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A New Approach to the Computer Aided Calculation of Points on the Envelope Helical Surface Characteristics

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ABSTRACT

Presented calculations and the choice of variables $u, v \in [0,1]$ as curvilinear on the given basic surface φ enable the creation of a versatile programme for the graphical processing of the characteristics not only of the envelope helical, but also rotational surface Φ defined by the basic rotational surface φ. Basic surface φ can be generated from the basic curve c determined by (1) applying a class of revolutions about the coordinate axis x defined by the matrix $\mathbf{T_R}(v)$, for $v \in [0,1]$ (2), that is positioned in the space by the transformation $\mathbf{T}$ (3). The envelope helical surface Φ can be created by helical movement of the points on the characteristics. The axis o of the helical movement located into the coordinate axis z is the axis of the surface Φ.

Key words: envelope surface, envelope helical surface, envelope rotational surface, characteristics, meridian section, creative space

Novi pristup numeričkom računanju točaka na lukovima dodira ovojnice helikoidne plohe

SAŽETAK

Izloženi proračuni i izbor varijabli $u, v, \in [0,1]$ za krivolinijske koordinate na danoj temeljnoj plohi φ omogućuju stvaranje široko primjenjivog programa za grafičko procesuiranje lukova dodira, kako za ovojnicu helikoidne plohe tako i za racionalnu plohu Φ određenu temeljnom plohom φ. Temeljna ploha φ može biti generirana iz temeljne krivulje c (1) kao klasa rotacija oko koordinatne osi x, definirana matricom $\mathbf{T_R}(v)$ za $v \in [0,1]$ (2), koja je u prostoru smještena pomoću transformacije $\mathbf{T}$ (3). Ovojnica helikoidne plohe Φ može nastati helikoidnim gibanjem točaka luka dodira. Os o helikoidnog gibanja, smještena na koordinatnu os z, ujedno je i os plohe Φ.

Ključne riječi: ovojnica, ovojnica helikoidne plohe, ovojnica racionalne plohe, luk dodira, meridijanski presjek, prostor stvaranja

MSC: 65D17

An envelope surface Φ is the envelope of a 1-parametric system of surfaces created by a continuous movement of a basic surface φ.

Characteristics is a curve segment along which the envelope surface Φ touches the basic surface φ.

The same envelope surface Φ can be created by the continuous movement of characteristics. (Detailed description of envelope surfaces can be found in Velichová [3].)

Let us deal with an envelope helical surface Φ created by a helical movement (the movement is a class of geometric transformation concatenated from a revolution movement about the axis o and a translation movement in the direction of the vector collinear to the axis o of revolution) of a basic surface φ. An envelope rotational surface is a special type of the envelope helical surface with the helical movement pitch $|\mathbf{z}_v|$ ($\mathbf{z}_v$ is the translation vector corresponding to the angle of revolution equal to 2π) equal to zero.

In the paper (Szarková [2]), the helical and rotational surface Φ created from the conical or cylindrical surface φ were discussed. In the present paper, the basic surface φ will be a rotational surface created by revolving a plane curve segment located in the xz-plane about the coordinate axis x. We will work in the creative space with the homogeneous coordinates (in correspondence with Qiulin [1]).

By the helical movement of the spherical surface G we create the envelope surface called Archimedean serpentine. The characteristics is a principal circle h, that is the intersection of the basic spherical surface G and the plane ξ perpendicular to the tangent line to the helix (trajectory of the movement) of the surface G centre S.

Let G be a spherical surface from a set of spherical surfaces, that are tangent to the defined rotational surface φ in parallel circles. Let k be the tangent parallel circle on G. Intersection points ${}^1M$, ${}^2M$ of circles h and k are points of the characteristics e (Fig. 1.).

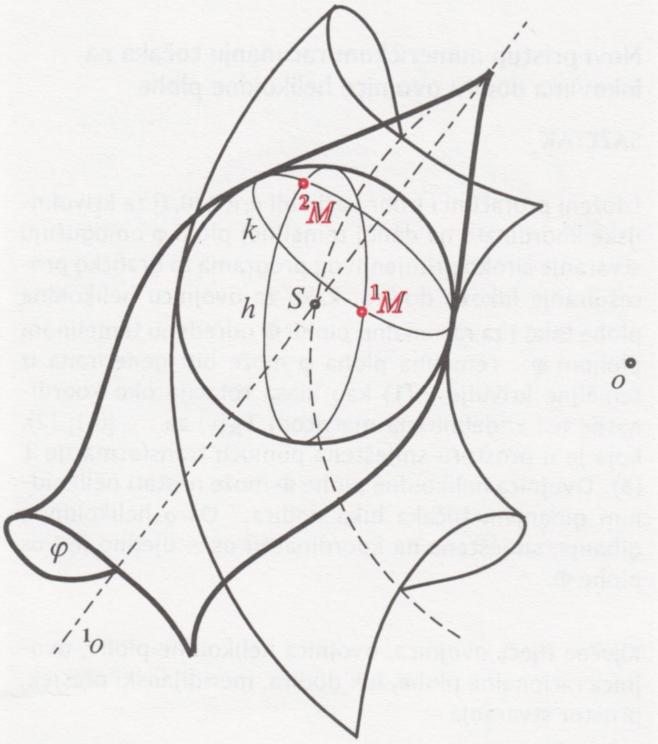

Fig. 1

Any rotational surface φ with the axis ${}^1o$ in general position can be transformed in such way that the axis of revolution will be in one of the coordinate axis and vice versa.

Let us create the basic surface in the basic position as the surface φ' with the axis ${}^1o'$ generated from the basic curve c located in the xz-plane, and defined by the vector function

$$ {}^1\mathbf{r}(u) = (x(u), 0, z(u), 1) \quad \text{for} \quad u \in [0,1] \tag{1}$$

applying a class of revolutions about the coordinate axis x defined by the matrix $\mathbf{T_R}(v)$,

$$\mathbf{T_R}(v) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos 2\pi v & \sin 2\pi v & 0 \\ 0 & -\sin 2\pi v & \cos 2\pi v & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{for} \quad v \in [0,1]. \tag{2}$$

Let φ' is positioned in the space by the transformation $\mathbf{T}$ (3),

$$\mathbf{T} = \mathbf{T_{oy}} \cdot \mathbf{T_{oz}} \cdot \mathbf{T_p}$$

$$= \begin{pmatrix} \cos\alpha\cos\beta & \sin\beta\cos\alpha & -\sin\alpha & 0 \\ -\sin\beta & \cos\beta & 0 & 0 \\ \cos\beta\sin\alpha & \sin\alpha\sin\beta & \cos\alpha & 0 \\ x_p & y_p & z_p & 1 \end{pmatrix}, \tag{3}$$

where $\mathbf{T_{oz}}$ is the revolution about the coordinate axis z by the angle α,

$\mathbf{T_{oy}}$ is the revolution about the coordinate axis y by the angle β,

$\mathbf{T_p}$ is the translation with the direction vector $\mathbf{v} = (x_p, y_p, z_p, 0)$.

The basic surface is now moved from the basic position φ' and located to the general position φ in the space determined by the chosen constant values of α, β, $\mathbf{v}$.

To simplify the results, the whole scene will be transformed by the transformation $\mathbf{T}'$ (in the transposed form) (4) so that we could work with the rotational surface φ', with the axis ${}^1o'$ congruent to the coordinate axis x:

$$\mathbf{T}' = \mathbf{T_p} \cdot \mathbf{T_{oz}} \cdot \mathbf{T_{oy}} \tag{4}$$

$$= \begin{pmatrix} c_\alpha c_\beta & s_\beta c_\alpha & -s_\alpha & -x_p c_\alpha c_\beta - y_p s_\alpha c_\beta + z_p s_\beta \\ -s_\beta & c_\beta & 0 & x_p s_\beta - y_p c_\beta \\ c_\beta s_\alpha & s_\alpha s_\beta & c_\alpha & -x_p c_\beta s_\alpha - y_p s_\alpha s_\beta + z_p c_\alpha \\ 0 & 0 & 0 & 1 \end{pmatrix}^{\mathrm{T}},$$

where $s_\alpha = \sin\alpha$, $c_\alpha = \cos\alpha$, $s_\beta = \sin\beta$ and $c_\beta = \cos\beta$; $\mathbf{T}'$ and $\mathbf{T}$ are inverse transformations.

The axis o of the helical movement located into the coordinate axis z is the axis of the envelope surface $\boldsymbol{\Phi}$. Let the axis o be defined by the point $O\,(0,0,0,1)$ and the direction vector $\mathbf{s}(0,0,z_0,0)$, where z_0 is the reduced pitch $z_0 = \frac{|z_v|}{2\pi}$.

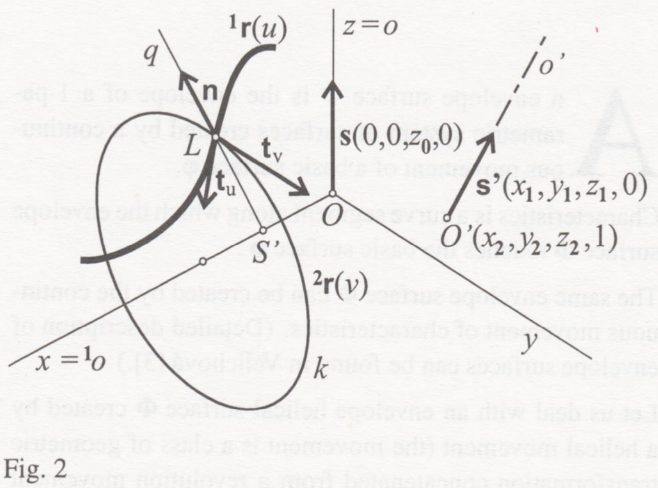

Fig. 2

The transformed axis o' will be defined by the point $O'(x_2, y_2, z_2, 1)$:

$$(x_2, y_2, z_2, 1) = (0,0,0,1) \cdot \mathbf{T}'$$

and by the direction vector $s'(x_1, y_1, z_1, 0)$:

$$(x_1, y_1, z_1, 0) = (0,0,z_0,0) \cdot \boldsymbol{T'}$$

The analytic representation of the parallel circle k on the surface $\boldsymbol{\varphi}'$ that is located in the plane parallel to the yz-plane and is incident to the point $L(u_0, 0) = \mathbf{r}(u_0) = (x(u_0), 0, z(u_0), 1)$, $u_0 \in [0,1]$ on the basic curve c is in the form

$$^{2}\mathbf{r}(v) = (x(u_0),\, r_k \cdot \sin(2\pi v),\, r_k \cdot \cos(2\pi v),\, 1) \qquad \text{for } v \in [0,1], \tag{5}$$

where the radius of the parallel circle $r_k = z(u_0)$.

The common tangent plane τ to the surface $\boldsymbol{\varphi}'$ and the auxiliary tangent spherical surface $\boldsymbol{G}$ in the point L on the basic curve c will be determined by the tangent vector $\mathbf{t}_u$ to the curve c in the point L and the tangent vector $\mathbf{t}_v$ to the v-isoparametric curve on the surface $\boldsymbol{\varphi}'$ determined by the first derivative of the function (5) in the point L. Tangent plane direction vectors

$$\mathbf{t}_u = {}^{1}\mathbf{r}'(u_0) = (x'(u_0), 0, z'(u_0), 0), \quad \text{where} \quad x'(u_0) \neq 0,$$
$$\mathbf{t}_v = {}^{2}\mathbf{r}'(0) = (0,1,0,0),$$

define normal vector

$$\mathbf{n} = \mathbf{t}_u \times \mathbf{t}_v = (-z'(u_0), 0, x'(u_0), 0).$$

The normal line q with the direction vector $\mathbf{n}$

$$x = x(u_0) - z'(u_0) \cdot l$$
$$y = 0$$
$$z = z(u_0) + x'(u_0) \cdot l, \qquad \text{for} \quad l \in [-\infty, \infty],\ u_0 \in [0,1]$$

intersects the coordinate axis x in the point S', which is the centre of the auxiliary tangent spherical surface $\mathbf{G}'$:

$$S'(x_{\mathbf{G}}, 0,0,1) = \left(\frac{x(u_0) \cdot x'(u_0) + z(u_0) \cdot z'(u_0)}{x'(u_0)}, 0, 0, 1 \right).$$

The point $S(x_S, y_S, z_S, 1)$ located on the axis ^{1}o is defined by the transformation $\mathbf{T}$ (3):

$$(x_S, y_S, z_S, 1) = (x_{\mathbf{G}}, 0, 0, 1) \cdot \mathbf{T}$$

The tangent line to the helix, which is the trajectory of the point S movement, is parallel to the basic line on the direction conical surface located in the point $P(y_S, -x_S, 0, 1)$. (The construction can be found in Velichová [3].)

The tangent line to the helix in the point S can be determined by the direction vector $\mathbf{t}(-y_S, x_S, (-1)^i \cdot z_0, 0)$. Constant value $i = 1$ is valid for the clockwise and $i = 2$ for the anticlockwise helical movement.

By the transformation $\mathbf{T}'$ (4) of the vector $\mathbf{t}$ we can obtain the coordinates of the direction vector $\mathbf{t}'(x_t, y_t, z_t, 0)$ of the tangent line to the helix of the point S' with axis o'

$$(x_t, y_t, z_t, 0) = (-y_S, x_S, (-1)^i \cdot z_0, 0) \cdot \mathbf{T}'.$$

Centre S' is incident to the plane ξ perpendicular to the vector $\mathbf{t}'$, and is determined implicitly by the equation

$$x_t(x - x_{\mathbf{G}}) + y_t \cdot y + z_t \cdot z = 0.$$

Plane ξ intersects the parallel circle k (6) located in the plane $\mu' // \mu$ (7)

$$(x - x(u_0))^2 + z^2 = z(u_0)^2 \tag{6}$$
$$x = x(u_0) \tag{7}$$

in the points $^{1}M'$, $^{2}M'$ of the constructed characteristics e' on the surface φ'. Their coordinates are defined as follows

$$\begin{aligned} {}^{j}x' &= x(u_0) \\ {}^{j}y' &= \frac{-x_t(x(u_0) - x_{\mathbf{G}})}{y_t} - z_t \cdot {}^{j}t \\ {}^{j}z' &= y_t \cdot {}^{j}t, \qquad \text{for} \quad j = 1,2;\ u_0 \in [0,1]. \end{aligned} \tag{8}$$

The parameter ^{j}t can be calculated from the quadratic equation (9) that is the solution of the equations (6)–(8)

$$t^2(y_t^2 + z_t^2) + t\left(2\frac{x_t \cdot z_t(x(u_0) - x_{\mathbf{G}})}{y_t}\right) + \left(\frac{-x_t(x(u_0) - x_{\mathbf{G}})}{y_t}\right)^2 - (z(u_0))^2 = 0 \tag{9}$$

for the value $y_t \neq 0$.

If $D = \left((z(u_0))^2(y_t^2 + z_t^2) - x_t^2(x(u_0) - x_{\mathbf{G}})^2\right)$ is a negative number, the points of the characteristics do not exist.

For $D \geq 0$

$$^{j}t = \frac{x_t \cdot z_t(x(u_0) - x_{\mathbf{G}}) \pm y_t\sqrt{D}}{y_t(y_t^2 + z_t^2)} \qquad \text{for} \quad j = 1,2;\ u_0 \in [0,1].$$

It is necessary to eliminate the case of the tangent line to the helix of the point S' determined by the direction vector $\mathbf{t}'$ that is located in the xz-plane.

By transformation $\mathbf{T}$ (3) of the point $M'(x', y', z', 1)$ of the characteristics e' on the surface φ' we can obtain the coordinates of the point $M(x_M, y_M, z_M, 1)$ of the characteristics e

$$(x_M, y_M, z_M, 1) = (x', y', z', 1) \cdot \mathbf{T}.$$

The shape of the envelope surface can be better comprehended by its meridian section than by the characteristics, which is usually a space curve segment. Coordinates of the point $M^*(x^*, y^*, z^*, 1)$ located on the meridian section in the xz-plane can be obtained from the coordinates of the

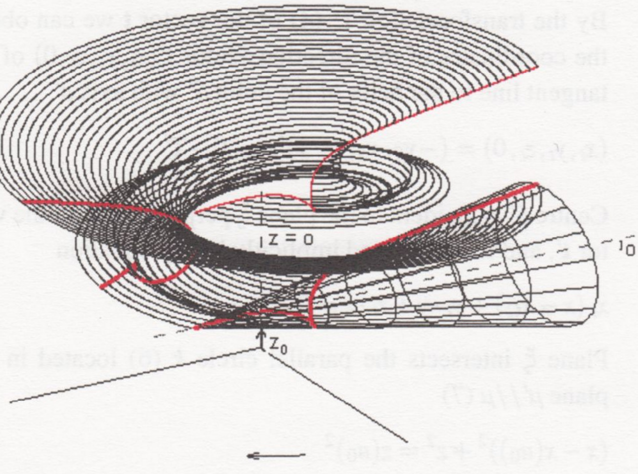

Fig. 3

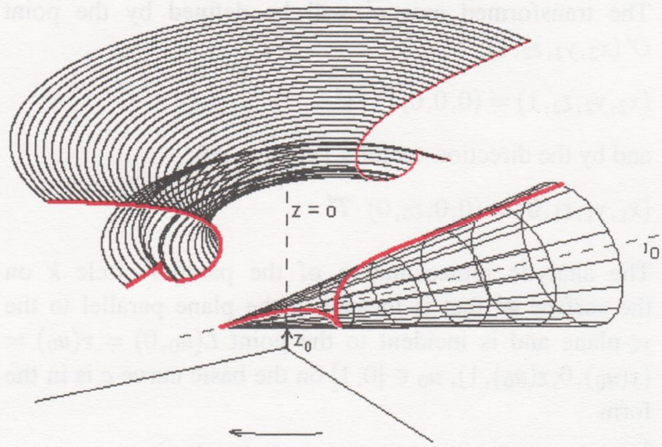

Fig. 4

point $M(x_M, y_M, z_M, 1)$ on the characteristics, as the solutions of the following equations

$$x^* = \pm\sqrt{x_M^2 + y_M^2}$$
$$y^* = 0$$
$$z^* = z_M + (-1)^i \cdot \omega \cdot z_0$$

where ω is the directed angle of the revolution about the axis o to the xz-plane oriented in the helical movement direction and constant value $i = 1$ is valid for the clockwise and $i = 2$ for the anticlockwise helical movement.

If the surface **φ** is a rotational conical or cylindrical surface, then we can create the envelope surface **Φ** by the presented method, but also by the method published in the paper (Szarková [2]). Graphical processing of the helical envelope surfaces created from the rotational surfaces by the two given different methods can be compared on illustrations in figures 3 and 4 presented in the paper (Szarková [2]) and figures 4 and 5 presented in this paper.

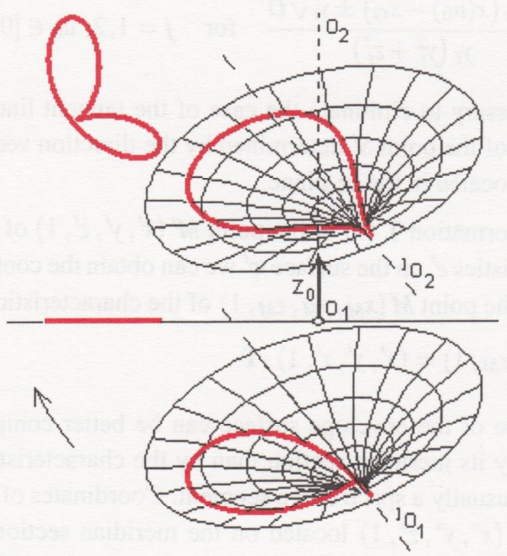

Fig. 5

In the paper (Szarková [2]) the created envelope helical surface **Φ** had been represented as the net of parts of characteristics in the Fig. 3, resp. as the net of meridian section segments in the Fig. 5. In this paper the same helical envelope surface **Φ** is represented as the net of helices that are trajectories of some points on characteristics in the Fig. 3 and on meridian section in the Fig. 4.

In the Fig. 5 the characteristics and the meridian section of the envelope helical surface **Φ** determined by the basic rotational surface **φ** with the basic curve in a cissoid is illustrated. The envelope helical surface created from the basic rotational surface **φ** with the basic curve in a Witch of Agnési is represented as the net of helices that are trajectories of the points on characteristics in the Fig. 6.

The formulas for the calculations are a bit complicated, but the computer processing is very fast. The general advantage of the computer approach to the given problematic dwells in its creative access. The interactive access available for the constructor helps him to realise the design of the elabo-

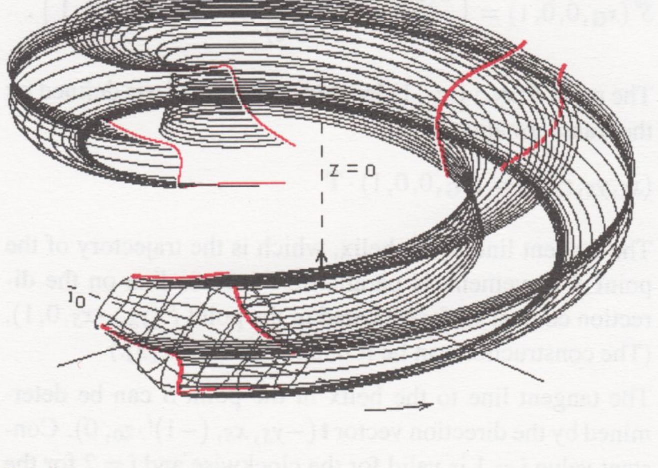

Fig. 6

rating machines in a short time, as it provides the possibility to fix the generating surface of the created envelope surface into the desired position. It is possible to exclude efficiently those positions, in which the envelope surface cannot be created (does not exit) or is of an unsuitable design.

Presented figures are examples of the characteristics, the meridian section and the basic surface **φ** of the envelope surface **Φ** projected in the Monge projection method and Axonometry and visualised as outputs of the programme (written by the author) on the screen.

References

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