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GLORIA BITTERFELD

Sinthesis of Grassmann Chain Mechanisms in the Plane Using the *Ausdehnungslehre* and *Mathematica*

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ABSTRACT

The Grassmannian mathematical system, well known as the Ausdehnungslehre (Linear Extension Theory) together with the symbolic computation program Mathematica is used to synthesise a class of planer mechanisms, named Grassmann mechanisms. The objective of investigating Grassmann mechanisms is to be able to compute easily the design parameters of the mechanism from given trajectory points.

Mechanisms in this class have moving links rotating on pivots and intersecting tracks. The paper reports on the simplest type of synthesis result of Grassmann Chain mechanisms using four given precision points for each mechanism apart to determine the design parameters of the mechanism of the class whose trajectory passes through those points.

Sinteza mehanizama Grassmanovog lanca u ravnini uz primjenu *Ausdehnungslehre* i *Mathematice*

SAŽETAK

Grassmannov matematički sustav, poznat kao *Ausdehnungslehre* zajedno sa programom za simbolićko računanje *Mathematica* koristi se za sintetiziranje klase ravninskih mehanizama, nazvanih Grassmannovim mehanizmima. Cilj istraživanja Grassmannovih mehanizama je omogućavanje lakšeg izračunavanja dizajna parametara mehanizma iz zadanih točaka putanje.

Mehanizmi u toj klasi imaju pomične veze koje rotiraju na pivotima i sjekuće staze. Članak izvještava o rezultatu najjednostavnijeg tipa sinteze mehanizama Grassmannovog lanca uz primjenu četiri zadane točke preciznosti za svaki dio mehanizma da bi se odredilli parametri dizajna mehanizma iz klase čije trajektorije prolaze kroz te točke.

1. Introduction

Mechanisms are less versatile than robots, but less expensive, and if a straightforward method for automatic mechanism synthesis could be developed, more manufacturing processes could be automated inexpensively. The concept of the *Ausdehungslehre* was first expounded in 1844 by Hermann Grassmann and forms a language which has a geometric interpretation as a "prescription to construct" [1]. The *Ausdehnungslehre* is a superset of the Victor Calculus, and hence has significant applications to engineering, particularly when implemented with a symbolic computational program like *Mathematica* [2]. This paper implements a synthesis of Grassmann Chain mechanisms constructed of two chains of mechanisms with moving links rotating on pivots and intersecting tracks. That is to start with a required motion and to determine the parameters of a mechanism to give this motion. A numerical example is presented.

To our knowledge there has been no previous work on synthesis of these of mechanisms, or on the use of the *Ausdehnungslehre* to develop theories in this area.

2. The *Ausdehnungslehre* and Geometric Duality

One of the important facets of Grassmannís theory is its geometrical interpretation. The progressive ($\wedge$) and the regressive ($\overline{\wedge}$) products may be interpreted geometrically as follows: the progressive product or *wedge* operator may be read as constructing higher order elements from lower order elements and the regressive products or *wedge-bar* operator as intersecting elements to form lower order ones [3]. The regressive product operation is based on the fundamental formula called The Rule of the Middle Factor [4].

Geometric duality in the plane can be represented as in Table 1 below [3]:

Progresive Product	Regresive Product
$L \equiv P_1 \wedge P_2$	$P \equiv L_1 \overline{\wedge} L_2$
$\pi \equiv P_1 \wedge P_2 \wedge P_3$	$l \equiv L_1 \overline{\wedge} L_2 \overline{\wedge} L_3$
$\pi \equiv L \wedge P$	$l \equiv P \overline{\wedge} L$

Table 1: Geometric Duality

Where:	**P**	denotes a point
	L	denotes a line
	π	denotes a plane
	1	denotes the unit scalar

The equivalence relation ≡ is used to affirm that these relations are true algebraically up to a scalar multiple. For some cases three lines do not intersect at a point.

3. Synthesis of Grassmann Chain Mechanisms - General Case

The simple class of planar mechanisms discussed in this paper is that which involves just fixed pivots and sliders, and which have a single degree of freedom.

Given any number of precision points for each mechanism of the chain it is required to find the chain mechanism which will describe the trajectories for each mechanism apart through those points. Their input is considered to be a point rotating with uniform angular velocity in a circle.

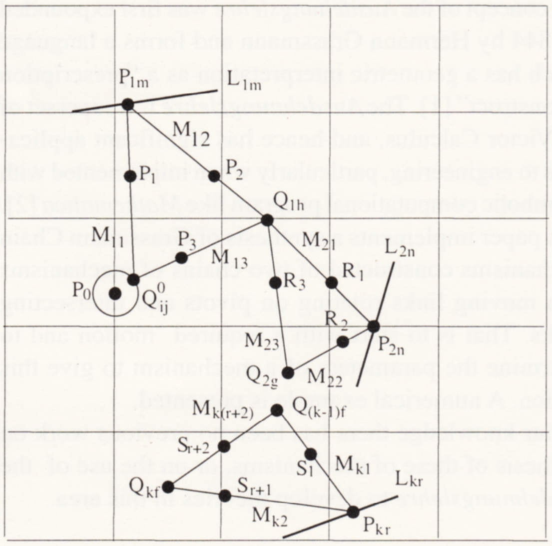

Figure 1: The Construction Scheme of Grassmann Chain Mechanism in the General Case

Regarding to the Figure 1 Grassmann Chain Mechanism notation is as follows:

Note that the first indices refer to the number of the link mechanism.

	First Link **Mechanism**	Second Link ...... **Mechanism**	k^{th} Link **Mechanism**
N° of tracks (order)	m	n	r
N° of pivots	m+2	n+2	r+2
Tracks	$L_{11},...,L_{1m}$	$L_{21}, ..., L_{2n}$	$L_{k1}, ..., L_{kr}$
Pivots	$P_1, ..., P_{m+2}$	$R_1, ..., R_{n+2}$	$S_1, ..., S_{r+2}$
Sliding points	$P_{11}, ..., P_{1m}$	$P_{21}, ..., P_{2n}$	$P_{k1}, ..., P_{kr}$
N° of precision points	h	g	f
Precision points	$Q_{11}, ..., Q_{1h}$	$Q_{21}, ..., Q_{2g}$	$Q_{k1}, ..., Q_{kf}$
N° of mobile links	m+2	n+2	r+2
Mobile links	$M_{11}, ..., M_{1(m+2)}$	$M_{21}, ..., M_{2(n+2)}$	$M_{k1}, ..., M_{k(r+2)}$

Table 2: The Notation of Grassmann Chain Mechanism

$\mathbf{P}_0$ — represents the center of the input circle

$\mathbf{Q}^0_{ij}$ — represents the intersection point on the circle made by the last mobile link passing through the last pivot of the first link mechanism corresponding to the precision point $\mathbf{Q}_{ij}$ of the link mechanism

$\mathbf{Q}_{(k-1)i}$ — represents the i^{th} precision point of the second last link mechanism of the chain mechanism

The proposed Chain Mechanism has only two chains mechanisms with one track and three pivots each.

3.1. The Synthesis of the First Link Mechanism of the Chain

The proposed synthesis process has two stages as follows [5]:

3.1.1. Stage 1 for the First Link Mechanism: Location of the Last Pivot and the Center of the Circle

The first stage is the synthesis of the center of the circle $\mathbf{P}_0$ and the last pivot $\mathbf{P}_3$. This part of the synthesis requires only four precision points as shown in Table 3. It is easy to observe, that the number of unknowns of this stage does not depend on the number of tracks and the other pivots of the mechanism and depends only on the unknown coordinates of the center of the circle $\mathbf{P}_0$ and the pivot $\mathbf{P}_3$.

Definition of the points $\mathbf{P}_0$ and $\mathbf{P}_3$	N° of unknowns of the first stage
$\mathbf{P}_0 = \Theta + x_0\,\mathbf{i} + y_0\,\mathbf{j}$	x_0, y_0
$\mathbf{P}_3 = \Theta + x_3\,\mathbf{i} + y_3\,\mathbf{j}$	x_3, y_3

Table 3: The Number of Unknowns of the First Stage of the Synthesis

If the mechanism is to be of 1 degree of freedom the problem is to find the center of the unit circle, the pivots and the position of the tracks.

With reference to Figure 1 select 4 precision points $\mathbf{Q}_{1h}$, where h=1, 2, 3, 4 and join with coresponding points $\mathbf{Q}^0_{ij}$ on the circle and pass through pivot P_3.

$$\mathbf{Q}_{1h} \wedge \mathbf{P}_3 \wedge \mathbf{Q}^0{}_{ij} = 0 \quad i=1 \text{ and } j=1, 2, 3, 4 \qquad (1)$$

Solve these equations for the coordinates of $\mathbf{P}_0$ (the center of the circle) and $\mathbf{P}_3$ (the last pivot of the first mechanism). The result allows two sets of solutions, so we may have two possibilities for $\mathbf{P}_0$ and $\mathbf{P}_3$.

Assume the selected order of the four given precision points $\mathbf{Q}_{11}\mathbf{Q}_{12}\mathbf{Q}_{13}\mathbf{Q}_{14}$ and select the corresponding rotating points at equal intervals on the circle circumference. We have an infinite number of possibilities for the selection of the position of the rotating points on the circle circumference. For each particular selection of the rotating points, the solution of the points $\mathbf{P}_0$ and $\mathbf{P}_3$ is changed. Therefore, there is an infinite number of solutions for the points $\mathbf{P}_0$ and $\mathbf{P}_3$. These solutions are related to the position of the respective rotating points. The selection of the rotating points gives the time of movement from one precision point to the another.

Suppose that those rotating points have the positions as follows:

$\mathbf{Q}^0_{11} = \Theta + [x_0 + R\ \mathrm{Cos}(0)]\ \mathbf{i} + [y_0 + R\ \mathrm{Sin}(0)]\ \mathbf{j};$

$\mathbf{Q}^0_{12} = \Theta + [x_0 + R\ \mathrm{Cos}(Pi/2)]\ \mathbf{i} + [y_0 + R\ \mathrm{Sin}(Pi/2)]\ \mathbf{j};$

$\mathbf{Q}^0_{13} = \Theta + [x_0 + R\ \mathrm{Cos}(Pi)]\ \mathbf{i} + [y_0 + R\ \mathrm{Sin}(Pi)]\ \mathbf{j};$

$\mathbf{Q}^0_{14} = \Theta + [x_0 + R\ \mathrm{Cos}(3Pi/2)]\ \mathbf{i} + [y_0 + R\ \mathrm{Sin}(3Pi/2)]\ \mathbf{j};$

It is important to notice that we have two solutions for $\mathbf{P}_0$ and $\mathbf{P}_3$. For each permutation of the four precision points the positions of the points $\mathbf{P}_0$ and $\mathbf{P}_3$ will be changed as is shown in Table 3. There is a correspondence between the precision points ($\mathbf{Q}_{1h}$) and the rotating points ($\mathbf{Q}^0_{ij}$) and for any changes in this correspondence, the solution for the center of the circle $\mathbf{P}_0$ and the pivot $\mathbf{P}_3$ will be changed.

N° of precision points	Permutations of the precision points out of four	N° of solutions for $\mathbf{P}_0$ and $\mathbf{P}_3$
4	4_{P4}	48
5	5_{P4}	240
6	6_{P4}	720

Table 4: Number of Possibilities for the Position of $\mathbf{P}_0$ and $\mathbf{P}_3$. Related to the Permutation of the Precision Points

The number of possibilities for the position of $\mathbf{P}_0$ and $\mathbf{P}_3$ is related to the number of precision points and the fact that there are two solutions in each case. If the number of the precision points is greater than four then, to the first four selected precision points add the other precision points and the number of possibilities of the position of $\mathbf{P}_0$ and $\mathbf{P}_3$ will be increase.

With reference to Table 4 for 4 precision points there will be 48 (2 × 24) changes, for precision points there will be 240 (2×120) changes and for 6 precision points there will be 720 (2×360) changes.

3. 1. 2. Stage 2 for the First Mechanism: Location of Pivots and Tracks

The second stage is the synthesis of the tracks and all other pivots. The number of tracks determines the number of pivots of the second stage. If m represents the number of tracks then, m+1 represents the number of pivots of the second stage of synthesis.

Table 5, 6 and 7 show the number of pivots and the orientation of the tracks required by stage 2 of the synthesis as a function of the number of precision points.

Table 5: Number of Precision Points Related to the Number of Pivots and the Tracks Oriented Anywhere in the Plane

N° of tracks and their positions	Definition of the tracks $\mathbf{L}_{1m}$	N° of pivots of the second stage	Definition of the pivots P	N° of unknown parameters	N° of precision points h
One track anywhere in the plane	$\mathbf{L}_{11} = (\Theta + xa\mathbf{i}) \wedge (\Theta + yb\mathbf{j})$	2	$\mathbf{P}_1 = \Theta + x_1\mathbf{i} + y_1\mathbf{j}$ $\mathbf{P}_2 = \Theta + x_2\mathbf{i} + y_2\mathbf{j}$	x_1, y_1 x_2, y_2 xa, yb	6
Two tracks anywhere in the plane	$\mathbf{L}_{11} = (\Theta + xa_1\mathbf{i}) \wedge (\Theta + yb_1\mathbf{j})$ $\mathbf{L}_{12} = (\Theta + xa_2\mathbf{i}) \wedge (\Theta + yb_2\mathbf{j})$	3	$\mathbf{P}_1 = \Theta + x_1\mathbf{i} + y_1\mathbf{j}$ $\mathbf{P}_2 = \Theta + x_2\mathbf{i} + y_2\mathbf{j}$ $\mathbf{P}_3 = \Theta + x_3\mathbf{i} + y_3\mathbf{j}$	x_1, y_1 x_2, y_2 x_3, y_3 xa_1, yb_1 xa_2, yb_2	10
Three tracks anywhere in the plane	$\mathbf{L}_{11} = (\Theta + xa_1\mathbf{i}) \wedge (\Theta + yb_1\mathbf{j})$ $\mathbf{L}_{12} = (\Theta + xa_2\mathbf{i}) \wedge (\Theta + yb_2\mathbf{j})$ $\mathbf{L}_{13} = (\Theta + xa_3\mathbf{i}) \wedge (\Theta + yb_3\mathbf{j})$	4	$\mathbf{P}_1 = \Theta + x_1\mathbf{i} + y_1\mathbf{j}$ $\mathbf{P}_2 = \Theta + x_2\mathbf{i} + y_2\mathbf{j}$ $\mathbf{P}_3 = \Theta + x_3\mathbf{i} + y_3\mathbf{j}$ $\mathbf{P}_4 = \Theta + x_4\mathbf{i} + y_4\mathbf{j}$	x_1, y_1 x_2, y_2 x_3, y_3 x_4, y_4 xa_1, yb_1 xa_2, yb_2 xa_3, yb_3	14
m		m+1		(2m+1)*2	(2m+1)*2

Table 6 shows the mechanism parameters if the track L_{1m} is parallel to one of the axes. This table could be considered as a particular case of Table 5.

If there are more than two tracks their orientation in the plane could vary. For example, the position of two tracks in the plane could be: both anywhere in the plan - Table 5, both parallel to one of the axes - Table 6, or one track parallel to an axis and one track anywhere in the plane - Table 7.

With reference to the Figure 1 the second stage is the synthesis the pivots P_1, P_2 and the position of the tracks L_{1m}. Assume that the mobile link from the rotating points Q^0_{ij} will pass through the pivot P_1, then will intersect the first track L_{11} in a sliding point P_{11}.

Proceeding around the loop shows that the final equation comes from the requirement that the points P_{1m}, P_{m+1} and Q_{1h} be collinear, that is

$$P_{1m} \wedge P_{m+1} \wedge Q_{1h} = 0 \qquad (2)$$

where h=1, 2, 3, 4, ..., is the number of precision points and m represents the number of tracks.

Table 8 shows the final equations for the general case of m tracks starting with one and two tracks.

Table 6: Number of Precision Points Related to the Number of Pivots and Tracks Oriented Parallel to One the Axes in the Plane

Nº of tracks and their positions	Definition of the tracks L_{1m}	Nº of pivots of the second stage	Definition of the pivots P	Nº of unknown parameters	Nº of precision points h
One track parallel to one of the axes	$L_{11} = (\Theta + yb\mathbf{j}) \wedge \mathbf{i}$	2	$P_1 = \Theta + x_1\mathbf{i} + y_1\mathbf{j}$ $P_2 = \Theta + x_2\mathbf{i} + y_2\mathbf{j}$	x_1, y_1 x_2, y_2 yb	5
Two tracks parallel to one of the axes	$L_{11} = (\Theta + yb_1\mathbf{j}) \wedge \mathbf{i}$ $L_{12} = (\Theta + yb_2\mathbf{j}) \wedge \mathbf{i}$	3	$P_1 = \Theta + x_1\mathbf{i} + y_1\mathbf{j}$ $P_2 = \Theta + x_2\mathbf{i} + y_2\mathbf{j}$ $P_3 = \Theta + x_3\mathbf{i} + y_3\mathbf{j}$	x_1, y_1 x_2, y_2 x_3, y_3 yb_1 yb_2	8
Three tracks parallel to one of the axes	$L_{11} = (\Theta + yb_1\mathbf{j}) \wedge \mathbf{i}$ $L_{12} = (\Theta + yb_2\mathbf{j}) \wedge \mathbf{i}$ $L_{13} = (\Theta + yb_3\mathbf{j}) \wedge \mathbf{i}$	4	$P_1 = \Theta + x_1\mathbf{i} + y_1\mathbf{j}$ $P_2 = \Theta + x_2\mathbf{i} + y_2\mathbf{j}$ $P_3 = \Theta + x_3\mathbf{i} + y_3\mathbf{j}$ $P_4 = \Theta + x_4\mathbf{i} + y_4\mathbf{j}$	x_1, y_1 x_2, y_2 x_3, y_3 x_4, y_4 yb_1 yb_2 yb_3	11
m		m+1		3m+2	3m+2

Table 7: Number of Precision Points Related to the Number of Pivots and Tracks Oriented One Parallel to One of the Axes and One Anywhere in the Plane

Nº of tracks and their positions	Definition of the tracks L_{11} and L_{12}	Nº of pivots of the second stage	Definition of the pivots P	Nº of unknown parameters	Nº of precision points h
Two tracks in the plane: one parallel to one of the axes and one anywhere in the plane	$L_{11} = (\Theta + yb_1\mathbf{j}) \wedge \mathbf{i}$ $L_{12} = (\Theta + xa_2\mathbf{i}) \wedge (\Theta + yb_2\mathbf{j})$	3	$P_1 = \Theta + x_1\mathbf{i} + y_1\mathbf{j}$ $P_2 = \Theta + x_2\mathbf{i} + y_2\mathbf{j}$ $P_3 = \Theta + x_3\mathbf{i} + y_3\mathbf{j}$	x_1, y_1 x_2, y_2 x_3, y_3 yb_1 xa_2, yb_2	9

Nº of tracks	Final Equation
one track	$F[1] = ((\mathbf{Q}^0_{ij} \wedge \mathbf{P}_1) \overline{\wedge} \mathbf{L}_{11}) \wedge \mathbf{P}_2 \wedge \mathbf{Q}_{1h} = \mathbf{P}_{11} \wedge \mathbf{P}_2 \wedge \mathbf{Q}_{1h}$
two tracks	$F[2] = ((((\mathbf{Q}^0_{ij} \wedge \mathbf{P}_1) \overline{\wedge} \mathbf{L}_{11}) \wedge \mathbf{P}_2) \overline{\wedge} \mathbf{L}_{12}) \wedge \mathbf{P}_3 \wedge \mathbf{Q}_{1h} = \mathbf{P}_{12} \wedge \mathbf{P}_3 \wedge \mathbf{Q}_{1h}$
m tracks	$F[m] = \mathbf{P}_{1m} \wedge \mathbf{P}_{m+1} \wedge \mathbf{Q}_{1h}$

Table 8: Final Equations for m Tracks

Note: If the second stage of synthesis has more than four unknown more than four precision points are required. The supplementary precision points firstly must satisfy the condition (1) to find their corresponding rotating points and secondly the condition (2) to find the mechanism parameters required by the second stage.

3. 2. THE DETERMINATION OF THE UNKNOWN PARAMETERS IN GRASSMAN CHAIN MECHANISM - GENERAL CASE

With determination of the unknown parameters method of Grassman Chain Mechanisms is an extension of the synthesis method for Grassmann Mechanisms.
Grassmann Chain Mechanisms may be constructed by two, three or any number of mechanisms joined in a chain. Each mechanism has an independent number of tracks and pivots (e. g. the first mechanism could have one track and tree pivots, the second mechanism could have two tracks and four pivots and the third mechanism could have again one track and three pivots).
The construction scheme for Grassmann Chain Mechanisms is shown in the Figure 1.
This part is called determination of unknown parameters, because the method used involves precision lines in addition to precision points. Precision lines are lines which the trajectory must touch or through which it must pass. In the examples, they will be represented as variable points with one of the coordinates fixed.
The determination of the unknown parameters of the second mechanism has also two stages: the first stage of this method calculates the location of the last pivot and second stage calculates the location of the pivots and tracks.

3. 2. 1. STAGE 1 FOR THE SECOND LINK MECHANISM: LOCATION OF THE LAST PIVOT

With reference to the Figure 1 the first stage is the determination of the last pivot $\mathbf{R}_3(\mathbf{R}_{n+2})$.
The Grassmannian expression for the first stage is:

$$\mathbf{Q}_{1h} \wedge \mathbf{R}_{n+2} \wedge \mathbf{Q}_{2g} = 0 \qquad (3)$$

where h=1,2 is the number of the precision point of the first link mechanism and g =1, 2, 3, 4, ..., is the number of the precision point of the second link mechanism.
Solve these equations for the coordinates of the pivot $\mathbf{R}_3(\mathbf{R}_{n+2})$.
This part of synthesis requires only two precision points as shown in Table 9.

Definition of the pivot $\mathbf{R}_{n+2}$	Nº of unknowns of the first stage
$\mathbf{R}_{n+2} = \Theta + x_{n+2}\,\mathbf{i} + y_{n+2}\,\mathbf{j}$	x_{n+2}, y_{n+2}

Table 9: Number of Unknowns of the First Stage

For the same reason as discussed in section 3.1.1. the number of unknowns to be solved in the first stage of the second link mechanism synthesis does not depend on the number of tracks and the other pivots of the mechanism.
To determine unknown parameters of the second mechanism it is necessary to have two precision points and four precision lines.

The method of the first stage:
The stage has only two unknowns. Assume the selected order of the two precision points ($\mathbf{Q}_{21}$ and $\mathbf{Q}_{22}$) and select the corresponding precision points from the first mechanism ($\mathbf{Q}_{11}$ and $\mathbf{Q}_{12}$). Apply equation (3) and solve for the co-ordinates of the pivot $\mathbf{R}_3(\mathbf{R}_{n+2})$.
Suppose there are given four precision lines:

$\mathbf{L}_{p1} = (\Theta + a\mathbf{j}) \wedge x_{Lp1}\mathbf{i}$, $\mathbf{L}_{p2} = (\Theta + b\mathbf{j}) \wedge x_{Lp2}\mathbf{i}$,
$\mathbf{L}_{p3} = (\Theta + c\mathbf{j}) \wedge x_{Lp3}\mathbf{i}$ and $\mathbf{L}_{p4} = (\Theta + d\mathbf{j}) \wedge x_{Lp4}\mathbf{i}$,

where: x_{Lp1}, x_{Lp2}, x_{Lp3} and x_{Lp4} are the x co-ordinates where the trajectory must pass or touch the precision lines.

Assume that the trajectory will pass or touch the precision lines in at least one point. Write this point each precision line with x co-ordinate unknown

$\mathbf{Q}_{Lp1} = \Theta + x_{Lp1}\mathbf{i} + a\mathbf{j}$, $\mathbf{Q}_{Lp2} = \Theta + x_{Lp2} + b\mathbf{j}$,
$\mathbf{Q}_{Lp3} = \Theta + x_{Lp3}\mathbf{i} + c\mathbf{j}$ and $\mathbf{Q}_{Lp4} = \Theta + x_{Lp4} + d\mathbf{j}$.

Apply the condition that the mobile link to the corresponding points from the previous trajectory ($\mathbf{Q}_{13}$, $\mathbf{Q}_{14}$, $\mathbf{Q}_{15}$ and $\mathbf{Q}_{16}$) passes through the last pivot $\mathbf{R}_3(\mathbf{R}_{n+2})$, and solve for their unknown co-ordinates.

$$\begin{aligned} \mathbf{Q}_{13} \wedge \mathbf{R}_3 \wedge \mathbf{Q}_{Lp1} &= 0 \\ \mathbf{Q}_{14} \wedge \mathbf{R}_3 \wedge \mathbf{Q}_{Lp2} &= 0 \\ \mathbf{Q}_{15} \wedge \mathbf{R}_3 \wedge \mathbf{Q}_{Lp3} &= 0 \\ \mathbf{Q}_{16} \wedge \mathbf{R}_3 \wedge \mathbf{Q}_{Lp4} &= 0 \end{aligned} \qquad (4)$$

Solve these equations for the x co-ordinates of the precision lines and find the precision points on the respective precision lines, $\mathbf{Q}_{23}$, $\mathbf{Q}_{24}$, $\mathbf{Q}_{25}$ and $\mathbf{Q}_{26}$.

The selection order of the precision points or the precision lines of the second link mechanism with the respective precision points of the first link mechanism determines the trajectory shape and the Chain Mechanism assembly. For each permutation of the precision points or the precision lines the position of the pivot $\mathbf{R}_3(\mathbf{R}_{n+2})$ will be changed as is shown in the Table 10.

3. 2. 2. STAGE 2 FOR THE SECOND LINK MECHANISM: LOCATION OF PIVOTS AND TRACKS

The second stage is the determination of the tracks and all other pivots. The number of tracks determines the number of pivots.

The method of the second stage:

With reference to Figure 1 the second stage is the determination of the pivots $\mathbf{R}_1, \ldots, \mathbf{R}_{n+1}$ and the position of the tracks $\mathbf{L}_{21}, \ldots, \mathbf{L}_{2n}$.

Assume that all mobile links from the precision points of the first mechanism $\mathbf{Q}_{1h}$ will pass through the pivot $\mathbf{R}_1$ and then insersect the track $\mathbf{L}_{2n}$ in a sliding point $\mathbf{P}_{2n}$.

The final equation comes from the requirement that the points $\mathbf{P}_{2n}$, $\mathbf{R}_{n+1}$ and $\mathbf{Q}_{2g}$ be collinear that is

$$\mathbf{P}_{2n} \wedge \mathbf{R}_{n+1} \wedge \mathbf{Q}_{2g} = \mathbf{0}$$

where g = 1, 2, 3, 4,..., is the number of the precision point of the second link mechanism and n = 1, 2, 3, 4,... is the number of the track of the second link mechanism. Table 11. shows the final equation for the general case of n tracks starting with one and two tracks.

Table 11: Final Equation for n Tracks

4. EXAMPLE USING FOUR PRECISION POINTS FOR EACH MECHANISM OF THE CHAIN MECHANISM

Referring to the Table 5 to synthesis the first mechanism it is necessary to have six precision points. As the result of that for the second mechanism it is necessary to have also two precision points and four precision lines. As the problem gave only four precision points for the first mechanisms and two precision points and two precision lines for the second mechanism it will be assume any two of the unknowns for each mechanism.

4.1. THE FIRST LINK MECHANISM

The first mechanism has the following precision points: $\mathbf{Q}_{11} = \Theta + 7\mathbf{i} - \mathbf{j}$, $\mathbf{Q}_{12} = \Theta + \mathbf{i}$, $\mathbf{Q}_{13} = \Theta + 2\mathbf{i} - 2\mathbf{j}$ and $\mathbf{Q}_{14} = \Theta + 5\mathbf{i} - 3\mathbf{j}$.

Assume the position of the track $\mathbf{L}_{11} = (\Theta + 3\mathbf{j}) \wedge \mathbf{i}$.

Referring to the synthesis method presented in the paragraph 3.1. the first mechanism has the following solutions:

$\mathbf{P}_0 = \Theta + 1.25\mathbf{i} - 1.25\mathbf{j}$, $\mathbf{P}_3 = \Theta + 3.25\mathbf{i} - 2.25\mathbf{j}$,

$\mathbf{P}_1 = \Theta + 2.52301\mathbf{i} - 0.427847\mathbf{j}$ and

$\mathbf{P}_2 = \Theta + 4.96599\mathbf{i} - 0.517007\mathbf{j}$.

The first mechanism will plot the trajectory as is show in the figure below.

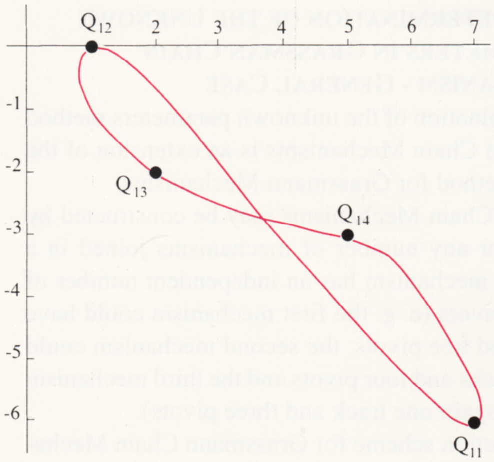

Figure 2: The Trajectory of the First Link Mechanism

4. 2. THE SECOND LINK MECHANISM

The second mechanism has the following precision points: $\mathbf{Q}_{21} = \Theta + 9\mathbf{i} - 4\mathbf{j}$, $\mathbf{Q}_{22} = \Theta + 7\mathbf{j}$ and the following precision lines:

$\mathbf{L}_{p1} = \Theta + x_{Lp1}\mathbf{i} - 6\mathbf{j}$ and $\mathbf{L}_{p2} = \Theta + x_{Lp2}\mathbf{i} - 5\mathbf{j}$.

Assume the position of the track $\mathbf{L}_{21} = (\Theta + 10\mathbf{j}) \wedge \mathbf{i}$.

Stage 1: Select the precision points $\mathbf{Q}_{21}$ and $\mathbf{Q}_{22}$. For each of the precision points the mobile link must join them and their corresponding precision points on the previous trajectory ($\mathbf{Q}_{11}$ and $\mathbf{Q}_{12}$) and pass through the

Table 10: Number of Changes in the Position of $\mathbf{R}_3$ ($\mathbf{R}_{n+2}$) Related to the Permutation of the Precision Points

N° of precision points	Permutation of the precision points out of two	Changes of the position of $\mathbf{R}_{n+2}$
6	$^6P_2$	30

Table 11: Final Equation for n Tracks

Number of tracks	Final Equation
one track	$F[1] = ((\mathbf{Q}_{1i} \wedge \mathbf{R}_{1)} \barwedge \mathbf{L}_{21)} \wedge \mathbf{R}_2 \wedge \mathbf{Q}_{2i} = \mathbf{P}_{2n} \wedge \mathbf{R}_2 \wedge \mathbf{Q}_{2i}$
two tracks	$F[2] = ((((\mathbf{Q}_{1i} \wedge \mathbf{R}_1) \barwedge \mathbf{L}_{21}) \wedge \mathbf{R}_2) \barwedge \mathbf{L}_{22}) \wedge \mathbf{R}_3 \wedge \mathbf{Q}_{2i} = \mathbf{P}_{22} \wedge \mathbf{R}_3 \wedge \mathbf{Q}_{2i}$
n - tracks	$F[n] = \mathbf{P}_{2n} \wedge \mathbf{R}_{n+1} \wedge \mathbf{Q}_{2i=0}$

last pivot $\mathbf{R}_3$. Apply the equation (3) and the problem gives the solution for $\mathbf{R}_3$.

$$\mathbf{Q}_{11} \wedge \mathbf{R}_3 \wedge \mathbf{Q}_{21} = \mathbf{0}$$
$$\mathbf{Q}_{12} \wedge \mathbf{R}_3 \wedge \mathbf{Q}_{22} = \mathbf{0}$$

The problem has one solution for $\mathbf{R}_3$: $\mathbf{R}_3 = \Theta - \mathbf{i} - 14\mathbf{j}$.
Select any point on each precision line and write this point with x co-ordinate unknown. Then apply the Grassmannian equation (4) and solve for their unknown co-ordinates.
In this example the Grassmannian equations are:

$$\mathbf{Q}_{13} \wedge \mathbf{R}_3 \wedge \mathbf{Q}_{L,p1} = \mathbf{0}$$

$$\mathbf{Q}_{14} \wedge \mathbf{R}_3 \wedge \mathbf{Q}_{L,p2} = \mathbf{0}$$

The problem gives the solutions for the points $\mathbf{Q}_{L,p1}$ and $\mathbf{Q}_{L,p2}$. These points may now be consider precision points and named: $\mathbf{Q}_{23} = \Theta + \mathbf{i} - 6\mathbf{j}$, and $\mathbf{Q}_{24} = \Theta + 3.90909\mathbf{i} - 5\mathbf{j}$.

Stage 2: Calculates the co-ordinates of the pivots $\mathbf{R}_1$ and $\mathbf{R}_2$ by applying the equation F[1] from the Table 11.

$$((\mathbf{Q}_{11} \wedge \mathbf{R}_{1)} \overline{\wedge} \mathrm{L}_{21}) \wedge \mathbf{R}_2 \wedge \mathbf{Q}_{21} = \mathrm{P}_{21} \wedge \mathbf{R}_2 \wedge \mathbf{Q}_{21}$$
$$((\mathbf{Q}_{12} \wedge \mathbf{R}_{1)} \overline{\wedge} \mathrm{L}_{21}) \wedge \mathbf{R}_2 \wedge \mathbf{Q}_{22} = \mathrm{P}_{22} \wedge \mathbf{R}_2 \wedge \mathbf{Q}_{22}$$
$$((\mathbf{Q}_{13} \wedge \mathbf{R}_{1)} \overline{\wedge} \mathrm{L}_{21}) \wedge \mathbf{R}_2 \wedge \mathbf{Q}_{23} = \mathrm{P}_{23} \wedge \mathbf{R}_2 \wedge \mathbf{Q}_{23}$$
$$((\mathbf{Q}_{14} \wedge \mathbf{R}_{1)} \overline{\wedge} \mathrm{L}_{21}) \wedge \mathbf{R}_2 \wedge \mathbf{Q}_{24} = \mathrm{P}_{24} \wedge \mathbf{R}_2 \wedge \mathbf{Q}_{24}$$

The solution for the pivots $\mathbf{R}_1$ and $\mathbf{R}_2$ is:

$\mathbf{R}_1 = \Theta - 4.47433\mathbf{i} - 3.83747\mathbf{j}$ and

$\mathbf{R}_2 = \Theta + 10.7921\mathbf{i} - 4.55935\mathbf{j}$.

The second mechanism will plot the trajectory as is shown in the figure below.

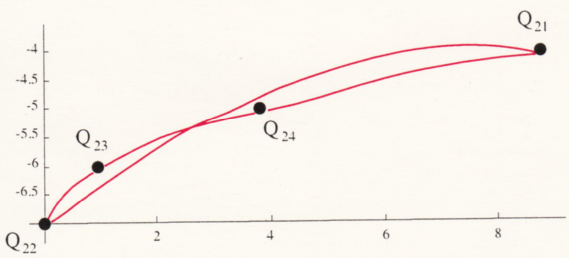

Figure 3: Trajectory of the Second Link Mechanism

5. Conclusions

The paper has shown a method of synthesis of a class of mechanisms constructed by moving links, rotating on fixed points and intersecting fixed lines using a Grassmannian mathematical formulation and the symbolic computational programme Mathenatica.

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Dr. Gloria Bitterfeld
12 Brooks St., South-Oakleigh, 3167
Victoria, Australia
Fax: 61 3 9532 7969
e-mail:stavrou@swin.edu.au