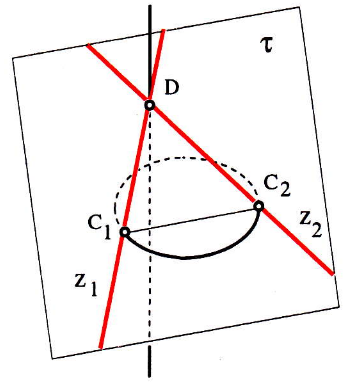

Fig.3

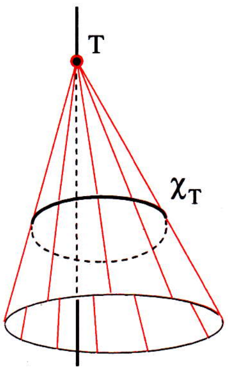

Fig.4

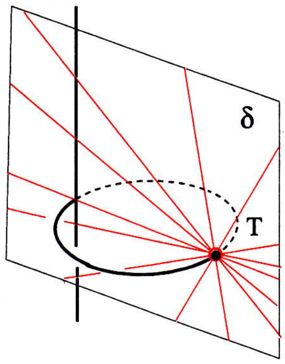

Fig.5

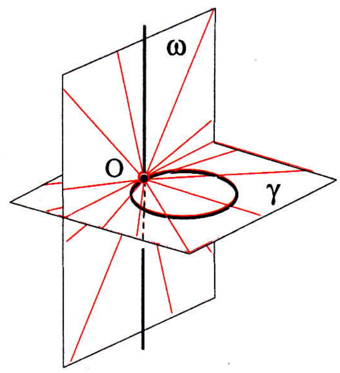

Fig.6

Singular points of $K_1^2(c,d)$ (points through which ∞^1 rays of the congruence pass) lie on the c and d. The rays of $K_1^2(c,d)$ which are incident with a point $T \in d$, $T \neq O$, generate a 2nd degree cone χ_T determined by the vertex T, and the conic c (Fig.4). The rays of $K_1^2(c,d)$ which are incident with a point $T \in c$, $T \neq O$, form the pencil of lines (T) in the plane δ determined by T and d (Fig.5). The rays of $K_1^2(c,d)$ which are incident with a point O form two pencils of lines (O) in the planes γ and ω (Fig.6).

Singular planes of $K_1^2(c,d)$ (planes which contain ∞^1 rays of the congruence) are the planes of the pencil $[d]$ and the plane γ.

Inversions with respect to a quadric Ψ, in the projective space P^3, mean transformations $i_\Psi : P^3 \rightarrow P^3$ where corresponding points A and $i_\Psi(A)$ are conjugate points with respect to Ψ. If lines wich pass through A and $i_\Psi(A)$ form $K_1^2(c,d)$, inversion is called the *quartic inversion in space*. It is Cremona transformation with singular points on the curves d, c and e^6, where e^6 is the 6th order space curve of the double points of the parabolic involutions inducted by Ψ on the rays of $K_1^2(c,d)$ [1].

It is proved [1, p.192] that for every plane ϕ, $i_\Psi(\phi)$ is the 4th order surface, with the double line d, which contains c and e^6. These surfaces, according to Sturm[8, p.315], belong to the class of nth order surfaces with a multiple line with the (n-2)th order of multiplicity. According to [4, p.1575] they form one of the four based types in Kummer [3] classification of quartics which pass through conics. It was proved by Meyer[4,p.1631-1636] that the 4th order surface with a double line d contains: sixteen simple lines, four pinch-points* on the double line and, besides ∞^1 conics which lie in the planes of the pencil $[d]$, other 128 conics which lie in 64 planes. It is shown [1] which points and lines in the plane ϕ were transformed into the mentioned lines, pinch- points and conics on the surface $i_\Psi(\phi)$.

Theorem 1.1.
The pedal surface Φ of $K_1^2(c,d)$ for the pole P is the image of the plane at infinity given by quartic inversion with respect to $K_1^2(c,d)$ and any sphere with the center P.

Proof. For every point $A \in P^3$, $i_\Psi(A)$ lies in the polar plane of A with respect to the quadric Ψ. For every point A^∞ at infinity the polar plane with respect to sphere with the center P is the plane through P which is perpendicular to every line which passes through A^∞. Therefore, if Ψ is a sphere with the center P , for every point $A^\infty \in \phi^\infty$, $i_\Psi(A^\infty)$ is the intersection of the ray of $K_1^2(c,d)$, which passes through A^∞ and the plane through P which is perpendicular to that ray.

* According to Salmon [6, p.300], the *pinch-points* of a surface locus are points on its double curve at which the two tangent planes coincide; at a pinch -point any section, except certain sections, has a cusp. These points Meyer[4] termed *Kuspidalpunkte* , and in [1] I translated them as *cuspidal points*. Now, I want to match the terms with [6].

Corollary 1.1.
The pedal surface Φ is the 4^{th} order surface with the double line d, and contains the absolute conic.

Proof. That is a direct consequence of the theorem 1.1. and properties of the quartic inversion. Namely, it is proved [1] that for every plane ϕ, quadric Ψ and $K_1^2(c,d)$, $i_\Psi(\phi)$ is the 4^{th} order surface with the double line d which cuts ϕ at two rays of $K_1^2(c,d)$ and the intersection conic of ϕ and Ψ. Since ϕ^∞ cuts a sphere in the absolute conic, Φ passes through it.

Corollary 1.2.
Besides the double line d on the pedal surface Φ only four, two or none simple real lines can exist.

Proof. According to [1] sixteen simple lines of $i_\Psi(\phi)$ are the images of the lines DC_1, DC_2 and DE_i, where $\{D\}=d\cap\phi$, $\{C_1,C_2\}=c\cap\phi$ and $\{E_i, i=1,..,6\}=e^6\cap\phi$. If Ψ is a sphere and ϕ the plane at infinity, points E_i lie on the absolute conic, so DE_i are imaginary lines. The point D is always real and C_1, C_2 are real and distinct, real and consecutive or a pair of imaginary points if generatrix c of $K_1^2(c,d)$ is a hyperbola, a parabola or an ellipse. It is clear from the definition of the transformation that a real point is corresponding with a real point and an imaginary with an imaginary point. Since it is true for points, it is also true for lines. Since every of the lines DC_1, DC_2 and DE_i is transformed into one conic which is an image of the point D, and the two lines in the plane of the pencil $[d]$ hence on the surface Φ four, two or none simple real lines exist if c is a hyperbola, a parabola or an ellipse.

Considering corollary 1.2. pedal surfaces of $K_1^2(c,d)$ could be classified by the numbers of simple lines of them.

type of *c*	number of real simple lines	type of $_F$
hyperbola	4	H
parabola	2	P
ellipse	0	E

Theorem 1.2.
Every plane $\delta\in[d]$, which cuts a conic c in the point C, cuts the pedal surface Φ of $K_1^2(c,d)$ for the pole P in the line d and the circle with the diameter $\overline{CP'}$ where P' is the normal projection of P on the plane δ.

Proof. According to [1] every line $t\subset\delta\in[d]$ is transformed into two conics. One of them is the image of the intersection point of t and d, which doesn't lie in δ, and the other is the image of t given by generalized quadratic inversion (this inversion was described in details by Niče[5]) in the plane δ for the pole C and with respect to the intersection conic of Ψ and δ. In conditions of the theorem the base conic of quadratic inversion is the circle with the center P' and t is the line at infinity in the plane δ. The image of t^∞ is the locus of intersections of perpendicular lines through C and P', which is the circle with the diameter $\overline{CP'}$.

This theorem will be the basis for the constructive elaboration of pedal surfaces.
If either of the generatrices c or d be the curve at infinity, the pedal surface Φ, according to [1], breaks up into the plane at infinity and one cubic surface. In other cases, since the plane ϕ^∞ is in general position with respect to the generatrices of $K_1^2(c,d)$, Φ could degenerate or break up only if $P\in d$ or $P\in c$.

2. THE PEDAL SURFACES OF A SPECIAL CONGRUENCE K_1^2

The purpose of this section is to investigate one class of pedal surfaces of the type E and find the way for their construction using Wolfram Research Mathematica®.
This special congruence is determined by the circle c which lies in the plane γ perpendicular to the line d. Since drawing in Mathematica® demands parametric equations of surfaces, $K_1^2(c,d)$ is connected with a right-handed Cartesian coordinate system (O,x,y,z). The position of generatrices of $K_1^2(c,d)$ and orientations of angles u and v are as in Fig.7.

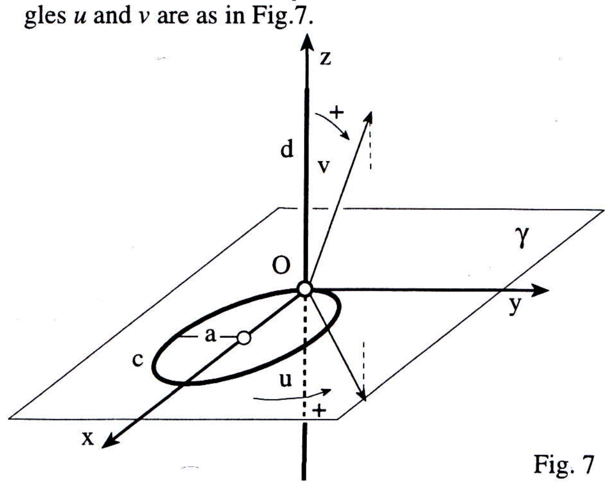

Fig. 7

Since
$d\equiv x=0,\ y=0$

$c\equiv(x-a)^2+y^2=a^2,\ z=0,\ a>0$ and if

$P\equiv(p,q,r)$

the pedal surface of $K_1^2(c,d)$ for the pole P is uniquely determined with four numbers a, p, q and r. It will be designated Φ $[a, p, q, r]$.
It is clear that between the pencil of planes $[d]$ and the semiclosed interval $[-0.5\pi, 0.5\pi)$ one-to-one mapping $u\leftrightarrow\delta(u)$ exists (Fig.8). According to theorem 1.2. every plane $\delta(u)$ cuts Φ $[a, p, q, r]$ into the circle $\overline{c}(u)$ with the diameter $\overline{CP'}$. In the plane $\delta(u)$ (O,t,z) is the Cartesian coordinate system where axis t, intersection of $\delta(u)$ and the plane xy, has positive orientation in the semiplane $x\geq 0$ (Fig.8).

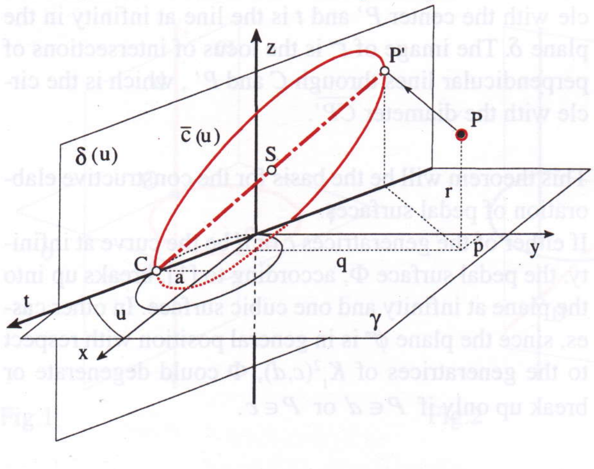

Fig. 8

Since

$t_C(u) = 2a\cos u$ and $t_{P'}(u) = p\cos u + q\sin u$

then

$t_S(u) = 0.5((2a+p)\cos u + q\sin u)$ and

$R(u) = 0.5\sqrt{((2a-p)\cos u - q\sin u)^2 + r^2}$.

Therefore,

$$t(v) = R(u)\sin v + t_S(u)$$
$$z(v) = R(u)\cos v + 0.5r \ , \ v\in[0,2\pi) \qquad (1)$$

are parametric equations of the circle $\overline{c}\,(u)$ in the coordinate system *(O,t,z)* in the plane $\delta(u)$.

Since

$x = t\cos u$ and $y = t\sin u$,

then

$$x(u,v) = \cos u(R(u)\sin v + t_S(u))$$
$$y(u,v) = \sin u(R(u)\sin v + t_S(u)) \qquad (2)$$
$$z(u,v) = R(u)\cos v + 0.5r,$$
$$u\in[-0.5\pi, 0.5\pi),\ v\in[0,2\pi)$$

are parametric equations of the surface Φ [*a, p, q, r*] in the coordinate system *(O,x,y,z)*. It is clear that *v-curves* are circles in the planes δ(*u*) of the pencil [*d*], and *u-curves* will be used only for drawing and they will not be analysed.

One of the parameters *a, p, q, r* could be eliminated from the equations (2), but trying to preserve the geometric interpretation of these numbers we use the form (2).

Now, Mathematica® can draw every surface Φ [*a, p, q, r*]. One example is shown in the Fig. 9.

```
In[1]:=
<<Graphics`ParametricPlot`
<<Graphics`Colors`
A[RGBColor[r_,g_,b_]]:=RGBColor[r,0,0]
A[GrayLevel[x_]]:=GrayLevel[1-x]
T[a_,p_,q_]:=.5(2a Cos[u]+p Cos[u]+q Sin[u])
R[a_,p_,q_,r_]:=.5Sqrt[(2a Cos[u]-p Cos[u]-q Sin[u])^2+r^2]
X[a_,p_,q_,r_]:=Cos[u](Sin[v] R[a,p,q,r]+T[a,p,q])
Y[a_,p_,q_,r_]:=Sin[u](Sin[v] R[a,p,q,r]+T[a,p,q])
Z[a_,p_,q_,r_]:=Cos[v] R[a,p,q,r]+.5r
ParametricPlot3D[{X[3,-2,4,1],Y[3,-2,4,1],Z[3,-2,4,1]},
{u,-Pi/2,Pi/2,Pi/40},{v,0,2Pi,Pi/12},ViewPoint->{1,4.5,-3},
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ColorOutput->A,Background->GrayLevel[.2]]
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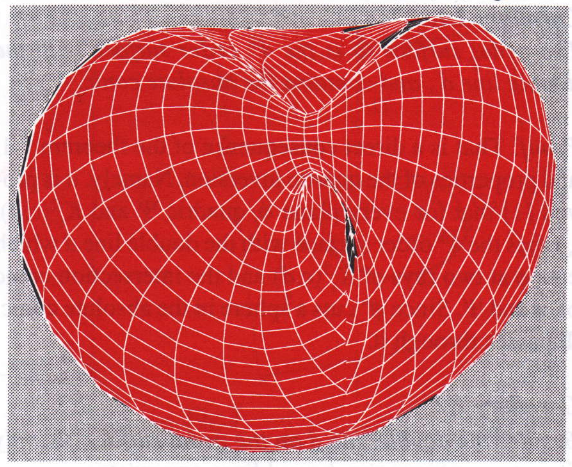

Fig.9

In order to make forms of surfaces clearer we can present only parts of them. Generatrices of $K_1^{\,2}(c,d)$ and a pole *P* can be drawn on Φ [*a, p, q, r*], too. By changing a viewpoint a surface could be seen in the best way. A few examples are shown in Fig. 10. Because of the loss of the four colours and thus reduced clarity of the pictures, some details (points, blackening of some lines) are added to Mathematica graphics.

The further investigation of the surfaces Φ [*a, p, q, r*] (the areas of the elliptic, parabolic or hyperbolic points; the real singular points which depend on the coordinates *p, q, r*; the Gaussian and geodesic curvature; a geodesic lines, etc.) which demands methods of differential geometry is beyond the concept of this paper, as it is said in the introduction.

3. SOME PROPERTIES OF THE SURFACES Φ [*a,p,q,r*]

In the cylindrical coordinate system *(O,t,u,z)* the system of *v*-curves i.e., circles $\overline{c}\,(u)$, is presented by equations:

$$(t - t_S(u))^2 + (z - 0.5r)^2 = R^2(u),$$
$$u\in[-0.5\pi, 0.5\pi) \qquad (3)$$

Proposition 3.1.

Φ[a,p,q,r] is symetric with respect to the plane z=0.5r.

Proof. For every center *S* of the $\overline{c}\,(u)$, $z_S = 0.5r$.

Proposition 3.2.

Circles $\overline{c}\,(u_1)$ *and* $\overline{c}\,(u_2)$ *meet the line d in the same points if and only if* $u_1 + u_2 = \arctan\dfrac{q}{p}$

Proof. If *t*=0 then from (3) the intersection points of $\overline{c}\,(u)$ and *d* are $(0,0,\pm\sqrt{R^2(u) - t_S^2(u)} + 0.5r)$.

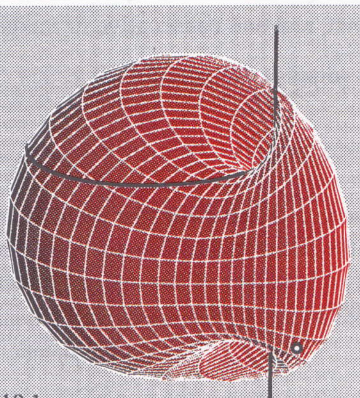

Fig. 10.1
Φ [3,-1,2,-4], viewpoint (-1,4,1.5)

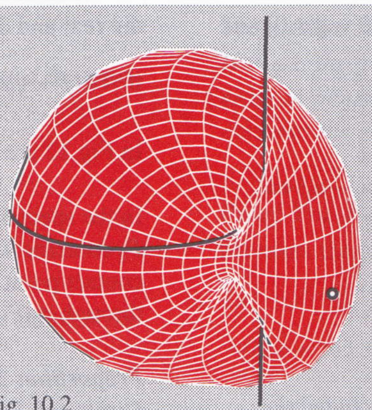

Fig. 10.2
Φ [3,-2,2,-1], viewpoint (-1,4,1.5)

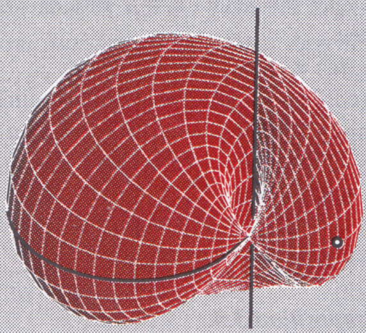

Fig. 10.3
Φ [3,-1,3,0], viewpoint (2,6,5)

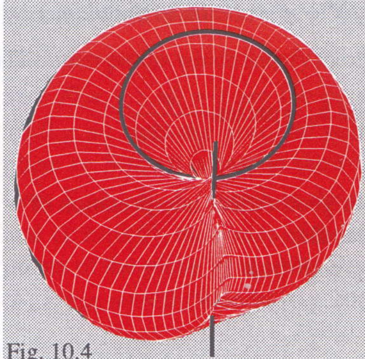

Fig. 10.4
Φ [3,2,2,-10], viewpoint (-3,0.25,5)

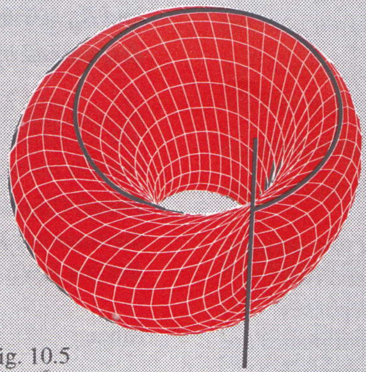

Fig. 10.5
Φ [3,4,2,-4], viewpoint (-3,1,4)

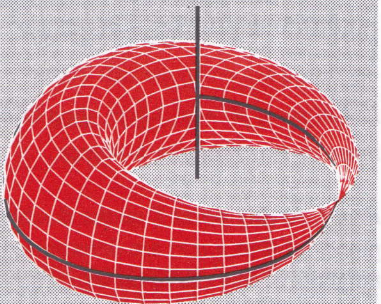

Fig. 10.6
Φ [3,4,2,0], viewpoint (6,1,4)

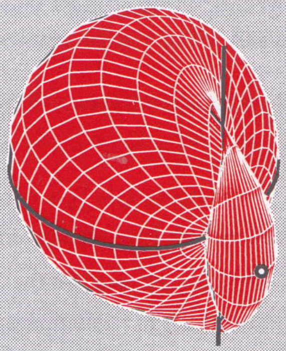

Fig. 10.7
Φ [3,-1,0,0], viewpoint (-2,3,4)

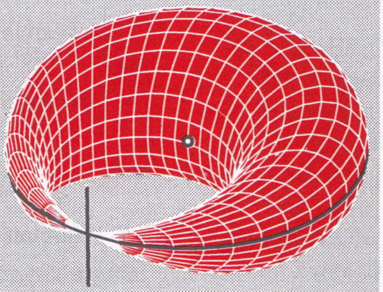

Fig. 10.8
Φ [3,3,0,0], viewpoint (-5,-3,4)

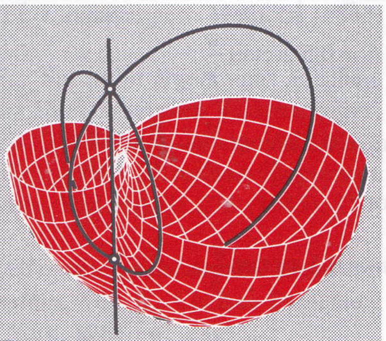

Fig. 11.
$\Phi[3,-2,2,0.5]$, $u_1=\pi/8$, $u_2=-3\pi/8$

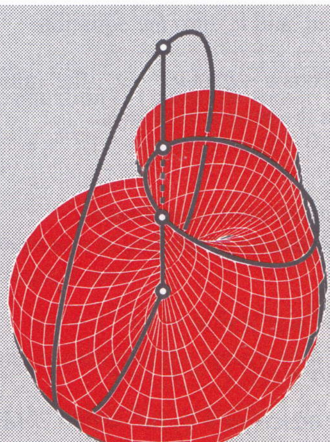

Fig. 12.1
$\Phi[3,-2,2,4]$, $u_I=-\pi/8$, $u_{II}=3\pi/8$

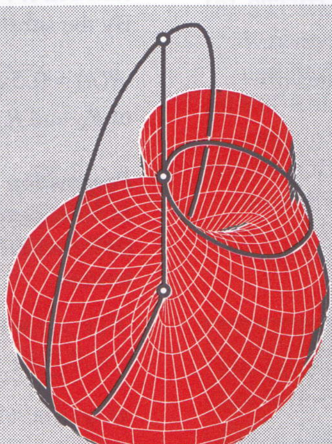

Fig. 12.2
$\Phi[3,-2,2,\, 2\sqrt{6(\sqrt{2}-1)}\,]$, $u_I=-\pi/8$, $u_{II}=3\pi/8$

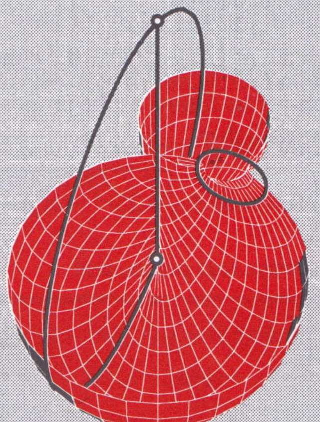

Fig. 12.3
$\Phi[3,-2,2,1]$, $u_I=-\pi/8$, $u_{II}=3\pi/8$

If Φ[a,p,q,r] doesn't break up** (i.e., $p \neq 0 \vee q \neq 0$) and $\delta(u_1) \neq \delta(u_2)$ (i.e., $u_1 \neq u_2+k\pi$, $k\in Z$) then

${}_1z_2(u_1) = {}_1z_2(u_2) \Leftrightarrow$

$p(\cos 2u_1 - \cos 2u_2) = q(\sin 2u_2 - \sin 2u_1) \Leftrightarrow$

$p\sin(u_1+u_2)\sin(u_1-u_2) = q\cos(u_1+u_2)\sin(u_1-u_2) \Leftrightarrow$

$$u_1 + u_2 = \arctan\frac{q}{p}.$$

In Fig.11 is one example.

Proposition 3.3.
If the special case ($p > 0$, $q = r = 0$) *is excluded the pinch-points of* Φ[*a,r,q,r*] are

$$\left(0,0,0.5r \pm\sqrt{0.25r^2 - a(p\pm\sqrt{p^2+q^2})}\right)$$

If $4a(p+\sqrt{p^2+q^2}) \lesseqgtr r^2$, then Φ[*a,p,q,r*] *has four, three or two real pinch-points.*

Proof. At any point T on the double line d two tangent planes of Φ[*a,p,q,r*] are determined by d and two tangents of any curve on Φ[*a,p,q,r*] which passes through T. Therefore at $T \in d$ the two tangent planes are the planes of two circles $\overline{c}(u)$ which pass through T. From the proposition 3.2. it is clear that the two tangent planes coincide for the angles u_I and u_{II} if

(*i*) $u_I = 0.5 \arctan\frac{q}{p}$,

$u_{II} = 0.5 \arctan\frac{q}{p} - \frac{\pi}{2}$ if $\frac{q}{p} \geq 0$,

or

(*ii*) $u_I = 0.5 \arctan\frac{q}{p}$,

$u_{II} = 0.5 \arctan\frac{q}{p} + \frac{\pi}{2}$ if $\frac{q}{p} < 0$

Since the planes $\delta(u_I)$ and $\delta(u_{II})$ are the tangent planes at the pinch-points, by the substitution of u_I and u_{II} into the equations

${}_1z_2 = \pm\sqrt{R^2(u) - t_S^2(u)} + 0.5r$ the coordinates of pinch-points follow as the proposition says.
Since $a>0$ then

$${}_1P_2\left(0,0,0.5r\pm\sqrt{0.25r^2 - a(p-\sqrt{p^2+q^2})}\right)$$

are always real points, and

$${}_3P_4\left(0,0,0.5r\pm\sqrt{0.25r^2 - a(p+\sqrt{p^2+q^2})}\right)$$

are real and distinct, real and consecutive or imaginary if $4a(p+\sqrt{p^2+q^2}) \lesseqgtr r^2$.

In Fig.12 are three examples.

In the special case ($p > 0$, $q = r = 0$) the points P_1 and P_2 coincide with the point O, and O is not a cusp but a contact point of two branches of the any plane section through it (Fig.10.8).

Proposition 3.4.
Circles of the system (3) *with extreme radii* $R_m = 0.5r$ *and* $R_M = 0.5\sqrt{(2a-p)^2+q^2+r^2}$ *lie in the planes determined by angles*

$$u_m = \arctan\frac{2a-p}{q} \text{ and } u_M = \arctan\frac{q}{p-2a}.$$

Proof. Considering our geometrical interpretation of the function R it is clear that
$R(u_1) \leq R(u_2) \Leftrightarrow R^2(u_1) \leq R^2(u_2)$.

Since

$$(R^2(u))' = 0 \Leftrightarrow \begin{Bmatrix} (2a-p)\sin u + q\cos u = 0 \\ (2a-p)\cos u - q\sin u = 0 \end{Bmatrix} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} u = \arctan\dfrac{q}{p-2a} \\ u = \arctan\dfrac{2a-p}{q} \end{cases}$$

and

$$(R^2)''(\arctan\frac{q}{p-2a}) = -2((2a-p)^2+q^2) < 0,$$

$$(R^2)''(\arctan\frac{2a-p}{q}) = 2((2a-p)^2+q^2) > 0$$

then u_M is the maximum and u_m is the minimum of the function R^2.
By the substitution of u_M and u_m into the equation

$R(u) = 0.5\sqrt{((2a-p)\cos u - q\sin u)^2 + r^2}$ the values of R_M and R_m follow as the proposition says.

Proposition 3.5.
In homogeneous Cartasian point coordinates (*x:y:z:w*) *the general equation of* Φ[*a,p,q,r*] *can be written*

$$\begin{aligned} F(x,y,z,w) = (x^2+y^2+z^2)(x^2+y^2) - \\ -((2a+p)x+qy+rz)(x^2+y^2)w + \\ +2ax(px+qy)w^2 = 0 \qquad (4) \end{aligned}$$

** The cases when these pedal surfaces break up will be elaborated later in the paper.

Proof.

(i) If $x^2+y^2 \neq 0$, and w=1 the equation (4) and the equation

$$x^2+y^2+z^2-(2a+p)x-qy-rz+ \\ +\frac{2ax(px+qy)}{x^2+y^2}=0 \qquad (5)$$

are equivalent.
Replacing x by $t\cos u$ and y by $t\sin u$ the preciding equation takes the form

$$t^2+z^2-((2a+p)\cos u+q\sin u)t-rz+ \\ +2a\cos u(p\cos u+q\sin u)=0 \qquad (6)$$

which is the expanded form of the equation (3).

(ii) The other points which satisfy the equations $x^2+y^2=0$ and $F(x,y,z,w)=0$ are the points of the line z, i.e. $(0{:}0{:}z{:}w)$, and the absolute points $(1{:}\pm i{:}0{:}0)$ of the plane xy. Since it is easy to show that the partial derivatives F_x, F_y, F_z, F_w vanish in the every point $(0{:}0{:}z{:}w)$ and $(1{:}\pm i{:}0{:}0)$, the points of the line z, and the absolute points of the plane xy are singular points of the surface $F(x,y,z,w)=0$ (Woods[10,p.206]).

Proposition 3.6.

The plane at infinity cuts the surface $\Phi[a,p,q,r]$ in the absolute conic and the pair of isotropic lines through the point at infinity of the line d.

Proof. For w=0 the equation (4) takes the form

$(x^2+y^2+z^2)(x^2+y^2)=0$, and

$x^2+y^2+z^2=0, w=0$

are the equations of the absolute conic, and

$x^2+y^2=0, w=0$

are the equations of the isotropic lines through the point (0:0:1:0).

Proposition 3.7.

If $P\in d$ the pedal surface $\Phi[a,p,q,r]$ breaks up into one sphere and the isotropic planes through d.

Proof. For $P\in d$, i.e., $p=0 \wedge q=0$, and $w=1$ the equation (4) takes the form

$$(x^2+y^2)((x-a)^2+y^2+(z-0.5r)^2- \\ -a^2-0.25r^2)=0. \qquad (7)$$

Since $x^2+y^2=0$ is the equation of the isotropic planes through d, and

$(x-a)^2+y^2+(z-0.5r)^2-a^2-0.25r^2=0$

is the equation of the sphere with the center $(a,0,0.5r)$ and the radius $\sqrt{a^2+0.25r^2}$, $\Phi[a, p, q, r]$ breaks up as the proposition says.

Proposition 3.8.

If $P\in c$ the pedal surface $\Phi[a,p,q,r]$ doesn't degenerate in general case. But, for $P(2a,0,0)$ it degenerates into the generatrices of $K_1^2(c,d)$.

Proof. For $P\in c$, i.e., $q=\pm\sqrt{p(2a-p)}$, r=0, equation (4) takes the form

$$(x^2+y^2+z^2)(x^2+y^2)- \\ -((2a+p)x\pm\sqrt{p(2a-p)}y)(x^2+y^2)w+ \qquad (8) \\ +2ax(px\pm\sqrt{p(2a-p)}y)w^2=0$$

which in general presents a proper surface. But, for $p=2a$, q=0 and w=1 the equation (4) takes the form

$$(x^2+y^2-2ax)^2+z^2(x^2+y^2)=0. \qquad (9)$$

All real points which satisfy this equation are points on the conic $c\equiv x^2+y^2-2ax=0, z=0$ and the line $d\equiv x=0, y=0$.

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