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Density of Interior Points Distribution in Modelling of Hyperpatches

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ABSTRACT

The paper deals with the modelling of hyperpatches (solid cells) on the basis of their creative representations and with the calculations of some intrinsic geometric properties of hyperpatches. The analytic representation of the hyperpatch in the form of a vector function of three variables provides the possibility to calculate and to control geometric properties and the density of distribution (homogeneous or non-homogeneous) of the hyperpatch interior points.

Key words: hyperpatch modelling, geometric properties, density of interior points

Gustoća raspodjele unutarnjih točaka u modeliranju hiperdijelova

SAŽETAK

Rad se bavi modeliranjem hiperdijelova (punih ćelija) na osnovi njihovih kreativnih reprezentacija i proračunima za neka svojstva unutarnje geometrije hiperdijelova. Analitička reprezentacija hiperdijela u formi vektorske funkcije od tri varijable omogućuje izračunavanje i kontrolu geometijskih svojstava i gustoće raspodjele (homogene ili nehomogene) za unutarnje točke hiperdijela.

Ključne riječi: modeliranje hiperdijelovima, geometrijska svojstva, raspodjela unutarnjih točaka

MSC: 65D17, 68U07

1 Introduction

Solid modelling is an important part of geometric modelling (described in [1], [2]). There exist different approaches to this problem, based on different representations of the modelled objects. Forms of the distinguished representations are determined by the areas of application of the generated objects, that are composite solids composed from several solid cells – hyperpatches. In the hyperpatch modelling, the geometry of the hyperpatch interior points can be considered on the basis of the intrinsic geometric properties, which are equivalent to the intrinsic geometric properties of surfaces. The intrinsic geometric properties of a hyperpatch are determined by the partial derivatives of the analytic representation of the hyperpatch, a vector function in three variables, and can be calculated from the coefficients of the hyperpatch fundamental forms. They can be extracted directly from the hyperpatch basic figure, which is a part of the input data structure for the computer processing.

In the relevance to the form of the basic figure, the geometry of the distribution of hyperpatch interior points can be defined implicitly or explicitly.

2 Basic relations

Let $\mathbf{K} = (\mathbf{U}, \mathbf{G})$ be a Creative space described in details in [3]. A hyperpatch (a solid cell) S that is a three-parametric subset of the extended Euclidean space $_\infty E^3$ can be created on the basis of its creative law, which is in **K** synthetically represented by the creative representation, an ordered pair (U,g), where the basic figure U∈ **U** and the generating principle g∈ **G** are such, that applying the generating principle g on the basic figure U the hyperpatch S can be created. There are available three different forms of the generating principle g (geometric transformation T, a class of geometric transformations $T(u)$ defined on the interval, or any interpolation $I(u)$), that can be applied to the suitable basic figure, according to [4].

Creative representation of a hyperpatch, an ordered pair (U,g), can be expressed in the six different forms:

(a surface patch, a class of geometric transformations)

(a solid cell, a geometric transformation)

(a net of boundary surface patches, an interpolation)

(a sequence of surface patches, an interpolation)

(a grid of boundary edge curves, an interpolation)

(a grid of points, an interpolation).

The possibility to describe and to control the feature of the non-uniform distribution of points in a hyperpatch is provided in the case of its modelling as an interpolated figure, using the creative representation in which the generating principle is an interpolation. These are the last four types of the creative representations. The intrinsic geometric properties of the created hyperpatch are explicitly predetermined in the basic figure, which is in a form of the ordered set of separate geometric figures (points, curve segments or surface patches) related to the hyperpatch, or vectors that can be tangent vectors to the isoparametric curve segments, twist vectors to the isoparametric surface patches and the density vectors in the given hyperpatch points. These geometric figures are represented analytically by their vector functions appearing as the elements of the analytic representation of the entire basic figure, a map of the created hyperpatch. In the map – matrix, there are all elements distributed in the appropriate order and predetermine the intrinsic geometric properties of the created hyperpatch, the curvature of the isoparametric curve segments (edge curves) and the isoparametric surface patches (face surfaces), or the non-homogeneity of the interior points' distribution and density.

Industrial design and CAGD of non-homogeneous hyperpatches on the base of their creative representations can lead to more complex results with respect to their applicability in simulation of some physical processes as magnetism, electricity, heating, pressure or deformations of solids, where the possibility to control the distribution of the interior solid points appears.

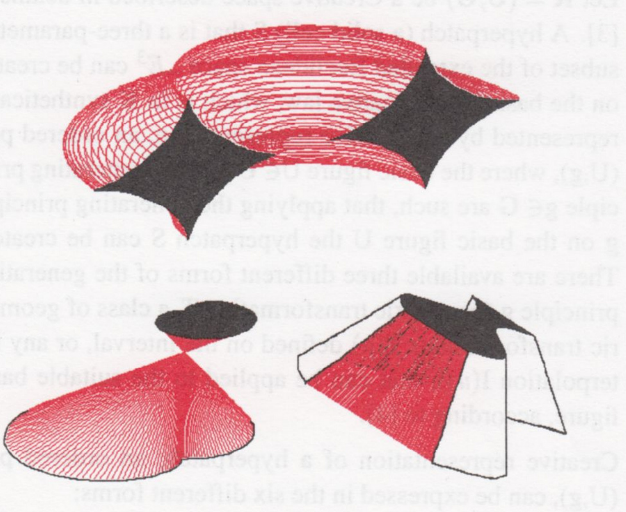

Fig. 1

In the first two types of the hyperpatch creative representations the created hyperpatches are homogeneous, and there can be distinguished several types of created solids with respect to the different types of generating principles in the form of the classes of geometric transformations. In the Fig. 1 there are illustrations of some solids generated by different classes of geometric transformations, a solid of revolution created from the basic region with the boundary in an asteroidal curve, a conical solid created from the region with the boundary in the form of a Limacon of Pascal subdued to the class of scalings with the centre in the conical solid vertex, while the scaling ratio $h > 1$. Frustum of the conical solid is created from the basic curve segment in the form of the sinusoidal curve segment subdued to a class of translations, and scalings with the given centre and the ratio $h < 1$.

In these two types of the created hyperpatches the intrinsic geometry is defined implicitly by the basic figure and the generating principle, and hyperpatch intrinsic geometric properties can be calculated from their analytic representations by means of the differential geometry. The partial derivatives of the vector function representing analytically the created hyperpatch with respect to all three variables and the mixed partial derivatives of the second and the third order can be used to determine the coefficients of thre first and the second fundamental forms of the created hyperpatch, and the density of distribution of the hyperpatch interior points.

The analytic representation of a hyperpatch – solid cell S is a vector function

$$\mathbf{r}(u,v,w) = (x(u,v,w),\, y(u,v,w),\, z(u,v,w),\, h(u,v,w))$$

defined on the region $\Omega = [0,1]^3$ (where x, y, z, h are homogeneous coordinate functions of three variables that are at least C^3 continuous on the region Ω, while the partial derivatives of the function $h(u,v,w)$ with respect to the all three variables u, v, w are equal to 0 on $[0,1]^3$), which is a local homeomorphic mapping of the region Ω on the hyperpatch S. Composite solids can be obtained as compositions of several elementary hyperpatches – solid cells.

There exist 3 isoparametric systems of surface patches (exactly one of the parameters u, v, w is constant) forming a net of isoparametric surface patches of a hyperpatch. Boundary surface patches (facets) correspond to the constant values of parameters equal to 0 or 1. If there are two of the parameters u, v, w equal to some constant values, we can speak about isoparametric curve segments of a hyperpatch, if the values are equal to 0 or 1, about boundary isoparametric curve segments (edges). Two isoparametric surface patches from different systems intersect in an isoparametric curve segment, two isoparametric curve segments from different systems intersect in a hyperpatch point. In the hyperpatch point there are all three parameters constant and we denote them as parametric (curvilinear) coordinates of the hyperpatch point. Points with parametric coordinates (not) equal to 0 or 1 are (interior) exterior points of the hyperpatch (in details in [2], [5]).

A hyperpatch boundary according to [2] consists of 6 boundary surface patches – face surfaces of a hyperpatch, 12 boundary curve segments – edge curves of a hyperpatch, and 8 corner points – vertices of a hyperpatch.

3 Hyperpatches created by classes of geometric transformations

Let the hyperpatch S be created from a basic figure in the form of a surface patch U analytically represented by the point function

$$\mathbf{p}(u,v) = (\xi(u,v), \psi(u,v), \zeta(u,v), \eta(u,v))$$

defined on the region $\Phi = [0,1]^2$, where ξ, ψ, ζ, η are homogeneous coordinate functions of two real variables that are at least C^3 continuous on the region Φ (while the partial derivatives of the coordinate function $\eta(u,v)$ with respect to the both variables u and v are equal to zero on Φ), which is a local homeomorphic mapping of the region Φ on the surface patch U.

Geometric transformation that can be applied as a generating principle to the basic figure in the form of a hyperpatch is represented by a regular square matrix of rank 4 with real numbers as its elements

$$\mathrm{T} = (a_{ij}) \quad \text{for} \quad i,j = 1,2,3,4.$$

Analytic representation of the class of geometric transformations that is a suitable generating principle applicable to the basic figure in the form of a surface patch is a matrix function derived from the matrix T in the form

$$\mathrm{T}(w) = (a_{ij}(w)) \quad \text{for} \quad i,j = 1,2,3,4,$$

which is a continuous function of one real variable defined on the interval $I = [0,1]$, with the values in the set of all regular square matrices with real elements representing single geometric transformations.

Analytic representation of the created hyperpatch is a point function

$$\begin{aligned}\mathbf{r}(u,v,w) &= \mathbf{p}(u,v).\mathrm{T}(w)\\ &= (\xi(u,v), \psi(u,v), \zeta(u,v), \eta(u,v)).\mathrm{T}(w).\end{aligned}$$

defined and differentiable on the region $\Omega = \Phi \times I = [0,1]^3$.

Partial derivatives and the total differentials of the point function are in forms

$$\begin{aligned}\mathbf{r}_u(u,v,w) &= \mathbf{p}_u(u,v).\mathrm{T}(w)\\ &= (\xi_u(u,v), \psi_u(u,v), \zeta_u(u,v), 0).\mathrm{T}(w),\\ \mathbf{r}_v(u,v,w) &= \mathbf{p}_v(u,v).\mathrm{T}(w)\\ &= (\xi_v(u,v), \psi_v(u,v), \zeta_v(u,v), 0).\mathrm{T}(w),\\ \mathbf{r}_w(u,v,w) &= \mathbf{p}(u,v).\mathrm{T}'(w)\\ &= (\xi(u,v), \psi(u,v), \zeta(u,v), \eta(u,v)).\mathrm{T}'(w),\end{aligned}$$

$$\begin{aligned}d\mathbf{r} &= \mathbf{r}_u du + \mathbf{r}_v dv + \mathbf{r}_w dw\\ &= \mathbf{p}_u(u,v).\mathrm{T}(w)du + \mathbf{p}_v(u,v).\mathrm{T}(w)dv + \mathbf{p}(u,v).\mathrm{T}'(w)dw\\ &= (\mathbf{p}_u(u,v)du + \mathbf{p}_v(u,v)dv).\mathrm{T}(w) + \mathbf{p}(u,v).\mathrm{T}'(w)dw\\ &= d\mathbf{p}.\mathrm{T}(w) + \mathbf{p}(u,v).\mathrm{T}'(w)dw,\end{aligned}$$

$$\begin{aligned}\mathbf{r}_{uv}(u,v,w) &= \mathbf{p}_{uv}(u,v).\mathrm{T}(w)\\ &= (\xi_{uv}(u,v), \psi_{uv}(u,v), \zeta_{uv}(u,v), 0).\mathrm{T}(w),\\ \mathbf{r}_{uw}(u,v,w) &= \mathbf{p}_u(u,v).\mathrm{T}'(w)\\ &= (\xi_u(u,v), \psi_u(u,v), \zeta_u(u,v), 0).\mathrm{T}'(w),\\ \mathbf{r}_{vw}(u,v,w) &= \mathbf{p}_v(u,v).\mathrm{T}'(w)\\ &= (\xi_v(u,v), \psi_v(u,v), \zeta_v(u,v), 0).\mathrm{T}'(w),\\ \mathbf{r}_{uvw}(u,v,w) &= \mathbf{p}_{uv}(u,v).\mathrm{T}'(w)\\ &= (\xi_{uv}(u,v), \psi_{uv}(u,v), \zeta_{uv}(u,v), 0).\mathrm{T}'(w),\end{aligned}$$

$$\begin{aligned}d^2\mathbf{r} &= \mathbf{r}_{uu}du^2 + \mathbf{r}_{vv}dv^2 + \mathbf{r}_{ww}dw^2\\ &\quad + 2\mathbf{r}_{uv}dudv + 2\mathbf{r}_{uw}dudw + 2\mathbf{r}_{vw}dvdw\\ &= \mathbf{p}_{uu}(u,v).\mathrm{T}(w)du^2 + \mathbf{p}_{vv}(u,v).\mathrm{T}(w)dv^2\\ &\quad + \mathbf{p}(u,v).\mathrm{T}''(w)dw^2 + 2\mathbf{p}_{uv}(u,v).\mathrm{T}(w)dudv\\ &\quad + 2\mathbf{p}_u(u,v).\mathrm{T}'(w)dudw + 2\mathbf{p}_v(u,v).\mathrm{T}'(w)dvdw\\ &= (\mathbf{p}_{uu}(u,v)du^2 + 2\mathbf{p}_{uv}(u,v)dudv\\ &\quad + \mathbf{p}_{vv}(u,v)dv^2)\mathrm{T}(w) + 2(\mathbf{p}_u(u,v)dudw\\ &\quad + \mathbf{p}_v(u,v)dvdw).\mathrm{T}'(w) + \mathbf{p}(u,v).\mathrm{T}''(w)dw^2\\ &= d^2\mathbf{p}.\mathrm{T}(w) + 2d\mathbf{p}.\mathrm{T}'(w) + \mathbf{p}(u,v).\mathrm{T}''(w)dw^2.\end{aligned}$$

4 Interpolation hyperpatches

In the modeling of interpolation hyperpatches, the analytic representation of a tri-cubic hyperpatch is in a form

$$\begin{aligned}\mathbf{r}(u,v,w) &= \mathbf{a}_{333}u^3v^3w^3 + \mathbf{a}_{332}u^3v^3w^2 + \ldots + \mathbf{a}_{100}u + \mathbf{a}_{000}\\ &= \sum_{i=0}^{3}\sum_{j=0}^{3}\sum_{k=0}^{3} F_i(u)F_j(v)F_k(w)\mathbf{b}_{ijk} \text{ for } (u,v,w) \in [0,1]^3\end{aligned}$$

where $F_i(u), F_j(v), F_k(w)$ are cubic interpolation polynomials.

Geometric coefficients $\mathbf{b}_{ijk}$ define explicitly geometric properties of the hyperpatch and they form a three-dimensional matrix of the type 4x4x4, the map of the hyperpatch. The elements of this map are quadruples of homogeneous coordinates of the hyperpatch points (real points in $_\infty E^3$), tangent vectors to the hyperpatch edges, twist vectors to the hyperpatch faces, and vectors defining the distribution of points inside the hyperpatch – density vectors (points in $_\infty E^3$ at infinity).

Let the basic figure of a hyperpatch be an ordered grid of 64 real points in $_\infty E^3$. The basic grid of points defines implicitly also the curvature of edges and faces, and density of the hyperpatch, i. e. distribution of the interior points.

Analytic representation of the created hyperpatch is a point function

$$\mathbf{r}(u,v,w) = \sum_{k=0}^{3}\sum_{i=0}^{3}\sum_{j=0}^{3} B_{ijk}F_i(u)F_j(v)F_k(w)$$

$$\text{for} \quad (u,v,w) \in [0,1]^3.$$

The first partial derivatives of the point function are in the forms

$$\frac{\delta \mathbf{r}}{\delta u}(u,v,w) = \mathbf{r}_u(u,v,w) = \sum_{k=0}^{3}\sum_{i=0}^{3}\sum_{j=0}^{3} B_{ijk}F_i'(u)F_j(v)F_k(w),$$

$$\frac{\delta \mathbf{r}}{\delta v}(u,v,w) = \mathbf{r}_v(u,v,w) = \sum_{k=0}^{3}\sum_{i=0}^{3}\sum_{j=0}^{3} B_{ijk}F_i(u)F_j'(v)F_k(w),$$

$$\frac{\delta \mathbf{r}}{\delta w}(u,v,w) = \mathbf{r}_w(u,v,w) = \sum_{k=0}^{3}\sum_{i=0}^{3}\sum_{j=0}^{3} B_{ijk}F_i(u)F_j(v)F_k'(w)$$

and determine for the curvilinear coordinates $(a,b,c) \in [0,1]^3$ tangent vectors to the isoparametric curve segments in the regular point $P(a,b,c) = \mathbf{r}(a,b,c)$.

The mixed second partial derivatives of the point function are in the forms

$$\frac{\delta^2 \mathbf{r}}{\delta u \delta v}(u,v,w) = \mathbf{r}_{uv}(u,v,w)$$

$$= \sum_{k=0}^{3}\sum_{i=0}^{3}\sum_{j=0}^{3} B_{ijk}F_i'(u)F_j'(v)F_k(w),$$

$$\frac{\delta^2 \mathbf{r}}{\delta u \delta w}(u,v,w) = \mathbf{r}_{uw}(u,v,w)$$

$$= \sum_{k=0}^{3}\sum_{i=0}^{3}\sum_{j=0}^{3} B_{ijk}F_i'(u)F_j(vs)F_k'(w),$$

$$\frac{\delta^2 \mathbf{r}}{\delta v \delta w}(u,v,w) = \mathbf{r}_{vw}(u,v,w)$$

$$= \sum_{k=0}^{3}\sum_{i=0}^{3}\sum_{j=0}^{3} B_{ijk}F_i(u)F_j'(v)F_k'(w)$$

and determine for the curvilinear coordinates $(a,b,c) \in [0,1]^3$ the twist vectors to the isoparametric surface patches in the regular point $P(a,b,c)$.

The mixed third partial derivative of the point function is in the form

$$\frac{\delta^3 \mathbf{r}}{\delta u \delta v \delta w}(u,v,w) = \mathbf{r}_{uvw}(u,v,w)$$

$$= \sum_{k=0}^{3}\sum_{i=0}^{3}\sum_{j=0}^{3} B_{ijk}F_i'(u)F_j'(v)F_k'(w)$$

and determines for the curvilinear coordinates $(a,b,c) \in [0,1]^3$ the density vector in the regular point $P(a,b,c)$.

The total differentials of the point function are in forms

$$d\mathbf{r} = \sum_{k=1}^{3}\sum_{i=1}^{3}\sum_{j=1}^{3} B_{ijk}(F_i'(u)F_j(v)F_k(w) + F_i(u)F_j'(v)F_k(w) + F_i(u)F_j(v)F_k'(w)),$$

$$d^2\mathbf{r} = \sum_{k=1}^{3}\sum_{i=1}^{3}\sum_{j=1}^{3} B_{ijk}(F_i''(u)F_j(v)F_k(w)d^2u + F_i(u)F_j''(v)F_k(w)d^2v + +F_i(u)F_j(v)F_k''(w)d^2w + 2(F_i'(u)F_j'(v)F_k(w)dudv + F_i'(u)F_j(v)F_k'(w)dudw + F_i(u)F_j'(v)F_k'(w)dvdw) + F_i'(u)F_j'(v)F_k'(w)dudvdw).$$

5 Hyperpatch first differential form

We can describe geometrically not only the boundary of the hyperpatch as a three-dimensional region in the extended Euclidean space $_\infty E^3$, but also the geometry of the hyperpatch interior points.

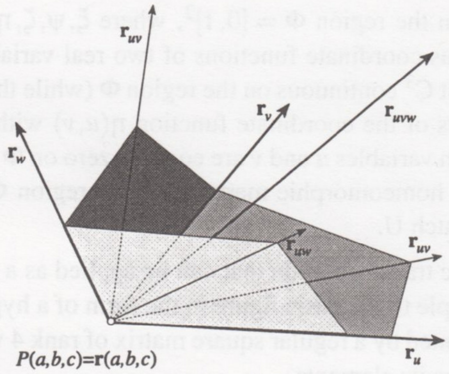

Fig. 2

At any regular point $\mathbf{r}(a,b,c) = P(a,b,c)$ of the hyperpatch there is defined a unique tangent trihedron Γ^{abc} formed by the three tangent planes to the isoparametric surface patches of the hyperpatch in this point (Fig. 2). Each of the tangent planes is defined by two tangent vectors to the isoparametric curve segments in their common point $P(a,b,c)$

$\tau_{uv}(\mathbf{r}_u(a,b,c), \mathbf{r}_v(a,b,c)),$
$\tau_{uv}(\mathbf{r}_u(a,b,c), \mathbf{r}_w(a,b,c))\, \tau_{uv}(\mathbf{r}_v(a,b,c), \mathbf{r}_w(a,b,c)).$

The tangent planes intersect in the common regular point $P(a,b,c)$, each two of them having a pierce line in the tangent line to the isoparametric curve segment

$$\tau_{uv} \cap \tau_{uw} = t_u, \tau_{uv} \cap \tau_{vw} = t_v, \tau_{uw} \cap \tau_{vw} = t_w.$$

Twist vectors $\mathbf{r}_u(a,b,c), \mathbf{r}_v(a,b,c), \mathbf{r}_w(a,b,c)$ to the isoparametric surface patches characterize implicitly their geometric shape, curvature, convexity or concavity.

Any change in the hyperpatch interior point distribution is reflected in the change of the density vector $\mathbf{r}_{uvw}(a,b,c)$. This geometric characteristics will be denoted as density of the interior point distribution. In some applications the density vectors in the vertices of the hyperpatch are defined directly in the hyperpatch basic figure as the geometric coefficients $\mathbf{b}_{ijk}$, and define thus the interior points distribution implicitly. The density vectors in the interior points can be explicitly calculated from the third mixed partial derivative of the hyperpatch point function.

Definition 1.

Density of the interior point distribution is defined by the density function

$\mathbf{r}_{uvw}(u,v,w) = (x_{uvw}(u,v,w), y_{uvw}(u,v,w), z_{uvw}(u,v,w), 0)$

that is a vector function defined on the region $\Omega = [0,1]^3$. *Value of the density function in the hyperpatch regular point* $P(a,b,c)$ *is the density vector oriented towards the interior of the tangent trihedron.*

Loci of the density function expresses the homogeneity of the interior point distribution with respect to the coordinate planes. Constant density function defines a homogeneous distribution in the hyperpatch with respect to all coordinate axes.

Definition 2.

Let $\mathbf{e}^1 = (1,0,0,0), \mathbf{e}^2 = (0,1,0,0)$ *and* $\mathbf{e}^3 = (0,0,1,0)$ *be the unit vectors in the direction of the coordinate axes. The scalar product*

$s^i = \mathbf{r}_{uvw}(u,v,w).\mathbf{e}^i, \quad i = 1,2,3$

is denoted as the ratio of the hyperpatch homogeneity with respect to the coordinate plane $(\mathbf{e}^j\mathbf{e}^k), i \neq j \neq k, \quad j,k = 1,2,3$. *For* $s^i = const$ *we speak about homogeneous distribution with respect to the coordinate plane* $(\mathbf{e}^j\mathbf{e}^k)$. *Hyperpatch is said to be homogeneous, if it is homogeneous with respect to all coordinate planes.*

Definition 3.

Hyperpatch first differential form denoted as $\Phi_1(u,v,w)$ *is the square of the first total differential of the hyperpatch analytic representation, the point function*

$\mathbf{r}(u,v,w) = (x(u,v,w), y(u,v,w), z(u,v,w), h(u,v,w))$

at least C^3 *continuous on the region* $\Omega = [0,1]^3$

$\Phi_1(u,v,w) = (d\mathbf{r})^2 = (\mathbf{r}_u du + \mathbf{r}_v dv + \mathbf{r}_w dw)^2.$

Hyperpatch first differential form can he expressed as the sum of the first fundamental forms $\varphi_1(u,v)$, $\varphi_1(u,w)$, $\varphi_1(v,w)$ of the hyperpatch 3 isoparametric surface patches subtracted by the sum of the first fundamental forms of the hyperpatch isoparametric curves $\varphi = \mathbf{r}_u^2 du^2 + \mathbf{r}_v^2 dv^2 + \mathbf{r}_w^2 dw^2$:

$$\begin{aligned}\Phi_1(u,v,w) &= (\mathbf{r}_u du + \mathbf{r}_v dv + \mathbf{r}_w dw)^2 = \\ &= (\mathbf{r}_u^2 du^2 + 2\mathbf{r}_u\mathbf{r}_v du dv + \mathbf{r}_v^2 dv^2) \\ &\quad + (\mathbf{r}_u^2 du^2 + 2\mathbf{r}_u\mathbf{r}_w du dw + \mathbf{r}_w^2 dw^2) \\ &\quad + (\mathbf{r}_v^2 dv^2 + 2\mathbf{r}_v\mathbf{r}_w dv dw + \mathbf{r}_w^2 dw^2) \\ &\quad - (\mathbf{r}_u^2 du^2 + \mathbf{r}_v^2 dv^2 + \mathbf{r}_w^2 dw^2) \\ &= \varphi_1(u,v) + \varphi_1(u,w) + \varphi_1(v,w) - \varphi.\end{aligned}$$

Let us denote coefficients of the uv-isoparametric surface patch first fundamental form by $E = \mathbf{r}_u^2, F = \mathbf{r}_u\mathbf{r}_v, G = \mathbf{r}_v^2$, and similarly coefficients of the uw-isoparametric surface patch first fundamental form by $E = \mathbf{r}_u^2, F^\star = \mathbf{r}_u\mathbf{r}_v, G^\star = \mathbf{r}_w^2$, and coefficients of the vw-isoparametric surface patch first fundamental form by $G = \mathbf{r}_v^2, F^{\star\star} = \mathbf{r}_v\mathbf{r}_w, G^\star = \mathbf{r}_w^2$.

Then we can write the following formula

$$\begin{aligned}\Phi_1(u,v,w) = &-(Edu^2 + Gdv^2 + G^\star dw^2) \\ &+ (Edu^2 + 2Fdudw + Gdv^2) \\ &+ (Edu^2 + 2F^\star dudw + G^\star dw^2) \\ &+ (Gdv^2 + 2F^{\star\star} dvdw + G^\star dw^2).\end{aligned}$$

The discriminant of the hyperpatch first differential form is a positive number expressed in a hyperpatch regular point as the value of the determinant

$$D = \det\begin{pmatrix} E & F & F^\star \\ F & G & F^{\star\star} \\ F^\star & F^{\star\star} & G^\star \end{pmatrix} = [\mathbf{r}_u\mathbf{r}_v\mathbf{r}_w]^2,$$

where $[\mathbf{r}_u\mathbf{r}_v\mathbf{r}_w]$ is the mixed triple scalar product of the first partial derivatives of the hyperpatch point function with respect to the variables u, v, w.

The first differential form of the hyperpatch created from the basic surface patch by the class of geometric transformations can be calculated as

$$\begin{aligned}(d\mathbf{r})^2 &= (d\mathbf{p}.\mathrm{T}(w) + \mathbf{p}(u,v).\mathrm{T}'(w)dw)^2 \\ &= (d\mathbf{p}.\mathrm{T}(w))^2 + 2d\mathbf{p}.\mathbf{p}(u,v).\mathrm{T}(w).\mathrm{T}'(w)dw \\ &\quad + (\mathbf{p}(u,v).\mathrm{T}'(w)dw)^2 \\ &= \varphi_1(u,v).\mathrm{T}^2(w) \\ &\quad + 2\sqrt{\varphi_1(u,v)}\mathbf{p}(u,v).\mathrm{T}(w).\mathrm{T}'(w)dw \\ &\quad + \mathbf{p}^2(u,v).\mathrm{T}'^2(w)dw^2.\end{aligned}$$

6 Volume and density of distribution of the interior points

The volume of the hyperpatch represented analytically by the point function

$\mathbf{r}(u,v,w) = (x(u,v,w), y(u,v,w), z(u,v,w), h(u,v,w))$

defined and at leat C^3 continuous on the region $\Omega = [0,1]^3$, is the value of the triple integral

$$V = \iiint_\Omega \sqrt{D(u,v,w)}dudvdw = \iiint_\Omega |(\mathbf{r}_u\mathbf{r}_v\mathbf{r}_w)|dudvdw$$

where $D(u,v,w)$ is the function of the discriminant of the hyperpatch first differential form.

The mixed triple scalar product of the tangent vectors to the isoparametric curve segments in the regular point of the hyperpatch can be expressed in the following forms

$$s = [\mathbf{r}_u \mathbf{r}_v \mathbf{r}_w] = (\mathbf{r}_u \times \mathbf{r}_v).\mathbf{r}_w = [\mathbf{r}_v \mathbf{r}_w \mathbf{r}_u]$$
$$= (\mathbf{r}_v \times \mathbf{r}_w).\mathbf{r}_u = [\mathbf{r}_w \mathbf{r}_u \mathbf{r}_v] = (\mathbf{r}_w \times \mathbf{r}_u).\mathbf{r}_v$$

and can be calculated from the coefficients of the first fundamental forms of related surface patches determined by pairs of the isoparametric curve segments of the hyperpatch, $\varphi_1(u,v), \varphi_1(u,w), \varphi_1(v,w)$ (Fig.3).

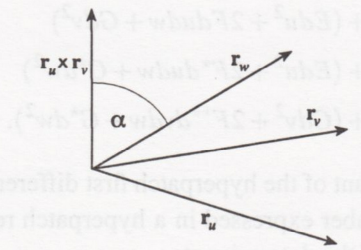

Fig. 3

The angle that forms the vector product of tangent vectors $\mathbf{r}_u$and $\mathbf{r}_v$ (normal vector to the isoparametric surface patch) with the third tangent vector $\mathbf{r}_w$ is denoted α, while

$$|\mathbf{r}_u \times \mathbf{r}_v| = \sqrt{EG - F^2},$$

$$\alpha = |\angle(\mathbf{r}_u \times \mathbf{r}_v), \mathbf{r}_w|, \quad \cos\alpha = \frac{s}{\sqrt{EG - F^2}.\sqrt{G^\star}}.$$

The similar relations are valid also for other two tangent vectors and their related angles β and γ with the vector products $\mathbf{r}_v \times \mathbf{r}_w$ and $\mathbf{r}_w \times \mathbf{r}_u$

$$|\mathbf{r}_v \times \mathbf{r}_w| = \sqrt{GG^\star - F^{\star\star 2}},$$

$$\beta = |\angle(\mathbf{r}_v \times \mathbf{r}_w), \mathbf{r}_u|, \quad \cos\beta = \frac{s}{\sqrt{GG^\star - F^{\star\star 2}}.\sqrt{E}},$$

$$|\mathbf{r}_w \times \mathbf{r}_u| = \sqrt{EG^\star - F^{\star 2}},$$

$$\gamma = |\angle(\mathbf{r}_w \times \mathbf{r}_u), \mathbf{r}_v|, \quad \cos\gamma = \frac{s}{\sqrt{EG^\star - F^{\star 2}}.\sqrt{G}}.$$

We can calculate (for $\alpha, \beta, \gamma \neq \frac{\pi}{2}$)

$$h = \frac{1}{cos^2\alpha} + \frac{1}{cos^2\beta} + \frac{1}{cos^2\gamma}$$
$$= \frac{1}{s^2}((EG - F^2)G^\star + (GG^\star - F^{\star\star 2})E + (EG^\star - F^{\star 2})G)$$
$$= \frac{1}{s^2}(3EGG^\star - G^\star F^2 - EF^{\star\star 2} - GF^{\star 2})$$

and express the mixed triple scalar product s for the calculation of the hyperpatch volume V in the case of the nonzero value of h

$$|s| = \sqrt{\frac{(3EGG^\star - G^\star F^2 - EF^{\star\star 2} - GF^{\star 2})}{h}},$$

$$h = \frac{1}{cos^2\alpha} + \frac{1}{cos^2\beta} + \frac{1}{cos^2\gamma},$$

$$V = \int_0^1 \int_0^1 \int_0^1 |s| dudvdw.$$

Changing the position of the point on the hyperpatch for variable curvilinear coordinates $(u,v,w) \in \Omega = [0,1]^3$ we receive a function in three variables defined on Ω. If the value of angle α, β, or γ equals to 0 or π, we speak about weakly dense distribution of points in the direction of the concerned tangent vector.

For the constant values of all angles $\alpha = \beta = \gamma = 0$ or π, and therefore constant function $h(u,v,w) = 3$, the mixed triple scalar product of the tangent vectors to the isoparametric curve segments of the hyperpatch can be calculated from the expression

$$s = \sqrt{\frac{(3EGG^\star - G^\star F^2 - EF^{\star\star 2} - GF^{\star 2})}{3}}$$
$$= \sqrt{EGG^\star - \frac{1}{3}(G^\star F^2 + EF^{\star\star 2} + GF^{\star 2})}$$

and we speak about weakly dense distribution of hyperpatch interior. Geometric interpretation is obvious: if all angles are of the same zero value, the tangent vectors form an orthogonal reper, and the tangent trihedron is rightangled.

For the extreme values $\alpha, \beta, \gamma = \frac{\pi}{2}$ the number h streams to infinity, $\lim_{h \to \infty} s = 0$, and we speak about dense distribution of points in the direction of the related tangent vector.

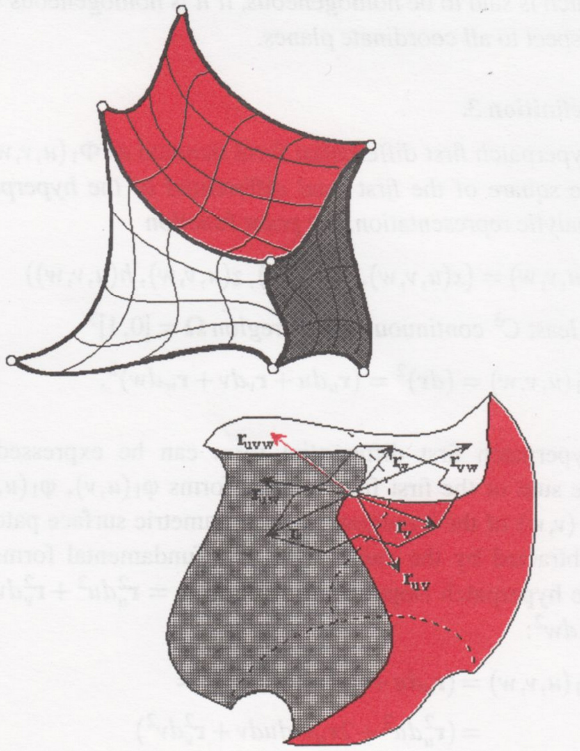

Fig. 4

In the Fig.4 there is an illustration of the Bezier approximation hyperpatch determined by the grid of $4 \times 4 \times 4$ real points in the space. An interpolation hyperpatch created from the basic grid of 8 real corner points – vertices, 24 tangent vectors to the isoparametric boundary curve segments – edges, 24 twist vectors to the boundary isoparametric surface patches – facets and 8 density vectors in the vertices determining the distribution of the interior points in the neighbourhood of the corresponding vertices is illustrated, while the data vectors (tangent, twist and density) are depicted in the figure in one of the vertices.

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