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DIFFERENTIAL EQUATION OF A LOXODROME ON THE SPHEROID

Diferencijalna jednadžba loksodrome na sferoidu

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Summary

A deeper insight into the principles of navigation leads to the understanding of spheroid models, elliptic integrals etc. Thanks to sophisticated modern math software, these problems can easily be solved complying with navigational practise. In addition, some texts present navigation subjects mixing spherical and spheroid models which results in discrepancy been unexplained. Plane and Mercator sailings shall be based upon the spheroidal model or the spherical model without mixing them, thus resulting in error in calculated distance or mid-latitude. The difference in latitude parts must be used for spheroid, in place of the difference in latitude, to avoid inconsistency. Differential equation of a loxodrome on the spheroid presented here aims to understand mathematical postulates of navigation.

Key words: navigation, geodesy

Sažetak

Načela navigacije temelje se na poznavanju sferoidnih modela, eliptičnih integrala itd. Uz pomoć računala moguće je lako riješiti probleme navigacijske prakse. Uz to, u nekim tekstovima iz navigacije, zbog nerazlikovanja sfernih i sferoidnih modela, dolazi do nejasnoća koje uzrokuju pogreške u računanju udaljenosti i srednje ispravljene širine. Za točno rješenje u računu se mora koristiti lukom elipse umjesto lukom kružnice. Diferencijalna jednadžba loksodrome na sferoidu ima za cilj razumijevanje matematičkih postavki navigacije.

Ključne riječi: navigacija, geodezija.

1. Introduction

Uvod

The Earth may be approximated by an ellipsoid of revolution (spheroid) as the eccentricity of its meridian ellipse is small (e = 0.08181919 and semi-major axis a = 6 378 137 m, for WGS84). The equation of the spheroid (ellipsoid of revolution) expressed in Cartesian coordinates (x, y, z) with origin at the centre, O, of the ellipsoid, x-axis lies in the plane of the zero meridian and axis of revolution along the z-axis, equals to:

$$(x^2+y^2)/a^2 + z^2/c^2 = 1 \qquad (1)$$

where semi-minor axis c = $a\sqrt{1-e^2}$. Geodetic coordinates are defined with respect to a reference ellipsoid (spheroid) i.e. an ellipse rotated about its minor axis and used as a model for the earth. From any point on the Earth's surface a normal (perpendicular) to this reference ellipsoid can be drawn. The geodetic (geographical) latitude, φ_G, is then the angle between the ellipsoidal equator and this normal, and the geodetic (geographical) longitude, λ_G, is simply the angle between an arbitrary zero meridian (reference plane) and the plane containing both the normal and the minor axis. Every ellipsoid has an associated Cartesian system. It's considered that the centres of the two systems coincide, that the minor axis of the ellipsoid lies along the z axis and that the x axis lies in the plane of the zero meridian. The transformation can then simply be carried out using:

$$x = (a \cos \varphi_G \cos \lambda_G)/\sqrt{1-e^2 \sin^2 \varphi_G} \qquad (2)$$

$$y = (a \cos \varphi_G \sin \lambda_G)/\sqrt{1-e^2 \sin^2 \varphi_G} \qquad (3)$$

$$z = a(1-e^2) \sin \varphi_G/\sqrt{1-e^2 \sin^2 \varphi_G} \qquad (4)$$

a – spheroid semi-major axis; e – spheroid eccentricity.

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2. Loxodrome on a Mercator chart

Loksodroma na Merkatorovoj karti

Loxodrome is a Latin synonym for rhumb, and has come to be used more as a geometric term – the course is a rhumb, the curve is a loxodrome. The loxodrome (rhumb line) is the curve of the constant course. It cuts all meridians of a rotating surface, in this case ellipsoid of revolution (spheroid), at the same angle. It is a spiral that approaches Earth's poles asymptotically. The loxodrome is the path taken when a compass is kept pointing in a constant direction. On a stereographic polar projection the loxodrome is shown as a logarithmic spiral. On the Mercator chart the loxodrome is a straight line with the equation as follows:

$$x = \pm \tan\alpha \; y + C \tag{5}$$

$$y = \ln\left[(\tan(\pi/4+|\varphi|/2))\,((1- e\sin|\varphi|)/(1+ e\sin|\varphi|))^{e/2}\right] \tag{6}$$

In the case of calculating distance (s),

$$s = \sec\alpha \int_{\varphi_1}^{\varphi_2} a(1-e^2)\,(1-e^2\sin^2\varphi)^{-3/2}\,d\varphi \tag{7}$$

α – azimuth (course);

a – equatorial radius in, nautical (for sphere) or geographical (for spheroid), miles.

A harmonic series expansion for the above integral, confined for the first two terms, gives a solution for meridional distance thus,

$$s = a\sec\alpha\,[(1-\frac{1}{4}e^2)\Delta\varphi - \frac{3}{8}e^2(\sin 2\varphi_2 - \sin 2\varphi_1)] \tag{8}$$

3. Differential Equation of a Loxodrome

Diferencijalna jednadžba loksodrome

The spheroid is represented by the following vector equation:

$$\vec{r}\,(\Theta, \lambda) = (a\sin\Theta\cos\lambda,\; a\sin\Theta\sin\lambda,\; c\cos\Theta) \tag{9}$$

where:

$\Theta = 90 - \psi$, the complement of geocentric latitude ψ, $(0 \le \Theta \le \pi)$, and

$$\tan\psi = (1-e^2)\tan\varphi \tag{10}$$

λ - the geographical (geocentric) longitude $(-\pi \le \lambda \le \pi)$, and a – spheroid semi-major axis; c – spheroid semi-minor axis.

The coefficients of the first differential form for the spheroid (Gaussian basic magnitudes of the first order) are:

$$E = (\partial x/\partial\Theta)^2 + (\partial y/\partial\Theta)^2 + (\partial z/\partial\Theta)^2 = a^2\cos^2\Theta + c^2\sin^2\Theta = a^2(1-e^2\sin^2\Theta), \tag{11}$$

$$G = (\partial x/\partial\lambda)^2 + (\partial y/\partial\lambda)^2 + (\partial z/\partial\lambda)^2 = a^2\sin^2\Theta, \text{ and} \tag{12}$$

$$F = (\partial x/\partial\Theta)(\partial x/\partial\lambda) + (\partial y/\partial\Theta)(\partial y/\partial\lambda) + (\partial z/\partial\Theta)(\partial z/\partial\lambda) = 0. \tag{13}$$

The first differential form for the spheroid equals to:

$$ds^2 = E\,d\Theta^2 + 2F\,d\Theta\,d\lambda + G\,d\lambda^2 \tag{14}$$

and finally:

$$ds^2 = a^2(\sin^2\Theta\,d\lambda^2 + (1-e^2\sin^2\Theta)\,d\Theta^2) \tag{15}$$

The angle between a loxodrome and meridian is defined by the angle between their tangents at the point of intersection which can be determined using the following expression:

$$\cos\alpha = \sqrt{E}\,d\Theta / \sqrt{E d\Theta^2 + G d\lambda^2} \tag{16}$$

resulting in differential equation of a loxodrome on the spheroid:

$$\cos^2\alpha\,((1-e^2\sin^2\Theta)\,d\Theta^2 + \sin^2\Theta\,d\lambda^2) = (1-e^2\sin^2\Theta)\,d\Theta^2 \tag{17}$$

yielding a general form for the equation of a loxodrome on the spheroid:

$$\lambda = \pm \tan\alpha \int_{\Theta_1}^{\Theta_2} \mathrm{cosec}\,\Theta\,\sqrt{1-e^2\sin^2\Theta}\,d\Theta \tag{18}$$

Inserting e = 0, equation of a loxodrome on a sphere is obtained:

$$\lambda = \pm \tan\alpha \ln\tan(\pi/4 + \varphi/2) + C; \text{ C – constant of integration} \tag{19}$$

4. Length of a Loxodrome on the Spheroid

Duljina loksodrome na sferoidu

The length of an arc of any curve on the surface of the spheroid is determined using the following formula:

$$ds = a\sqrt{\sin^2\Theta\,d\lambda^2 + (1-e^2\sin^2\Theta)\,d\Theta^2} \tag{20}$$

hence,

$$s = a\sec\alpha \int_{\Theta_1}^{\Theta_2} \sqrt{1-e^2\sin^2\Theta}\,d\Theta \tag{21}$$

A binomial expansion for the above integral, confined for the first two terms, provides a more precise solution for meridional distance thus,

$$s = a\sec\alpha\,[(1-\frac{1}{4}e^2)\Delta\Theta - \frac{1}{8}e^2(\sin 2\Theta_2 - \sin 2\Theta_1)] \tag{22}$$

With e = 0, semi-major axis 'a' becomes radius 'r' and ψ coincides with φ , we get the arc length of the loxodrome on a sphere:

$$s = r\,(\varphi_2 - \varphi_1) / \cos\alpha \tag{23}$$

4. Conclusion

Zaključak

Spheroid model for Loxodrome (Rhumb Line) calculation may prove useful for understanding mathematics of navigation. For this model geodetic or geographic mile (1855.3248m) in use can easily be converted into international nautical mile (1852m). Geocentric latitude yields geodetic (geographical) latitude for zero eccentricity. The integral in equation (18) and (21) represents an incomplete elliptic integral of the second kind. It cannot be expressed in terms of elementary functions, but can be expressed in terms of the inverses of the Jacobian elliptic functions and evaluated that way. For navigational purpose numerical integration, using math software or tabular techniques will suffice.

References

Literatura

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