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# Triangles from Central Points

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**ABSTRACT**
The paper deals with the problem of determining which central points $X$ of the triangle $ABC$ have the property that segments $AX$, $BX$, and $CX$ are the sides of a triangle. We shall prove that only thirteen out of hundred and one central points from Kimberling's list have this property. Moreover, the convex hull of ten among these points always consists only of the points having the above mentioned properties.

**Trokuti iz središnjih točaka**
**SAŽETAK**
U članku se promatra problem pronalaženja centralnih točaka $X$ trokuta $ABC$ sa svojstvom da su dužine $AX$, $BX$ i $CX$ stranice nekog trokuta. Pokazuje se da samo trinaest od sto i jedne centralne točke Kimberling-ove liste imaju to svojstvo. Nadalje, konveksna ljuska deset od tih točaka se uvijek sastoji samo od točaka s istim svojstvom.

## 1. INTRODUCTION

One of the basic problems in triangle geometry is to decide when three given segments are sides of a triangle. The opening chapter of the book *Recent Advances in Geometric Inequalities* by Mitrinović, Pečarić, and Volenec [5] gives an extensive survey of results on this question.

The present article is looking for ways of associating to a triangle $ABC$ a point $P$ of the plane such that segments $AP$, $BP$, and $CP$ are always sides of a triangle.

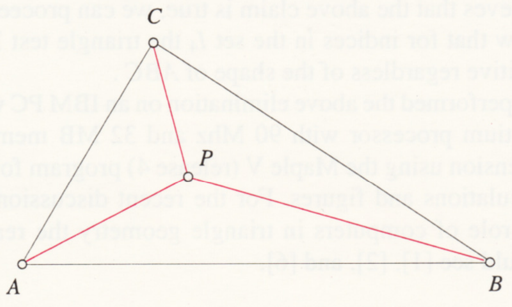


Fig. 1:
When segments $AP$, $BP$, and $CP$ are sides of a triangle?

The circumcenter $O$ and the centroid $G$ are easy examples of such points $P$. Indeed, segments $AO$, $BO$, and $CO$ having equal length are sides of an equilateral triangle while the segments $AG$, $BG$, and $CG$ being two thirds of medians are sides of a triangle (see [5, p. 20]).

Since $O$ and $G$ are just two of centers or central points of a triangle $ABC$ listed in Table 1 of [3], we can state a problem that we completely answer in this paper.

***Problem.***
*For what natural numbers* **i** *less than 102 will the central point $X_i$ of the triangle ABC from the Kimberling's list have the property that $AX_i$, $BX_i$, and $CX_i$ are sides of a triangle?*

Our main result is the following theorem.

***Theorem.***
*From 101 central points $X_i$ of the triangle ABC from Kimberling's Table 1, only values 2, 3, 8, 9, 10, 20, 21, 22, 40, 63, 71, 72, and 75 of the index* **i** *have the property that $AX_i$, $BX_i$, and $CX_i$ are sides of a triangle regardless of the shape of ABC. For the central point $X_{101}$ the only exception are isosceles triangles.*

Let $T$ denote a function that maps each triple $(a, b, c)$ of real numbers to a number
$2a^2b^2 + 2a^2c^2 + 2b^2c^2 - a^4 - b^4 - c^4$.
Since
$T(a,b,c) = (a+b+c)(b+c-a)(a-b+c)(a+b-c)$,
it is clear that positive real numbers $a$, $b$, and $c$ are sides of a triangle if and only if $T(a, b, c) > 0$. Let $T_i$ be a short notation for $T(AX_i, BX_i, CX_i)$, where $X_i$ is the $i$-th central point of $ABC$ and $i = 1,....,101$.

## 2. PLACEMENT OF *ABC*

We shall position the triangle $ABC$ in the following fashion with respect to the rectangular coordinate system in order to simplify our calculations. The vertex $A$ is the origin with coordinates (0,0), the vertex $B$ is on the $x$-axis and has coordinates $(rh, 0)$, and the vertex $C$ has coordinates $(g\,r\,u/k, 2\,f\,g\,r/k)$, where
$h = f + g$, $\quad k = fg - 1$, $\quad u = f^2 - 1$, $\quad v = g^2 - 1$,
$\varphi = f^2 + 1$, $\quad \psi = g^2 + 1$, $\quad \Phi = f^4 + 1$ and $\Psi = g^4 + 1$.
The three parameters $r$, $f$, and $g$ are the inradius and the cotangents of half of angles at vertices $A$ and $B$. Without loss of generality, we can assume that both $f$ and $g$ are larger than 1 (i. e., that angles $A$ and $B$ are acute).
Nice features of this placement are that all central points from Table 1 in [3] have rational functions in $f$, $g$, and $r$ as coordinates and that we can easily switch from $f$, $g$, and $r$ to side lengths $a$, $b$, and $c$ and back with substitutions

$$a = \frac{r\,f\,(g^2+1)}{k},\quad b = \frac{r\,g\,(f^2+1)}{k},\quad c = r\,h,$$

$$f = \frac{(b+c)^2 - a^2}{\sqrt{T(a,b,c)}},\quad g = \frac{(a+c)^2 - b^2}{\sqrt{T(a,b,c)}},\quad r = \frac{\sqrt{T(a,b,c)}}{2(a+b+c)}.$$

Moreover, since we use the Cartesian coordinate system, computation of distances of points and all other formulas and techniques of analytic geometry are available and well-known to widest audience. A price to pay for these conveniences is that symmetry has been lost and some expressions are complicated and awkward to print.

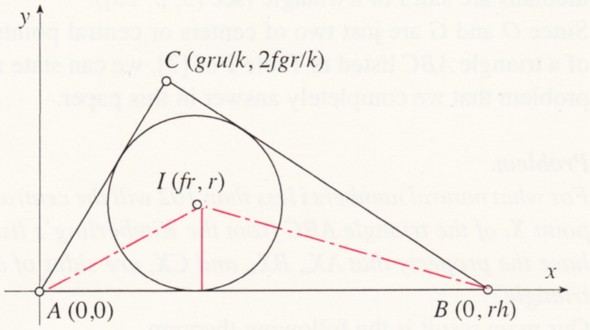


Fig. 2: Parameters are the intriadius and the cotangens of half of angles at vertices $A$ and $B$.

The third advantage of the above position of the base triangle is that we can easily find coordinates of a point with given trilinears. More precisely, if a point $P$ with coordinates $x$ and $y$ has projections $P_a$, $P_b$, and $P_c$ onto the sidelines $BC$, $CA$, and $AB$ and $\lambda = PP_a/PP_b$ and $\mu = PP_b/PP_c$, then

$$x = \frac{g\,h(\varphi\,\mu + u)\,r}{f\,\psi\,\lambda\,\mu + g\,\varphi\,\mu + h\,k},\quad y = \frac{2\,f\,g\,h\,r}{f\,\psi\,\lambda\,\mu + g\,\varphi\,\mu + h\,k}.$$

This formulas will greatly simplify our exposition because there will be no need to give explicitly coordinates of points but only its first trilinear coordinate. For example, we write $X_6[a]$ to indicate that the symmedian point $X_6$ has trilinears equal $a{:}b{:}c$. Then we use the above formulas with $\lambda = a/b$ and $\mu = b/c$ to get the coordinates

$$\left(\frac{(f\,u\,v + 2\,g\,\Phi)\,g\,h\,r}{2(f^2\,\Psi + f\,g\,u\,v + g^2\,\Phi)}, \frac{f\,g\,h^2 k\,r}{f^2\,\Psi + f\,g\,u\,v + g^2\,\Phi}\right)$$

of $X_6$ in our coordinate system.

## 3. CURVE DETERMINED BY THE FUNCTION $T$

Let $P$ be a point in the plane of the triangle $ABC$ with coordinates $p$ and $q$. We can easily find that $t_{ABC} = T(AP, BP, CP)$ is

$3k^4(p^2+q^2)^2 - 4k^3 r(p^2+q^2)(k_1 p + k_2 q) -$
$- 2k^2 r^2 (k_3 p^2 - k_4 p\,q - k_5 q^2) +$
$+ f^2 r^3 k_6 k_7 (4kvp - 8gkq - r\,k_6 k_7)$,

where

$k_1 = f\,v + 2g\,u$, $\quad k_2 = 2f\,g$,
$k_3 = f^2\Psi - 2g^2\Phi - 2f\,g\,u\,v - 2f^2 g^2$, $\quad k_4 = 8f^2 g\,v$,
$k_5 = f^2\Psi + 2f\,g^3 v + 2g^2\Phi - 10f^2 g^2 - 2f\,g\,u$,
$k_6 = f\,v - 2g$ and $k_7 = g\,h + k$.

It follows that $t_{ABC} = 0$ is the equation of an algebraic curve of order four which represents the boundary of two regions in the plane of $ABC$. The first region includes the vertices $A$, $B$, and $C$ and has the property that a point $P$ belongs to it if and only if segments $AP$, $BP$, and $CP$ are not sides of a triangle. The second region which we denote by $T_{ABC}$ is the complement of the first and has the property that a point $P$ belongs to it if and only if segments $AP$, $BP$, and $CP$ are sides of a triangle. Hence, the second region gives the solution to the first question in the introduction. The boundary of $T_{ABC}$ (i. e., the curve $T_{ABC} = 0$) is drawn in Figure 3.

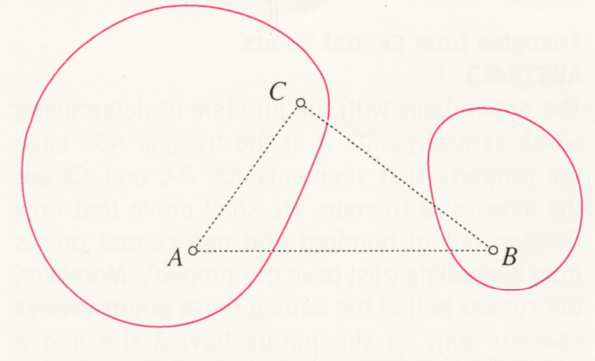


Fig. 3: Graph of the curve $t_{ABC} = 0$ that is the boundary of the region consisting of all points $P$ such that segments $AP$, $BP$ and $CP$ are sides of a triangle.

Our problem is thus equivalent with the problem of determining which central points from the Kimberling's Table 1 are in the region $T_{ABC}$ for every triangle $ABC$.

## 4. ELIMINATION OF 87 CENTRAL POINTS

An easy task is to eliminate 87 central points $X_i$ by exhibiting a triangle for which $T_i \le 0$. In fact, only three triangles with $r = 1$ and $(f, g)$ equal to

$$t_1 = (2,5), \qquad t_2 = (2,20), \qquad t_3 = \left(\frac{101}{100}, \frac{102}{100}\right)$$

will suffice. Indeed, $T_i \le 0$ for the triangle $t_j$ and $i \in I_j$, where $j = 1, 2, 3$,

$I_0 = \{1,\ldots,101\}$,
$I_2 = \{18, 26, 35, 45, 50, 69, 73, 76, 77, 78, 87\}$,
$I_3 = \{29, 48, 49, 64, 66, 67, 70, 74, 84, 92, 93, 98\}$,
$I_4 = \{2, 3, 8, 9, 10, 20, 21, 22, 40, 71, 72, 75, 101\}$, and
$I_1 = I_0 - I_2 - I_3 - I_4$.

The above statement is simple to state but the reader should be aware that there is a lot of work behind it because we must know coordinates of each central point from Kimberling's list. Under the assumption that one believes that the above claim is true, we can proceed to show that for indices in the set $I_4$ the triangle test $T_i$ is positive regardless of the shape of $ABC$.

We performed the above elimination on an IBM PC with Pentium processor with 90 Mhz and 32 MB memory extension using the Maple V (release 4) program for all calculations and figures. For the recent discussion on the role of computers in triangle geometry the reader should see [1], [2], and [6].

## 5. $X_2$ - CENTROID

$X_2[1/a]$ is the intersection of medians which join vertices with midpoints of opposite sides. Hence,

$$T_2 = \frac{16}{9}\,\frac{r^4 g^2 f^2 h^2}{k^2}$$

is always positive.

**6. $X_3$ - CIRCUMCENTER**

$X_3[\cos A]$ is the intersection of perpendicular bisectors of sides. It follows that

$$T_3 = \frac{3}{256}\,\frac{r^4(f^2+1)^4(g^2+1)^4}{k^4}$$

is always positive.

**7. $X_8$ - NAGEL POINT**

$X_8[(b+c-a)/a]$ is the intersection of lines $AA_{ea}$, $BB_{eb}$, and $CC_{ec}$, where $A_{ea}$, $B_{eb}$, and $C_{ec}$ are projections of excenters $A_e$, $B_e$, and $C_e$ onto sidelines $BC$, $CA$, and $AB$, respectively. One can easily find

$$T_8 = \frac{16\,r^4(h^2(k^2-k+1)-3k^3)}{k^2}$$

Since $h^2 \geq 4(k+1)$ and $k^2 - k + 1 > 0$, we get that the third factor of $T_8$ is larger than $k^3 + 4$. Hence, $T_8 > 0$.

**8. $X_9$ - MITTENPUNKT**

$X_9[b+c-a]$ is the point of concurrence of the symmedians of the excentral triangle $A_eB_eC_e$. Recall that symmedians are obtained by reflecting the medians about the corresponding interior angle bisectors.

In the standard way we discover that

$$T_9 = \frac{r^4 S_9}{16\,k^4(h^2+k^2+k)^4}, \text{ where}$$

$$S_9 = \sum_0^6 p_i\,k^{6-i}(k+1)^{6-i}h^{2i},\quad p_i = \sum_0^4 q_{ij}\,k^{4-j}, \text{ and}$$

$$\left[q_{ij}\right] = \begin{bmatrix} 0 & 0 & -16 & 0 & 0 \\ 0 & -8 & -32 & -32 & 0 \\ 3 & -24 & -8 & -96 & -16 \\ 12 & -20 & 32 & 8 & -64 \\ 18 & 4 & 26 & 168 & -8 \\ 12 & 12 & -12 & 100 & 16 \\ 3 & 4 & -14 & 4 & 3 \end{bmatrix}$$

It is not clear how one can argue that the polynomial $S_9$ is always positive. But, the following miraculous method will accomplish this goal.

Write $S_9$ in terms of $f$ and $g$. We get a polynomial $U_9$ with 97 terms. Since both $f$ and $g$ are larger than 1, we shall replace them with $1+f$ and $1+g$, where new variables are positive. This substitution will give us a new polynomial $V_9$ with 279 terms only 9 of which have negative coefficients. If all coefficients were positive, we would be done. In order to get rid of these 9 troublesome terms, we must perform two more substitutions that reflect cases $f \geq g$ and $g \geq f$. Hence, if we replace $f$ with $g + u'$ for $u' \geq 0$, from $V_9$ we shall get a polynomial $P_9$ in $g$ and $u'$ with 353 terms and all coefficients positive. Similarly, if we substitute $g$ with $f + v'$ for $v' \geq 0$, from $V_9$ we shall get a polynomial $Q_9$ in $f$ and $v'$ also with 353 terms and all coefficients positive. This concludes our proof that $T_9 > 0$.

**9. $X_{10}$ - SPIEKER CENTER**

$X_{10}[(b+c)/a]$ is the incenter of the medial triangle $A_mB_mC_m$ whose vertices are midpoints of sides. It follows that

$$T_{10} = \frac{r^4 S_{10}}{16\,k^4}, \text{ where}$$

$$S_{10} = (k+3)(3k+1)(k-1)^2h^4 - 2k^2(4k^3-7k^2-38k-31)h^2 - k^4(4k+7)(4k+3).$$

In order to prove that $T_{10} > 0$ we apply the method of proof for $X_9$. Polynomials $U_{10}$, $V_{10}$, and $P_{10}$, are of medium size having 29, 71, and 77 terms.

**10. $X_{21}$ - SCHIFFLER POINT**

$X_{21}[(b+c-a)/(b+c)]$ is the point of concurrence of Euler lines of triangles $BCX_1$, $CAX_1$, and $ABX_1$, where $X_1$ is the incenter of $ABC$. Recall that the line joining the centroid and the circumcenter of a scalene triangle $ABC$ is called the *Euler line of ABC*.

It follows that

$$T_{21} = \frac{16r^4(k+1)^2(f^2h^2-k^2)^2(g^2h^2-k^2)^2(h^2+k^2+4k)}{k^3(3h^2+3k^2+8k)^4}$$

so $T_{21}$ is clearly always positive.

**11. $X_{63}$ - ISOGONAL CONJUGATE OF THE CRUCIAL POINT**

$X_{63}[b^2+c^2-a^2]$ is the point of intersection of the line joining $X_1$ (incenter) with $X_{21}$ (Schiffler point) and the line joining $X_8$ (Nagel point) with $X_{20}$ (De Longchamps point - reflection of the orthocenter at the circumcenter). In the usual way we find that

$$T_{63} = \frac{32r^4\,S_{63}}{k^3(h^2+k^2+4k)^4},$$

where $S_{63}$ is a polynomial of degree 10 in $h$ with coefficients polynomials in $k$ of degree at most 11. The polynomial $U_{63}$ has 72 terms while $V_{63}$ has 162 terms and all coefficients positive. Hence, $T_{63} > 0$.

**12. CENTRAL POINT $X_{71}$**

$X_{71}[a(b+c)(b^2+c^2-a^2)]$ is the point of intersection of the line joining $X_4$ (orthocenter) with $X_9$ (Mittenpunkt) and the line joining $X_6$ (Grebe-Lemoine or symmedian point) with $X_{31}$ (2nd Power point). As above, we find that

$$T_{71} = \frac{r^4\,S_{71}}{16k^4(h^4+k^4+4h^2k^2+7h^2k+3k^3+2k^2)^4},$$

where $S_{71}$ is a polynomial of degree 20 in $h$ with coefficients polynomials in $k$ of degree at most 22. The polynomial $U_{71}$ has 265 terms while $V_{71}$ has 615 terms and only 8 negative coefficients. The polynomials $P_{71}$ and $Q_{71}$ both have 821 terms and all coefficients positive. Hence, $T_{71} > 0$.

**13. CENTRAL POINT $X_{72}$**

$X_{72}[(b+c)(b^2+c^2-a^2)]$ is the point of intersection of the line joining $X_1$ (incenter) with $X_6$ (Grebe-Lemoine point) and the line joining $X_4$ (orthocenter) with $X_8$

(Nagel point). By standard procedure we find that

$$T_{72} = \frac{r^4 S_{72}}{k^2 (f^2+1)^2 (g^2+1)^2}, \text{ where}$$

$S_{72} = (k^2 + 28k + 4)\, h^6 + k\,(k^3 - 4k^2 - 65k - 80)\, h^4 +$
$+ 2k\,(2k^4 - 6k^3 - 55k^2 - 104k - 64)\, h^2 +$
$+ k^4\,(5k + 8)\,(4k^2 + 11k + 8).$

The polynomial $U_{72}$ has 38 terms while $V_{72}$ has 78 terms and all coefficients positive. Hence, $T_{72} > 0$.

## 14. $X_{75}$ - Isogonal conjugate of the 2nd Power point

$X_{75}[1/a^2]$ is the point of intersection of the line joining $X_7$ (Gergonne point) with $X_8$ (Nagel point) and the line joining $X_{10}$ (Spieker center) with $X_{76}$ (3rd Brocard point - isogonal conjugate of the 3rd Power point). By usual method we find that

$$T_{75} = \frac{r^4 f^2 g^2 h^2 S_{75}}{k^2 (h^2k^2 + 3h^2k + k^3 + h^2 + k^2)^4},$$

where $S_{75}$ is a polynomial of degree 10 in $h$ with coefficients polynomials in $k$ of degree at most 13. The polynomial $U_{75}$ has 109 terms while $V_{75}$ has 268 terms and only 15 negative coefficients. The polynomials $P_{75}$ and $Q_{75}$ both have 319 terms and all coefficients positive. Hence, $T_{75} > 0$.

## 15. Central point $X_{101}$

$X_{101}[a/(b-c)]$ is the point on the circumcircle in which intersect the line joining $X_{71}$ with $X_{74}$ (isogonal conjugate of the intersection of Euler line with line at infinity) and the line joining $X_{10}$ (Spieker center) with $X_{98}$ (Tarry point). In the usual way we find that

$$T_{101} = \frac{r^4 m_1 m_2 m_3 (f-g)^2 (fk-h)^2 (gk-h)^2}{16k^4 (h^4 + k^4 - h^2k^2 - 4h^2k + 2k^3 - 3h^2 + k^2)^2}$$

where

$m_1 = 3f^2g^2 + f^3g + 2f^2 + g^2 + fg,$

$m_2 = 3f^2g^2 + fg^3 + f^2 + 2g^2 + fg,$

$m_3 = (1+3k)h^2 + (1+k)k^2.$

From this it is obvious that $T_{101} > 0$ in all cases except when $ABC$ is isosceles.

## 16. $X_{40}$ - Circumcenter of the extriangle

$X_{40}[(b+c)((b-c)^2 + a(b+c) - a^2) - a^3]$ is the point of concurrence of the perpendiculars from the excenters to the respective sides. Then

$$T_{40} = \frac{r^4 S_{40}}{16k^4}, \text{ where}$$

$S_{40} = 3h^8 + (4k^2 - 24k - 8)\, h^6 -$
$- (14k^4 + 24k^3 - 56k^2 - 32k + 16)\, h^4 +$
$+ 4k^2\,(k+2)\,(k^3 + 12k^2 + 2k - 4)\, h^2 +$
$+ k^4\,(k^2 - 4k - 4)\,(3k^2 + 4k + 4).$

The polynomial $U_{40}$ has 41 terms while $V_{40}$ has 71 terms and only 8 negative coefficients. Consider $V_{40}$ as a polynomial of degree 8 in $g$. It is amazing that coefficients of $g^5$, $g^4$, and $g^3$ are polynomials in $f$ with all coefficients positive. Even greater miracle is that quadratic trinomials associated to the first three terms (corresponding to powers 8, 7, and 6 with $g^6$ factored out) and the last three terms (quadratic part of $V_{40}$) both have positive leading coefficients and negative discriminants. We conclude that $T_{40} > 0$.

## 17. $X_{20}$ - De Longchamps point

When we apply the above method of proof to De Longchamps point $X_{20}$ or to Exeter point $X_{22}$ it does not work so that we must do something else. The idea is to position the triangle $ABC$ so that the circumcenter $O$ is the origin, the vertex $C$ is $[r, 0]$ where $r$ denotes the circumradius, and positions of the vertices $A$ and $B$ are determined by parameters $f$ and $g$ which are tangents of half of angles that $OA$ and $OB$ make with the $x$-axis $OC$. Without loss of generality we can assume that $f > 0$ and $g > 0$. With this placement the triangle test $T_{20}$ is an expression that will clearly always be positive.

The coordinates of $A$ and $B$ are

$$\left(\frac{r(1-f^2)}{1+f^2}, \frac{2rf}{1+f^2}\right) \text{ and } \left(\frac{r(1-g^2)}{1+g^2}, \frac{2rg}{1+g^2}\right).$$

It follows that $X_{20}$ has coordinates

$$\left(\frac{r(k^2+4k-h^2)}{(fh-k)(gh-k)}, \frac{-2rh(k+2)}{(fh-k)(gh-k)}\right),$$

while

$$T_{20} = \frac{64r^4((h^2+3)(k+3)^2 + f^2g^2)}{(fh-k)^2(gh-k)^2}.$$

## 18. $X_{22}$ - Exeter point

$X_{22}[b^4 + c^4 - a^4]$ is the point of concurrence of lines $A_gA_t$, $B_gB_t$, and $C_gC_t$, where $A_g$ is the intersection different from $A$ of the median line $AA_m$ with the circumcircle and $A_t$ is the intersection of tangents to the circumcircle at $B$ and $C$ and with $B_g$, $C_g$, $B_t$ and $C_t$ defined analogously.

In order to prove that $T_{22} > 0$ we use the same placement of the base triangle $ABC$ as in the proof for $X_{20}$.

Since the equation of the circumcircle is $x^2 + y^2 = r^2$, one can find coordinates of $A_g$, $B_g$, and $C_g$ by solving quadratic equations. The coordinates of $A_t$, $B_t$, and $C_t$ are

$$(r, rg), \quad (r, rf), \quad \left(\frac{-rk}{k+2}, \frac{rh}{k+2}\right).$$

It follows that $X_{22}$ has coordinates

$$\left(\frac{r(k^2+4k-h^2)}{h^2+k^2-4k-8}, \frac{-2rh(k+2)}{h^2+k^2-4k-8}\right), \text{ while}$$

$$T_{22} = \frac{64r^4 S_{22}}{(h^2+k^2-4k-8)^4}, \text{ where}$$

$S_{22} = (k+3)^2 h^6 + 2(k+1)\,(k^3 + k^2 - 16k - 34)\, h^4 +$
$+ (k^4 - 4k^3 + 112k + 224)\,(k+1)^2 h^2 +$
$+ 16\,(k^3 - 12k - 20)\,(k+1)^3.$

If we consider $S_{22}$ as a polynomial in $f$ and $g$ we get a polynomial $W_{22}$ with 27 terms. When we replace $f$ with $g + u$ in $W_{22}$ we obtain a polynomial of order 8 in $u$ whose coefficients are polynomials in $g$ with all coefficients positive. Hence, $W_{22}$ is positive when $f \geq g$. In a similar way we see that it is also positive when $g \geq f$. Hence, $T_{22} > 0$.

## 19. New triangular triples

We can now compute distances $AX_i$, $BX_i$, and $CX_i$ for last twelve central points from our theorem to get the following corollary.

Most interesting points, lines, circles, curves,... associated with the triangle $ABC$ are expressions that involve symmetric functions of lengths $a$, $b$, and $c$ of sides $BC$, $CA$, $AB$ that we denote as follows.

$s = a + b + c$, $t = bc + ca + ab$, $m = abc$,
$s_a = -a + b + c$, $s_b = a - b + c$, $s_c = a + b - c$,
$m_a = bc$, $m_b = ca$, $m_c = ab$,
$d_a = b - c$, $d_b = c - a$, $d_c = a - b$,
$z_a = b + c$, $z_b = c + a$, $z_c = a + b$.

For each $k \geq 2$, $s_{ka}$, $s_{kb}$, and $s_{kc}$ are derived from $s_a$, $s_b$ and $s_c$ with the substitution $a = a^k$, $b = b^k$, $c = c^k$. In a similar fashion we can define analogous expressions using letters $m$, $d$, $t$ and $z$.

For an expression $f$, let $[f]$ denote a triple $(f, \varphi(f), \psi(f))$, where $\varphi(f)$ and $\psi(f)$ are cyclic permutations of $f$. For example, if $f = \sin A$ and $g = b + c$, then $[f] = (\sin A, \sin B, \sin C)$ and $[g] = (b + c, c + a, a + b)$.

Let us call a triple $[a]$ of real numbers *triangular* provided $a$, $b$, and $c$ are sides of a triangle.

**Corollary**

*If the triple $[a]$ is triangular, then the triples*

$$\left[\sqrt{a(a z_a + 2 d_a^2 - a^2)}\right], \left[\sqrt{2 a^2 z_{2a} - d_a^4 - a^4}\right],$$

$$\left[\sqrt{z_a(z_{2a} - m_a) + a(2 z_{2a} - m_a) - a^3}\right],$$

$$\left[\sqrt{a^6 - 3a^2 d_{2a}^2 + 2 d_{2a}^2 z_{2a}}\right],$$

$$\left[\sqrt{a m_a z_a + a^2(2 z_{2a} - m_a) - a^4 - d_a^2 z_{2a}}\right],$$

$$\left[\sqrt{d_{2a}^2 z_{2a}^2 - 2 a^2 d_{2a}^2 z_{2a} + 2 a^6 z_{2a} - a^8}\right],$$

$$\left[\sqrt{m_a (a^4 - 2 a^2 d_a^2 + d_{2a}^2)}\right],$$

$$\left[\sqrt{m_a (a^2 d_{2a}^2 + a^4 z_a^2 - a^6 - d_a^2 d_{2a}^2)}\right],$$

$$\Big[m_a(a^5 z_a^3 + a^6(2 z_{2a} - m_a) - a^8 - a^7 z_a - a^4 z_a^2(z_{2a} - 3m_a) + a^3 d_a^3 d_{2a} + a^2 d_{2a}^2 m_a - a d_a d_{2a}^3 - d_a^2 d_{2a}^2)^{\frac{1}{2}}\Big],$$

$$\Big[(3a d_a^2 m_a z_a + a^2 d_a^2(3m_a + z_{2a}) + a^3 m_a z_a + a^4(m_a + z_{2a}) - a^6 - d_a^4 z_a^2)^{\frac{1}{2}}\Big],$$

$$\left[a\sqrt{z_a^2(z_{2a} - m_a) - a m}\right], \textit{ and (with } ABC \textit{ not}$$

*isosceles)* $\left[m_a |d_a|\right]$ *are also triangular.*

## 20. Nagel point - Second proof

In this and the following sections we shall give alternative proofs for all fourteen central points using more traditional methods of proof. Of course, again we shall suppress most details because they are awkward to print.

We first find that $AX_8^2 = a(a s_a + 2d_a^2) / s$. It follows that

$$T_8 = \frac{T(a,b,c)(s^3 - 24m)}{s^3}$$ is positive since $s^3 \geq 27m$.

## 21. Mittenpunkt - Second proof

Since $AX_9^2 = (a^2(s_{2a} + z_{2a}) - d_a^4) / (s_2 - 2t)^2$, we obtain $T_9 = S_9 / q^4$, where $S_9$ is a polynomial
$(4R^2 - r^2)(12R^2 + 16rR + r^2)\sigma^4 -$
$- 2q^3 r(4R^2 - 8rR + r^2)\sigma^2 - q^6 r^2$, and $q = r + 4R$.

Here, $\sigma$ is the semi-perimeter, $r$ is the inradius, and $R$ is the circumradius. In order to derive this representation for $S_9$ we first write numerator and denominator of $T([AX_9])$ in terms of elementary symmetric polynomials $s$, $t$, and $m$ and then use the fact [5, p.7] that lengths of sides $a$, $b$, and $c$ are roots of the polynomial
$x^3 - 2\sigma x^2 + (\sigma^2 + qr)x - 4R\sigma$.

The conclusion that $S_9 > 0$ is argued as follows. Let

$$l_\pm = 2R^2 + 10R - r^2 \pm 2(R - 2r)\sqrt{R^2 - 2Rr}.$$

It is well-known (see [5, p. 2]) that $l_- \leq \sigma^2 \leq l_+$. Hence, it suffices to show that the polynomial $S_9$ is positive on the segment $[l_-, l_+]$. In other words, it suffices to show that for every number $p$ in the interior of the segment $[l_-, l_+]$, replacing $\sigma^4$ and $\sigma^2$ with $p^2$ and $p$ in $S_9$ we obtain positive value.

Any point $p$ from this segment different from $l_-$ and $l_+$ can be represented as $(l_- + k\, l_+) / (k + 1)$ for some positive real number $k$. When we compute $S_9$ at $p$ or at $l_-$ and $l_+$ and substitute $R = 2r + \varepsilon$, where $\varepsilon \geq 0$ (recall that $R \geq 2r$), we obtain $\lambda(M - N)$, where expressions $\lambda$, $M$, and $N$ are all positive. But, on can check that $M^2 - N^2$ is positive so that our claim follows.

## 22. Spieker center and De Longchamps point – Second proof

Since $AX_{10}^2 = (a s_{2a} + s(z_{2a} - m_a))/(4s)$, we have $T_{10} = S_{10} / 16$, where $S_{10}$ denotes the polynomial
$3\sigma^4 - 2r(4q - 35r)\sigma^2 - r^2(4q - r)(4q + 3r)$.

We continue as in the above proof to show that $S_{10} > 0$ for every triangle.

The identical approach applies to the central points $X_{20}$ (De Longchamps point), $X_{22}$ (Exeter point), $X_{40}$, $X_{63}$, $X_{71}$, $X_{72}$, and $X_{75}$.

## 23. Schiffler point – Second proof

This time
$AX_{21}^2 =$
$m z_a^2(a^2(s_{2a} - m_a) + a m_a z_a + s_b s_c z_{2a}) / (s(s^3 - 4st + 5m)^2)$,

so that the sign of $T_{21}$ depends only on the sign of $8r\,\sigma\,(r+R)$ which is clearly positive.

## 24. Central point $X_{101}$ – Second proof

Since $AX_{101}{}^2 = m_a{}^2\, d_a{}^2\, (s_4 + m\, s - a^3\, z_a - b^3\, z_b - c^3\, z_c)$, we immediately get

$$T_{101} = \frac{d_a^2\,(a\,s_a + m_a)\,d_b^2\,(a\,s_b + m_b)\,d_c^2\,(a\,s_c + m_c)}{(s^4 - 5\,s^2 t + 4t^2 + 6\,m\,s)^2}$$

is always positive unless $ABC$ is isosceles.

## 25. Convex hull of the polynomial group

The results in the previous sections might be quite inadequate to some readers because it is clear from the Section 3 that the region $T_{ABC}$ in the plane of the triangle $ABC$ consisting of all points $P$ such that segments $AP$, $BP$, and $C\,P$ are sides of a triangle is rather large while we have only found thirteen points that always belong to this region. In this section we shall improve our results by showing that the same technique applies to prove that the convex hull of the central points $X_2$, $X_3$, $X_8$, $X_9$, $X_{20}$, $X_{21}$, $X_{63}$, $X_{71}$, $X_{72}$, and $X_{75}$ always belongs to the region $T_{ABC}$. Notice that these are precisely the points for which our argument involving polynomials with all coefficients positive worked.

***Theorem***

*For any triangle ABC the convex hull of central points $X_2$, $X_3$, $X_8$, $X_9$, $X_{20}$, $X_{21}$, $X_{63}$, $X_{71}$, $X_{72}$, and $X_{75}$ consists only of points P with the property that the segments AP, BP, and CP are sides of a triangle.*

***Proof.***

We shall only give outlines for the proof that the segment $GO$ joining the centroid $G$ with the circumcenter $O$ and the triangle $GON$ with vertices $G$, $O$, and the Nagel point $N$ both lie in the region $T_{ABC}$. In a similar fashion one can show that any segment and any triangle on any two and on any three of the ten central points listed in the statement of the theorem have the same property. Of course, it is impossible to give all details of our proofs because in some cases we get polynomials with hundreds of terms so that without computers this approach is a hard task. Since most readers might make a standard mistake in thinking that everything about triangles belongs to elementary mathematics (whatever this might mean), it would be interesting to the author to see their "elementary" proofs of our results.

A point $P$ in the interior of the segment $GO$ has coordinates $[p, q]$, where

$p = r\,(3h\,k\,x + 2f\,v + 4g\,u)\,/\,(6k(x+1))$,

$q = r\,(8f\,g + 3(h^2 - k^2)\,x)\,/\,(12k(x+1))$ and $x$ is a positive real number.

When we substitute these values into the polynomial $t_{ABC}$ we obtain $r^4\, S_{GO}\,/\,(6912\,(x+1)^4)$, where $S_{GO} = \sum_{i=0}^{4} k_i\, x^i$ and each coefficient $k_i$ is a polynomial in $f$ and $g$. The replacement of $f$ with $1+f$ and $g$ with $1+g$ in $k_i$ for $i=0,\ldots,4$ leads to polynomials with all coefficients positive which completes our proof for this very simple case. For most other segments with ends among ten points from the statement we must also perform substitutions $f = g + u'$ and $g = f + v'$ (with $u'$, $v' \geq 0$) in order to get polynomials with all coefficients positive.

Let us now consider the triangle $GON$. An arbitrary point $P$ in its interior has coordinates $[p, q]$, where

$p = r\,(6\,(f\,\psi - 2g)\,x\,y + 2(f\,v + 2g\,u)\,x + 3h\,k\,y + 2f\,v + 4g\,u)\,/\,(6k\,(x+1)(y+1))$,

$q = r\,(24x\,y + 8f\,g\,x + 3(h^2 - k^2)\,y + 8f\,g)\,/\,(12k\,(x+1)(y+1))$,

and $x$ and $y$ are positive real numbers. When we substitute these values into the polynomial $t_{ABC}$ we obtain $r^4\, S_{GON}\,/\,(6912\,(x+1)^4\,(y+1)^4)$, where $S_{GON}$ is a polynomial of order 8 in $x$ and $y$ and whose coefficients $k_i$ ($i=0,\ldots,24$) are polynomials in $f$ and $g$. The replacement of $f$ with $1+f$ and $g$ with $1+g$ in $k_i$ for $i=0,\ldots,24$ leads to polynomials with almost all coefficients positive. However, after we perform substitutions $f = g + u'$ and $g = f + v'$ (with $u'$, $v' \geq 0$) we obtain polynomials with all coefficients positive which completes our proof. For other triangles the same strategy always applies but with far more complicated polynomials (with several hundreds of terms and very large coefficients).

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