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JELENA BEBAN - BRKIĆ

On the Focal Curve of Conic Pencils in I_2

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ABSTRACT

Within the classification of conic pencils in the isotropic plane, which has been carried out using methods of analytical geometry and linear algebra, it is specially interesting to observe the curve of isotropic focuses, which is shown to be a 3rd order curve. The properties of this very curve, for the discussed, most common subtype of conic pencils, are determined. It is shown that, referring to the selection of the fundamental points, it is possible to determine its shape, and to classify it according to the Newton's principle.

The discussed cases of conic pencils with its focal curves are illustrated with the figures programmed and drawn by *Mathematica®*.

Keywords: isotropic plane, fundamental points, pencil of conic sections, matrix of quadratic form, eigenvalues, focal curve, Newton's classification

O krivulji fokusa pramena konika u I_2
SAŽETAK

Pri klasifikaciji pramenova konika u izotropnoj ravnini, provedenoj metodama analitičke geometrije i linearne algebre, posebno je interesantno promatrati krivulju izotropnih fokusa za koju se pokazuje da je krivulja 3. reda. Određena su svojstva krivulje fokusa za promatrani, najopćenitiji, podtip pramena. Pokazano je da već prema odabiru temeljnih točaka možemo odrediti njen oblik i klasifikaciju prema Newtonovom principu.

Promatrani slučajevi su prikazani crtežima izrađenim pomoću programa *Mathematica®*.

Ključne riječi: izotropna ravnina, temeljne točke, pramen konika, matrica kvadratne forme, svojstvene vrijednosti, krivulja fokusa, Newtonova klasifikacija

INTRODUCTION

The pencil of conic sections (pencil of conics) is determined by two arbitrarily chosen conics of the pencil ([4], [12], [17]). These two conics have four common points that we call fundamental points of the pencil and mark them *A, B, C,* and *D*. Depending on the reality, position and multiplicity of fundamental points, various types of conic pencils can be distinguished. The classification in the projective and affine plane with respect to the position and multiplicity of fundamental points can be found for example in [4], [6].

The isotropic plane $I_2(R)$ is a real affine plane A_2 metrizied with real line $f \subset A_2$ and the point F incidental with it (see [13], [14]). The ordered pair $\{f, F\}$, $F \in f$, is called absolute figure of the plane $I_2(R)$.

Because of $I_2 \subset A_2 \subset P_2$, where P_2 is a real projective plane, all affine and projective properties of the conic pencils and of other geometric configurations in I_2 can be used.

In $I_2 \subset P_2$ the projective, homogeneous, $(x_0 : x_1 : x_2)$ and the accompanying affine coordinates $x = x_1/x_0$, $y = y_2/y_0$ are used. The absolute line f is defined by $x_0 = 0$, and the absolute point F by F (0:0:1). For the fundamental group of transformations one takes the mappings of the form

$$\bar{x} = c_1 + x$$
$$\bar{y} = c_2 + c_3 x + y, \quad \text{where } c_1, c_2, c_3 \in R,$$

which make the three-parametric *Motion group G_3 of isotropic plane* [13].

With the geometry of isotropic plane one can get acquainted through the works of N. M. Makarowa [11], B. Pavković [13], H. Sachs [14], K. Strubecker [18], and others. The conics of isotropic plane were first considered by N. M. Makarowa and K. Strubecker. The conic pencils of the most general type in I_2 have been worked by V. Ščurić-Čudovan [19] by means of synthetic method.

The classification of conic pencils in I_2 can be made with respect to the reality and multiplicity of fundamental points and regarding their position, as well as the position of fundamental straight lines according to the absolute figure $\{f, F\}$. For a more comprehensive classification, the curve of the centres and the curve of isotropic focuses (focal curve) are observed for each type of the conics' pencil. It is specially interesting to observe the focal curve which is going to be proved to be a 3[rd] order curve. Therefore this very curve is to be discussed in the present paper. An analysis of the most general subtype of conic pencils in I_2 will be made, and the classification of its cases will be carried out, using methods of analytical geometry and linear algebra. The properties of the focal curve for each discussed case are determined and, it is shown that, referring to the choice of the fundamental points, it is possible to determine its shape, and to classify it according to the Newton's principle [16].

The discussed cases of conic pencils with its focal curves are illustrated with the figures programmed and drawn by *Mathematica®*

1. Pencil of Conics

Fundamental points

Let the fundamental points A, B, C, and D be real, mutually different, providing that three of them do not lie on the same straight line. As it is the most general case, the pencils generated by such a selection of points, are called type **I**.

Fundamental lines

The fundamental points are joined by three singular, in three pairs of straight lines degenerated conics. Two arbitrarily chosen pairs, out of this three pairs of straight lines, are called fundamental straight lines.

Let the fundamental straight lines u, v, p and q be given in the equations

u... $a_u x + b_u y + c_u = 0$, $u{:=}CD$,
v... $a_v x + b_v y + c_v = 0$, $v{:=}AB$,
p... $a_p x + b_p y + c_p = 0$, $p{:=}AD$,
q... $a_q x + b_q y + c_q = 0$. $q{:=}BC$.

Main points

Three points of the intersection $K{=}AB\cap CD$, $L{=}AD\cap BC$, $M{=}AC\cap BD$, are called main points.

These are the centres of the singular conics of the pencil. The points K, L, M form a triangle that is self-polar for the conics of the pencil (see [12], [17]).

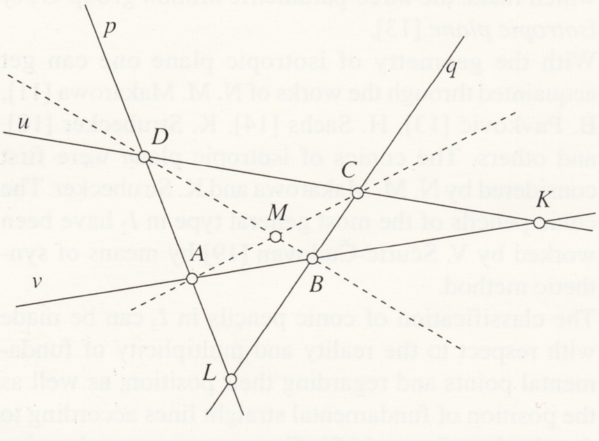

Fig.1: The most general position of the fundamental points, the accompanying fundamental lines and main points.

Fundamental conics

As fundamental conics two arbitrarily conics of the pencil can be chosen. We shall choose the two singular, in two pairs of fundamental straight lines degenerated conics.

Let the fundamental conics of a pencil be given in the equations

$F(x,y) \equiv a_1x^2 + 2b_1xy + c_1y^2 + 2d_1x + 2e_1y + f_1 = 0$, and
$G(x,y) \equiv a_2x^2 + 2b_2xy + c_2y^2 + 2d_2x + 2e_2y + f_2 = 0$,

where at least one of the numbers a_1, b_1, c_1, as well as one of the numbers a_2, b_2, c_2, is different from zero [3].

Their equations can be written in the form

$F(x,y) \equiv u\,v = 0$,
$G(x,y) \equiv p\,q = 0$,

where u, v, p, and q are the fundamental straight lines. The coefficients of the fundamental conics described by means of the coefficients of fundamental straight lines are

$a_1{=}a_u a_v$, $2b_1{=}a_u b_v{+}a_v b_u$, $c_1{=}b_u b_v$,
$2d_1{=}a_u c_v{+}a_v c_u$, $2e_1{=}b_u c_v{+}b_v c_u$, $f_1{=}c_u c_v$,
$a_2{=}a_p a_q$, $2b_2{=}a_p b_q{+}a_q b_p$, $c_2{=}b_p b_q$,
$2d_2{=}a_p c_q{+}a_q c_p$, $2e_2{=}b_p c_q{+}b_q c_p$, $f_2{=}c_p c_q$.

Pencil of Conics

The pencil of conic sections, defined by the fundamental conics $F(x,y) = 0$ and $G(x,y) = 0$, is given in the form

$H(x,y) \equiv \alpha\, F(x,y) + \beta\, G(x,y) = 0$,

and by introducing the parameter $\lambda{=}\beta/\alpha$, $\alpha \neq 0$, we have

$H(x,y) \equiv F(x,y) + \lambda G(x,y) = 0$, $\lambda \in R$.

The pencil $H(x,y) = 0$ can be written in the form

$$H(x,y) \equiv ax^2 + 2bxy + cy^2 + 2dx + 2ey + f = 0, \qquad (1.1)$$

where we have marked

$a = a_1 + \lambda a_2 = a_u a_v + \lambda a_p a_q$,
$2b = 2(b_1 + \lambda b_2) = a_u b_v + a_v b_u + \lambda(a_p b_q + a_q b_p)$,
$c = c_1 + \lambda c_2 = b_u b_v + \lambda b_p b_q$,
$2d = 2(d_1 + \lambda d_2) = a_u c_v + a_v c_u + \lambda\,(a_p c_q + a_q c_p)$,
$2e = 2(e_1 + \lambda e_2) = b_u c_v + b_v c_u + \lambda\,(b_p c_q + b_q c_p)$,
$f = f_1 + \lambda f_2 = c_u c_v + \lambda\, c_p c_q$.

For each $\lambda \in R$, the set of zeros of the polynomial $H(x,y) = F(x,y) + \lambda G(x,y)$ presents one of the second order curves if at least one of the numbers a, b, c is different from zero.

Focal curve

The notion of absolute plane points is connected with the notion of the focuses of second order curves. It is known that in the affine plane metrizied through absolute points (I_1, I_2), the tangents drawn from absolute points onto the second order curve intersect in the focuses of the curve (see [16], [17]).

In the isotropic plane in which there is only one absolute point F $(I_1 = I_2 = F,\ F \in R)$ the focuses are defined as real contact points of tangents drawn from the absolute point onto the second order curve. The geometric loci of the contact points of all tangents drawn from the absolute point onto the curves of conic pencil is the curve of isotropic focuses that we call focal curve.

2. Curves of the Conic Pencils

The type of the curve within the pencil (1.1) depends on the eigenvalues μ_1, μ_2 of the matrix

$$\sigma = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$$

of the quadratic form $ax^2 + 2bxy + cy^2$ within the polynomial $H = H(x,y)$ ([7], [8]).

The eigenvalues μ_1, μ_2 are obtained as the solutions of the equation

$\mu^2 - (a + c)\,\mu + a\,c - b^2 = 0,$

where

$a + c = tr\,\sigma = \mu_1 + \mu_2,$

and

$ac - b^2 = det\,\sigma = \mu_1\,\mu_2.$

It is known that the set of the second order polynomial zeros represent the curve of elliptical, parabolic or hyperbolic type depending on whether $det\sigma$ is greater, equal or less than zero ([3], [4], [8]).

We are observing the case when $det\,\sigma = 0$. Thereby the answer will be obtained of when and how many conics of the parabolic type are conveyed within the pencil.

$$det\,\sigma = \mu_1\mu_2 = ac - b^2 = = (a_1 + \lambda a_2)(c_1 + \lambda c_2) - (b_1 + \lambda b_2)^2,$$

that is

$$det\,\sigma = m\,\lambda^2 + n\,\lambda + p, \tag{2.1}$$

where we have put

$$m = a_2\,c_2 - b_2^{\,2} = a_p a_q\, b_p b_q - (a_p b_q + a_q b_p)/\,2,$$
$$n = a_1\,c_2 + a_2\,c_1 - 2b_1 b_2 = a_u a_v\, b_p b_q + a_p a_q\, b_u b_v - (a_u b_v + a_v b_u)(\,a_p b_q + a_q b_p), \tag{2.2}$$
$$p = a_1\,c_1 - b_1^{\,2} = a_u a_v\, b_u b_v - (a_u b_v + a_v b_u)^2/\,4.$$

Discriminante of a polynomial (2.1) is to be marked with D.

It can be derived that the determinant of the matrix σ will be equal zero in the following cases [10]:

i) for each $\lambda \in R$,
 if m=0, n=0, p=0;

ii) for neither $\lambda \in R$,
 if m=0, n=0, $p \neq 0$ or $m \neq 0$ i D<0;

iii) for $\lambda = -p/n$,
 if m=0, $n \neq 0$; (2.3)

iv) for $\lambda = -n/\,2m$,
 if $m \neq 0$ and D=0

v) for $\lambda_{1,2} = (-n \pm\sqrt{D}\,)/\,2m$,
 if $m \neq 0$ and D>0.

It can be seen from (2.3) that within the conic pencil all, two, one or none of the curves can be of parabolic type. In the most general case (2.3, v), two conics of parabolic type will exist that will separate the curves of hyperbolic from those of elliptical type.

3. List and Description of the Pencils of the Type I

Considering the pencils of conic sections of type **I**, the obtained results, according the position of the fundamental points, are given in the following proposition.

Proposition 1.

In the isotropic plane $I_2(R)$ referring to the group G_3 of isotropic motions there are 8 different subtypes of conic pencils determined with four real and mutually different points.

Proof. The proof is given in [2].

List and Description of the Pencils of the Type I

I.1. Fundamental points (A, B, C, D) are real finite points and no connecting straight line of fundamental points is an isotropic line, i. e. a line through the absolute point F.

I.1.1. Main points (K, L, M) are in the finiteness. Thereare two subcases.

I.1.2. One main point is on the absolute line f.

I.1.3. Two main points are on the absolute line f.

I.2. The fundamental points (A, B, C, D) are real finite points and one connecting straight line of the fundamental points is an isotropic line.

I.2.1. The main points (K, L, M) are in the finiteness There are two subcases.

I.2.2. One main point is on the absolute line f.

I.2.3. Two main points are on the absolute line f.

I.3. The fundamental points (A, B, C, D) are real finite points and two connecting straight lines of the fundamental points are isotropic lines.

I.3.1. One main point is on the absolute line f and it coincides with the absolute point F.

I.3.2. Two main points are on the absolute line f and one of them coincides with the absolute point F.

I.4. Two fundamental points are finite points and two on the absolute line f. The two fundamental points on the absolute line f do not coincide with the absolute point F.

I.5. Two fundamental points are finite points and two on the line f. The two fundamental points on the absolute line f do not coincide with the absolute point F. Connecting straight line of the fundamental finite points is an isotropic line.

I.6. Two fundamental points are finite points and two on the absolute line f. One of the two fundamental points on the absolute line f coincides with the point F.

I.7. Three fundamental points are finite points, and one is on the absolute line f not coinciding with the absolute point F.

I.7.1. No connecting straight line of the fundamental points is an isotropic line.

I.7.2. One connecting straight line of the fundamental points is an isotropic line.

I.8. Three fundamental points are finite points, and the point on the absolute line f coincides with the absolute point F.

4. Focal Curve

Without becoming less general, we can furtheron observe the normed equations of the fundamental lines and presume that the coefficients are

$a_u = a_v = a_p = a_q = 1$, (4.1)

if not defined otherwise.

Furthermore, using the group G_3 of motions of isotropic plane we can always map one of the fundamental lines (e.g. line v) onto the line $y = 0$, and one (e.g. line p) adjust so that it passes through the point of origin of the coordinate system. If it is not defined otherwise, we shall therefore presume furtheron that the coefficients are

$a_v = 0,\ b_v = 1,\ c_v = 0,\ c_p = 0.$ (4.2)

4.1. Pencil I.1.

The characteristic of the pencil - subtype **I**.1.: The fundamental points (A, B, C, D) are at finiteness and no connecting straight line of the fundamental points is an isotropic line.

The common characteristic for all the pencils of the discussed subtype, according the focal curve, is established by the following proposition.

Proposition 2. *Focal curve $k_f^{\,3}$ of each pencil-subtype I.1. has no double points.*

Proof. Let us presume otherwise, i.e. let $k_f^{\,3}$ have a double point T. That means that a common tangent of two curves of the pencils k_1 and k_2 ($k_1 \neq k_2$) is passing through the point T. Every two conics of the pencil have only 4 fundamental points in common. If $T \neq A, B, C,$ and D, hence the conics k_1 and k_2 have 5 mutual points, so $k_1 = k_2$. The point T cannot coincide with one of the fundamental points A, B, C, or D, since it is double, and the fundamental points are real and various.

Pencil I.1.1.

The characteristic of the pencil-case **I**.1.1.:

The main points (K, L, M) are in the finiteness.

Let the coordinates of the fundamental points be

$A(0,0)$, $B(b_1,0)$, $b_1>0$, $C(c_1,c_2)$, $c_1,c_2 \neq 0$,

$D(d_1,d_2)$, $d_1,d_2 \neq 0$, $b_1 \neq c_1 \neq d_1$, $c_2 \neq d_2$. (4.3)

The fundamental straight lines

$v := AB,\ p := AD,\ q := BC,\ u := CD$

are given in the equations:

$v \ldots\ y = 0.$

$p \ldots\ y = \varphi_p x$, where $\varphi_p = \dfrac{d_2}{d_1}$.

If we put $b_p = -\dfrac{1}{\varphi_p}$,

it follows $x + b_p y = 0.$

$q \ldots\ y = \varphi_q + v_q$,

where $\varphi_q = \dfrac{c_2}{c_1 - b_1}$ $v_q = -\dfrac{b_1 c_2}{c_1 - b_1}$.

If we put $b_q = -\dfrac{1}{\varphi_q}$, and $c_q = \dfrac{v_q}{\varphi_q} = -b_1$,

it follows $x + b_q y + c_q = 0.$

$u \ldots\ y = \varphi_u x + v_u$,

where $\varphi_u = \dfrac{d_2 - c_2}{d_1 - c_1}$ $v_u = \dfrac{c_2 d_1 - c_1 d_2}{d_1 - c_1}$.

If we put $b_u = -\dfrac{1}{\varphi_u}$, and $c_u = \dfrac{v_u}{\varphi_u} = \dfrac{c_2 d_1 - c_1 d_2}{d_2 - c_2}$,

it follows $x + b_u y + c_u = 0.$ (4.4)

The coordinates of the fundamental points described through b_p, b_q, c_q, b_u and c_u are

$A\,(0,0)$, $B\,(-c_q, 0)$,

$$C\left(\frac{b_u c_q - b_q c_u}{b_q - b_u}, \frac{c_u - c_q}{b_q - b_u}\right),\quad D\left(-\frac{b_p c_u}{b_p - b_u}, \frac{c_u}{b_p - b_u}\right),$$

where, according (4.3), we have

$b_q, b_u, b_p \neq 0$, $c_u, c_q \neq 0$ and $b_q \neq b_u$, $b_p \neq b_u$. (4.5)

The fundamental conic section determined by the lines u, v is given in the equation

$G\,(x,y) \equiv u\,v = 0$, that is

$G\,(x,y) \equiv (x + b_u y + c_u)\, y = xy + b_u y^2 + c_u y = 0.$

The quadratic form in the above equation has the matrix

$$\sigma = \begin{bmatrix} 0 & 1/2 \\ 1/2 & b_u \end{bmatrix},$$

and the determinant of the matrix is

$m = \det \sigma = -1/4.$

It is the matter of the hyperbolic type of a conic.

The fundamental conic section determined by the lines p, q is

$F\,(x,y) \equiv p\,q = 0$, that is

$$F\,(x,y) \equiv (x + b_p y)(x + b_q y + c_q) = \\ = x^2 + (b_p + b_q)xy + b_p b_q y^2 + c_q x + b_p c_q y = 0.$$

The quadratic form in the above equation has the matrix

$$\sigma = \begin{bmatrix} 1 & (b_p + b_q)/2 \\ (b_p + b_q)/2 & b_p b_q \end{bmatrix},$$

and the determinant of the matrix is

$p = \det \sigma = b_p b_q - (b_p + b_q)^2/4 = -(b_p - b_q)^2/4.$

According (4.4), $b_p = b_q$ would meant that the lines p and q are mutually parallel. This would implie that one main point is in the infinity, which is the case that does not belong to the pencils-subtype **I**.1.1. We can conclude that p is always less than zero, wherefrom it follows that it is the matter of the hyperbolic type of a conic.

The pencil generated with the curves F $(x,y) = 0$ and $G\,(x,y) = 0$ is

$H\,(x,y) \equiv F\,(x,y) + \lambda G\,(x,y) = 0$, that is

$$H\,(x,y) \equiv x^2 + (b_p + b_q + \lambda)xy + \\ + (b_p b_q + \lambda b_u)y^2 + c_q x + \\ + (b_p c_q + \lambda c_u)y = 0. \quad (4.6)$$

Proposition 3.
Conic pencils of the case **I**.*1.1. are up to the motions G_3 of the isotropic plane with five ivariants $\{\varphi_p, \varphi_q, v_q, \varphi_u, v_u\}$, i.e. $\{b_p, b_q, c_q, b_u, c_u\}$ fully determined.*

In order to determine how many and which curves of parabolic type are within this pencil, we need the determinant of quadratic form (2.1), i.e.
$det\ \sigma = m\lambda^2 + n\lambda + p$,
where *m, n, p* are the values given in relations (2.2).

$$det\ \sigma = 0 \Rightarrow m\lambda^2 + n\lambda + p = 0 \Rightarrow$$

$$\Rightarrow \lambda_{1,2} = \frac{-n \pm \sqrt{n^2 - 4mp}}{2m}. \qquad (4.7)$$

Which case it will be out of (2.3), depends on
$D = n^2 - 4mp$.
$D = (b_u - (b_p + b_q)/2)^2 - 4(-1/4)(-(b_p - b_q)^2/4)$
$= (b_u - b_p)(b_u - b_q)$.

According (2.3) it can be concluded that within the case **I**.1.1. there are three different subcases of conic pencils, depending on whether it is $D > 0$, $D < 0$ or $D = 0$.

FOCAL CURVE WITHIN THE PENCILS I.1.1.
We take the equation (4.6) of the pencil:

$$H(x,y) \equiv x^2 + (b_p + b_q + \lambda)xy +$$

$$+(b_p b_q + \lambda b_u)y^2 + c_q x + (b_p c_q + \lambda c_u)y = 0.$$

One computes

$$\frac{\partial H}{\partial y} \equiv (b_p + b_q + \lambda)x +$$

$$+2(b_p b_q + \lambda b_u)y + (b_p c_q + \lambda c_u) = 0,$$

and therefrom

$$\lambda = -\frac{(b_p + b_q)x + 2b_p b_q y + c_q b_p}{x + 2b_u y + c_u}.$$

Inserting thus received λ in $H(x,y) = 0$, the following focal curve is obtained :

$$k_f^3 \equiv x^3 + 2b_u x^2 y + ((b_p + b_q)b_u - b_p b_q)xy^2 +$$
$$+(c_u + c_q)x^2 + 2b_u c_q xy +$$
$$+(b_u b_p c_q - b_p b_q c_u)y^2 + c_u c_q x = 0. \qquad (4.8)$$

In order to define the obtained 3rd order curve more closely, let us go over to the homogeneous coordinates. Introducing $x = x_1/x_0$, $y = y_1/y_0$, we get

$$k_f^3 \equiv \frac{x_1^3}{x_0^3} + 2b_u \frac{x_1^2 x_2}{x_0^3} +$$

$$+((b_p + b_q)b_u - b_p b_q)\frac{x_1 x_2^2}{x_0^3} + (c_u + c_q)\frac{x_1^2}{x_0^2} +$$

$$+2b_u c_q \frac{x_1 x_2}{x_0^2} + (b_u b_p c_q - b_p b_q c_u)\frac{x_2^2}{x_0^2} + c_u c_q \frac{x_1}{x_0} = 0.$$

Looking for the points on the absolute straight line f, $f \equiv x_0 = 0$, one receives:

i) absolute point $F(0:0:1)$, on the line $x_1 = 0$;
ii) 2nd order equation
$x_1^2 + 2b_u x_1 x_2 + ((b_p + b_q)b_u - b_p b_q)x_2^2 = 0$,
and therefrom

$$\left(\frac{x_1}{x_2}\right)_{1,2} = -b_u \pm \sqrt{B}, \text{ where we have marked}$$

$$B = b_u^2 - (b_p + b_q)b_u + b_p b_q.$$

According i) and ii), there are three possibilities:

a) $B > 0$ ($D > 0$) $\Rightarrow$ There are three points at infinity:
1) absolute point $F(0:0:1)$,
2) point $T_1(0 : b_u - \sqrt{B} : -1)$,
3) point $T_2(0 : b_u + \sqrt{B} : -1)$.

b) $B < 0$ ($D < 0$) $\Rightarrow$ There is only one real point at infinity and that is the absolute point F $(0:0:1)$.

c) $B = 0$ ($D = 0$) $\Rightarrow$ There are two real points at infinity :
1) absolute point $F(0:0:1)$,
2) point $T_1(0 : b_u : -1)$.

PENCIL I.1.1. a)
The characteristic of the pencil - subcase **I**.1.1.a):
$D = (b_u - b_p)(b_u - b_q) > 0 \Rightarrow$
$(b_u > b_q$ and $b_u > b_p)$ or $(b_u < b_q$ and $b_u < b_p)$

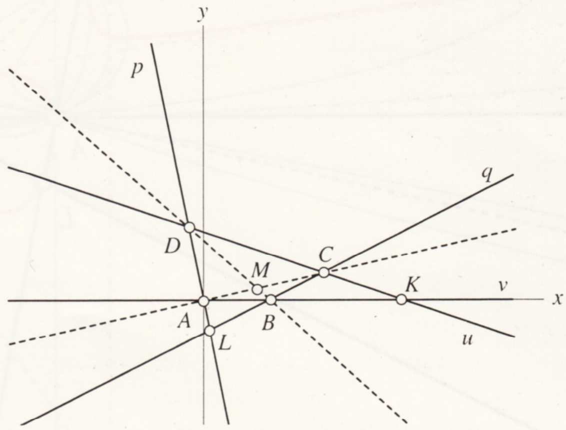

Fig.2: Fundamental points, the accompanying fundamental lines and main points within the pencil **I**.1.1.a), obtained by $b_u > b_q$ and $b_u > b_p$.

Since $D > 0$, it is the case (2.3, v), that is, there are exactly two conics of parabolic type within the pencil which are obtained when the values (4.7) are introduced into the pencil equation. They separate the conics of the hyperbolic from those of elliptic type. Thus, the conic pencil induces on the absolute straight line f one hyperbolic involution ([4], [12]).
Let us write the focal curve k_f^3 in the form
$a y^2 + b y + c = 0$,
where we have marked

$a = ((b_p + b_q)b_u - b_p b_q)\, x + (b_u b_p c_q - b_p b_q c_u),$
$b = 2b_u x^2 + 2b_u c_q x,$
$c = x^3 + (c_u + c_q)x^2 + c_u c_q x.$

The isotropic asymptote, i. e. the asymptote passing through the absolute point F, is obtained from the condition $a = 0$, that is

$$x = \frac{b_p(b_q c_u - b_u c_q)}{b_p(b_u - b_q) + b_q b_u}.$$

Since $D > 0 \Rightarrow B > 0$, the curve $k_f^{\,3}$ of isotropic focuses has three points at infinity. Therefrom, according to the Newton's classification of the 3rd order curves, $k_f^{\,3}$ belongs to the 1. group, i.e. among the curves having three asymptotes and three hyperbolic branches. The curves of this group are called "broken hyperbolas".

The basic forms of the curves from this group are determined with the roots of the auxiliary equation $D_4(x) = 0$, where $D_4(x) = b^2 - 4ac$ is the fourth degree polynomial. The calculation yields

$$D_4(x) = (b_u - b_q)\cdot(b_u - b_p)\cdot$$
$$\cdot x\,(x + c_q)\left(x + \frac{b_q c_u - b_u c_q}{b_q - b_u}\right)\left(x - \frac{b_p c_u}{b_u - b_p}\right),$$

wherefrom it can be seen that the roots of the equation $D_4(x) = 0$ are the abscissas of the fundamental points A, B, C, and D (4.5). As the points are real and various and within the observed pencil there are no isotropic lines, it holds.

Proposition 4.
The focal curve $k_f^{\,3}$ of all pencils of the subcase **I**.1.1.a) *consists of three hyperbolic branches and an oval, or of two hyperbolic branches and one straight line's branch.*

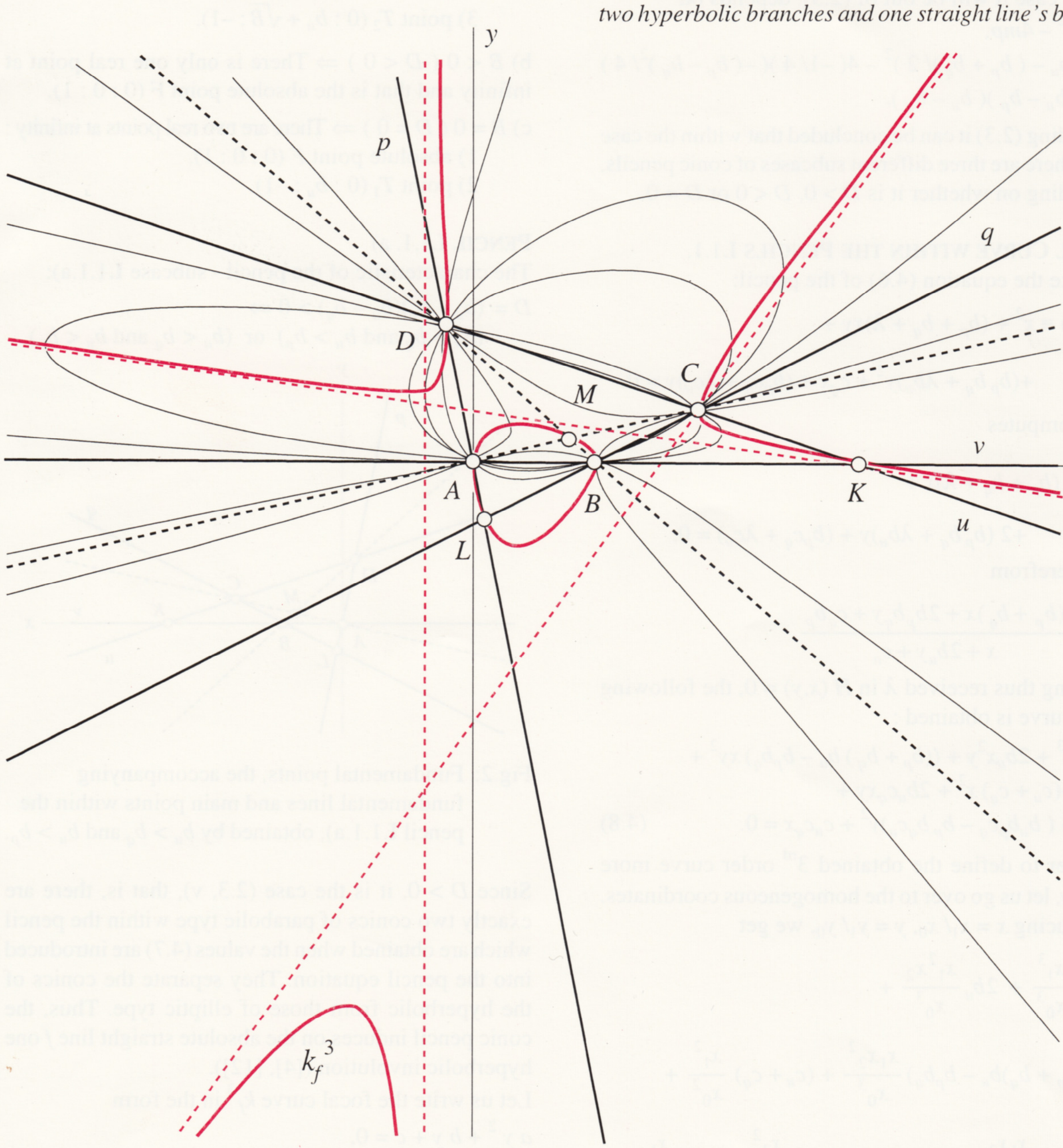

Fig.3: Pencil of conic sections - subcase **I**.1.1.a)

***Pencil* I.1.1. b)**

The characteristic of the pencil - subcase **I.**1.1.b):

$D = (b_u - b_p)(b_u - b_q) \Rightarrow$

$(b_q < b_u < b_p)$ or $(b_p < b_u < b_q)$

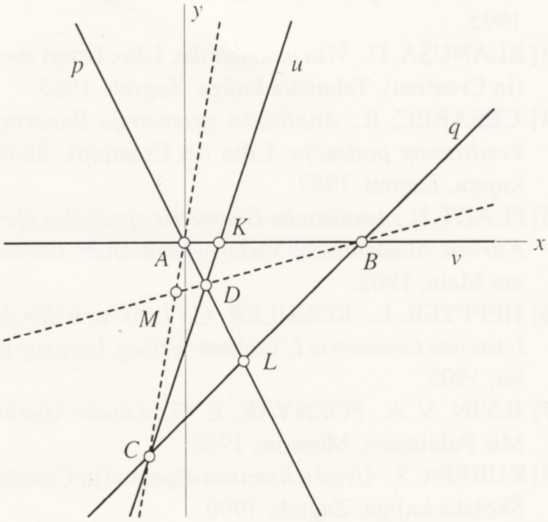

Fig.4: Fundamental points, the accompanying fundamental lines and main points within the pencil **I.**1.1.b), obtained by $b_q < b_u < b_p$.

Since $D < 0$, it is the case (2.3, ii), that is, no conics of parabolic type are within the pencil. As it is always $det\ \sigma < 0$, it can be concluded that the pencil consists only of hyperbolas. Thus, the conic pencil induces on the absolute straight line *f* one elliptic involution ([4], [12]).

Let us write the focal curve k_f^3 in the form

$a\,y^2 + b\,y + c = 0,$

where we have marked

$a = ((b_p + b_q)b_u - b_p b_q)\,x + (b_u b_p c_q - b_p b_q c_u),$

$b = 2b_u x^2 + 2b_u c_q x,$

$c = x^3 + (c_u + c_q)x^2 + c_u c_q x..$

The isotropic asymptote is obtained from the condition $a = 0$, that is

$$x = \frac{b_p(b_q c_u - b_u c_q)}{b_p(b_u - b_q) + b_q b_u}.$$

Since $D < 0 \Rightarrow B < 0$, the curve k_f^3 of isotropic focuses has only one real point at infinity. Therefrom, according to the Newton's classification of the 3. order curves, k_f^3 belongs to the 2. group, i.e. among the curves having one asymptote and one straightline's branch. The curves of this group are called "defective hyperbolas".

The basic forms of the curves from this group are determined with the roots of the auxiliary equation $D_4(x) = 0$, where $D_4(x) = b^2 - 4ac$ is the fourth degree polynomial. The calculation yields

$$D_4(x) = (b_u - b_q)(b_u - b_p)\cdot x\,(x + c_q)\left(x + \frac{b_q c_u - b_u c_q}{b_q - b_u}\right)\left(x - \frac{b_p c_u}{b_u - b_p}\right),$$

wherefrom it can be seen that the roots of the equation $D_4(x) = 0$ are the abscissas of the fundamental points *A*, *B*, *C*, and *D* (4.5). As the points are real and various and within the observed pencil there are no isotropic lines, it holds.

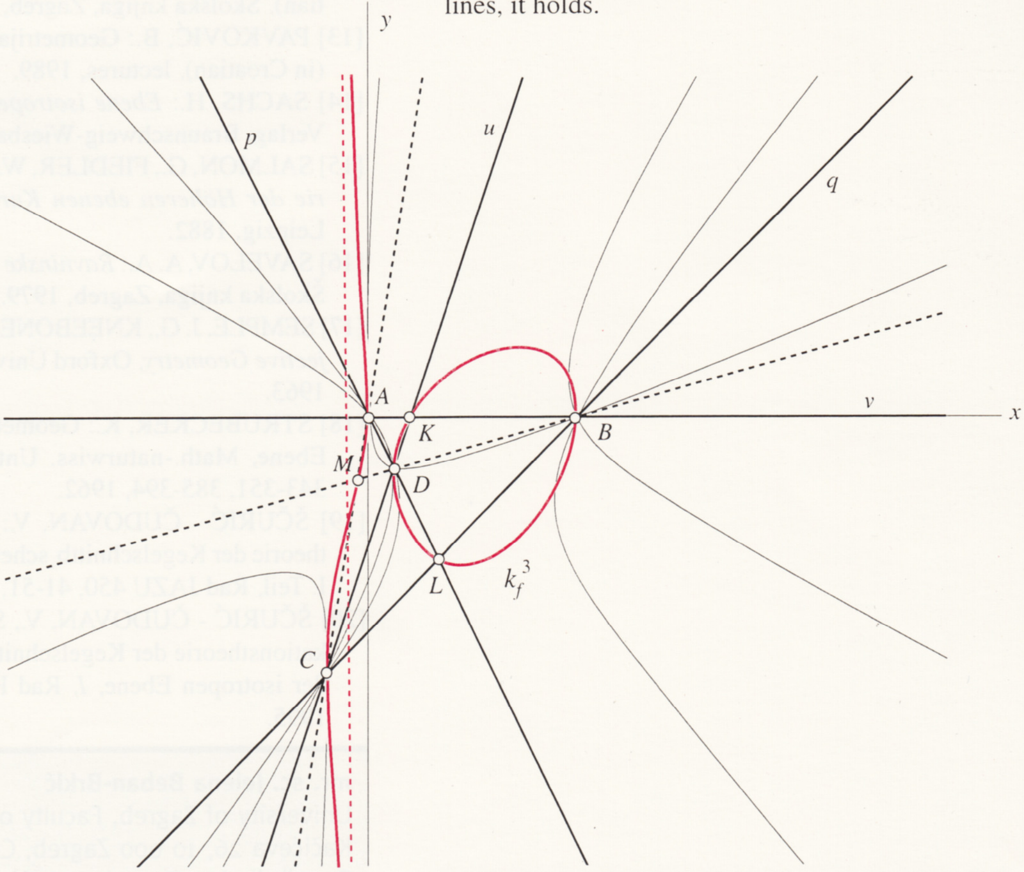

Fig.5: Pencil of conic sections - subcase **I.**1.1.b)

Proposition 5.
The focal curve k_f^3 of all pencils of the subcase **I**.1.1.b) *consists of one straightline's branch and an oval.*

PENCIL I.1.1. c)
The characteristic of the pencil - subcase **I**.1.1.c):
$D = (b_u - b_p)(b_u - b_q) = 0 \Rightarrow b_u = b_p$ or $b_u = b_q$.

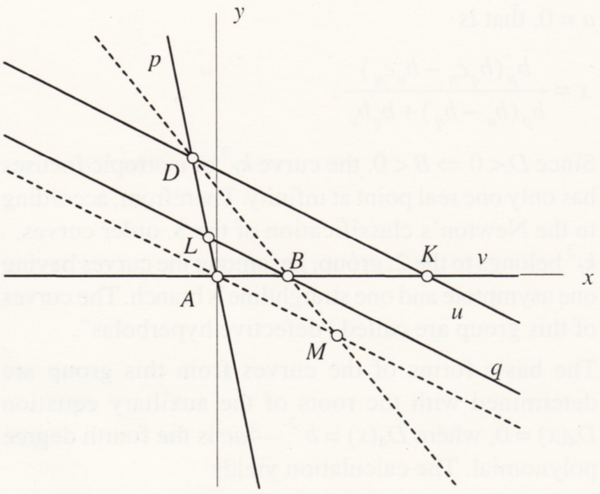

Fig. 6. Fundamental points, the accompanying fundamental lines and main points within the pencil **I**.1.1.c), obtained by $b_u = b_q$.

Since the lines u and p or u and q are mutually parallel, the fundamental point of intersection is in the infinity, so this case does not belong to the pencils - subtype **I**.1.

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mr. sc. Jelena Beban-Brkić
University of Zagreb, Faculty of Geodesy
Kačićeva 26, 10 000 Zagreb, Croatia
E-mail: jbeban@geodet.geof.hr