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MERSENNE $k-$FIBONACCI NUMBERS

Jhon J. Bravo and Carlos A. Gómez
Universidad del Cauca, Colombia and Universidad del Valle, Colombia

Abstract. For an integer $k \geq 2$, let $(F_n^{(k)})_n$ be the $k-$Fibonacci sequence which starts with $0, \ldots, 0, 1$ (k terms) and each term afterwards is the sum of the k preceding terms. In this paper, we find all $k-$Fibonacci numbers which are Mersenne numbers, i.e., $k-$Fibonacci numbers that are equal to 1 less than a power of 2. As a consequence, for each fixed k, we prove that there is at most one Mersenne prime in $(F_n^{(k)})_n$.

1. Introduction and preliminary results

For $k \geq 2$, we consider the *$k-$generalized Fibonacci sequence* or, for simplicity, the *$k-$Fibonacci sequence* $F^{(k)} := (F_n^{(k)})_{n \geq 2-k}$ given by the recurrence

$$F_n^{(k)} = F_{n-1}^{(k)} + F_{n-2}^{(k)} + \cdots + F_{n-k}^{(k)} \quad \text{for all} \quad n \geq 2,$$

with the initial conditions $F_{-(k-2)}^{(k)} = F_{-(k-3)}^{(k)} = \cdots = F_0^{(k)} = 0$ and $F_1^{(k)} = 1$. We shall refer to $F_n^{(k)}$ as the *nth $k-$Fibonacci number*. We note that this generalization is in fact a family of sequences where each new choice of k produces a distinct sequence. For example, the usual Fibonacci numbers are obtained for $k = 2$. For small values of k, these sequences are called Tribonacci ($k = 3$), Tetranacci ($k = 4$), Pentanacci ($k = 5$), Hexanacci ($k = 6$), Heptanacci ($k = 7$) and Octanacci ($k = 8$).

An interesting fact about the $k-$Fibonacci sequence is that the first $k+1$ non–zero terms in $F^{(k)}$ are powers of two, namely

$$F_1^{(k)} = 1 \quad \text{and} \quad F_n^{(k)} = 2^{n-2} \quad \text{for all} \quad 2 \leq n \leq k+1,$$

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while the next term is $F_{k+2}^{(k)} = 2^k - 1$. In fact, the inequality

$$F_n^{(k)} < 2^{n-2} \quad \text{holds for all} \quad n \geq k+2 \tag{1.1}$$

(see [1]). In general, Cooper and Howard in [6] proved the following nice formula:

LEMMA 1.1. *For $k \geq 2$ and $n \geq k+2$,*

$$F_n^{(k)} = 2^{n-2} + \sum_{j=1}^{\lfloor \frac{n+k}{k+1} \rfloor - 1} C_{n,j}\, 2^{n-(k+1)j-2},$$

where

$$C_{n,j} = (-1)^j \left[\binom{n-jk}{j} - \binom{n-jk-2}{j-2} \right].$$

In the above, we used the convention that $\binom{a}{b} = 0$ if either $a < b$ or if one of a or b is negative and denote $\lfloor x \rfloor$ the greatest integer less than or equal to x. For example, assuming that $k+2 \leq n \leq 2k+2$, Cooper and Howard's formula becomes the identity

$$F_n^{(k)} = 2^{n-2} - (n-k) \cdot 2^{n-k-3} \quad \text{for all} \quad k+2 \leq n \leq 2k+2. \tag{1.2}$$

Several authors have worked on problems involving generalized Fibonacci sequences. For instance, F. Luca ([11]) and D. Marques ([13]) proved that 55 and 44 are the largest repdigits (i.e., numbers with only one distinct digit in its decimal expansion) in the sequences $F^{(2)}$ and $F^{(3)}$, respectively. Moreover, D. Marques conjectured that there are no repdigits, with at least two digits, belonging to $F^{(k)}$, for $k > 3$. This conjecture was confirmed shortly afterwards by Bravo and Luca ([2]). Another conjecture (proposed by Noe and Post [16]) about coincidences between terms of these sequences was proved independently by Bravo–Luca ([3]) and Marques ([12]). We refer to [4, 9] for results on the largest prime factor of $F_n^{(k)}$.

A Mersenne number is a number of the form $M_m = 2^m - 1$, where m is a positive integer. A Mersenne prime is a Mersenne number that is prime. If r divides m, then $2^r - 1$ divides $2^m - 1$, so a Mersenne prime has a prime exponent. However, very few of the numbers of the form $2^p - 1$ (p prime) are prime. Mersenne numbers are the easiest type of numbers to be proved prime (because of the Lucas–Lehmer test), so are usually the primes on the list of largest known primes.

Mersenne primes have a deep connection to perfect numbers, which are numbers that are equal to the sum of their proper divisors. Historically, the study of Mersenne primes was motivated by this connection. In the 4th century BC, Euclid showed that if M is a Mersenne prime then $M(M+1)/2$ is a perfect number. Two millennia later, in the 18th century, Euler proved that all even perfect numbers have this form. No odd perfect numbers are known, and it is suspected that none exist.

In the present paper, we are interested in finding which $k-$Fibonacci numbers are Mersenne numbers, or equivalently, all $k-$Fibonacci numbers which consist of all 1's in base 2, and are therefore binary repunits. To be more precise, we study the Diophantine equation

$$F_n^{(k)} = 2^m - 1 \tag{1.3}$$

in positive integers n, k, m with $k \geq 2$.

It is important to mention that in the Fibonacci case, namely when $k = 2$, it is known that the intersection of Fibonacci sequence and Mersenne sequence is just $\{1, 3\}$. In this case, equation (1.3) reduces to find out when $F_n + 1$ is a power of 2. Here, almost every value of n can be eliminated by using the fact that the Fibonacci sequence is periodic modulo n for all n. Another way to determine whether the number $F_n + 1$ is a power of 2 is to use the known factorization $F_n + 1 = F_{(n+\delta)/2} L_{(n-\delta)/2}$, where $(L_n)_{n\geq 0}$ is the companion Lucas sequence of the Fibonacci sequence given by $L_0 = 2$, $L_1 = 1$ and $L_{n+2} = L_{n+1} + L_n$ for all $n \geq 0$, and $\delta \in \{-2, -1, 1, 2\}$ depends on the class of n modulo 4. However, similar divisibility properties for $F^{(k)}$ when $k \geq 3$ are not known and therefore it is necessary to attack the problem differently.

Since $F_1^{(k)} = F_2^{(k)} = 1$ and $F_{k+2}^{(k)} = 2^k - 1$, we see that the triples

$$(n, k, m) \in \{(1, k, 1), (2, k, 1), (k + 2, k, k)\} \tag{1.4}$$

are solutions of equation (1.3) for all $k \geq 2$. Solutions given by (1.4) will be called *trivial solutions*. In this paper, we prove the following theorem.

Theorem 1.2. *The Diophantine equation* (1.3) *has only trivial solutions.*

As immediate consequence of Theorem 1.2 we have the following corollary.

Corollary 1.3. *If M_p is a Mersenne prime in the $k-$Fibonacci sequence $F^{(k)}$, then $p = k$ and so $M_p = F_{p+2}^{(p)}$. Additionally, for each fixed k, there is at most one Mersenne prime in $F^{(k)}$. In particular, $M_2 = 3$, $M_3 = 7$, $M_5 = 31$ and $M_7 = 127$ are the only Mersenne primes in the sequences Fibonacci, Tribonacci, Pentanacci and Heptanacci, respectively.*

To prove our main result we use lower bounds for linear forms in logarithms (Baker's theory) to bound n and m polynomially in terms of k. When k is small, we use the theory of continued fractions by means of a variation of a result of Dujella and Pethő to lower such bounds to cases that allow us to treat our problem computationally. For large values of k, Bravo and Luca in [1, 2] developed some ideas for dealing with Diophantine equations involving $k-$Fibonacci numbers. However, when k is large and $m = n - 2$, the estimates given in [1,2] are not enough and therefore we need to get more accurate estimates to finish the job. For this aim, the formula of Cooper and Howard (Lemma 1.1) will play an important role.

Before proceeding further it may be mentioned that the characteristic polynomial of $F^{(k)}$, namely

$$\Psi_k(x) = x^k - x^{k-1} - \cdots - x - 1,$$

is irreducible in $\mathbb{Q}[x]$ and has just one zero real outside the unit circle. Throughout this paper, $\alpha := \alpha(k)$ denotes that single zero. The other roots are strictly inside the unit circle, so $\alpha(k)$ is a Pisot number of degree k. Moreover, it is also known that $\alpha(k)$ is located between $2(1-2^{-k})$ and 2, see [10, Lemma 2.3] or [17, Lemma 3.6]. To simplify notation, we shall omit the dependence on k of α.

We now consider the function $f_k(x) = (x-1)/(2+(k+1)(x-2))$, for an integer $k \geq 2$ and $x > 2(1-2^{-k})$. It is easy to see that the inequalities

$$(1.5) \qquad 1/2 < f_k(\alpha) < 3/4 \qquad \text{and} \qquad |f_k(\alpha^{(i)})| < 1, \qquad 2 \leq i \leq k$$

hold, where $\alpha := \alpha^{(1)}, \ldots, \alpha^{(k)}$ are all the zeros of $\Psi_k(x)$. So, by computing norms from $\mathbb{Q}(\alpha)$ to $\mathbb{Q}$, for example, we see that the number $f_k(\alpha)$ is not an algebraic integer. Proofs for this fact and (1.5) can be found in [5].

With the above notation, Dresden and Du showed in [7] that

$$(1.6) \qquad F_n^{(k)} = \sum_{i=1}^{k} f_k(\alpha^{(i)}){\alpha^{(i)}}^{n-1} \qquad \text{and} \qquad \left| F_n^{(k)} - f_k(\alpha)\alpha^{n-1} \right| < \frac{1}{2}$$

hold for all $n \geq 1$ and $k \geq 2$.

In addition to this, Bravo and Luca proved in [2] that

$$(1.7) \qquad \alpha^{n-2} \leq F_n^{(k)} \leq \alpha^{n-1} \quad \text{holds for all} \quad n \geq 1 \quad \text{and} \quad k \geq 2.$$

The observations in expressions (1.6) and (1.7) lead us to call α the *dominant zero* of $F^{(k)}$.

2. Linear forms in logarithms

In order to prove our main result, we need to use a Baker type lower bound for a nonzero linear form in logarithms of algebraic numbers, and such a bound, which plays an important role in this paper, was given by Matveev ([14]). We begin by recalling some basic notions from algebraic number theory.

Let η be an algebraic number of degree d with minimal primitive polynomial over the integers

$$a_0x^d + a_1x^{d-1} + \cdots + a_d = a_0 \prod_{i=1}^{d}(x - \eta^{(i)}),$$

where the leading coefficient a_0 is positive and the $\eta^{(i)}$'s are the conjugates of η. Then

$$h(\eta) = \frac{1}{d}\left(\log a_0 + \sum_{i=1}^{d} \log\left(\max\{|\eta^{(i)}|, 1\}\right)\right),$$

is called the *logarithmic height* of η. In particular, if $\eta = p/q$ is a rational number with $\gcd(p,q) = 1$ and $q > 0$, then $h(\eta) = \log\max\{|p|, q\}$.

Matveev ([14]) proved the following deep theorem.

Theorem 2.1 (Matveev's theorem). *Let $\mathbb{K}$ be a number field of degree D over $\mathbb{Q}$, $\gamma_1, \ldots, \gamma_t$ be positive real numbers of $\mathbb{K}$, and $b_1, \ldots, b_t$ rational integers. Put*

$$\Lambda := \gamma_1^{b_1} \cdots \gamma_t^{b_t} - 1 \qquad \textit{and} \qquad B \geq \max\{|b_1|, \ldots, |b_t|\}.$$

Let $A_i \geq \max\{Dh(\gamma_i), |\log\gamma_i|, 0.16\}$ be real numbers, for $i = 1, \ldots, t$. Then, assuming that $\Lambda \neq 0$, we have

$$|\Lambda| > \exp(-1.4 \times 30^{t+3} \times t^{4.5} \times D^2(1 + \log D)(1 + \log B)A_1 \cdots A_t).$$

To conclude this section, we give estimates for the logarithmic heights of some algebraic numbers. Let $\mathbb{K} = \mathbb{Q}(\alpha)$. Knowing that $\mathbb{Q}(\alpha) = \mathbb{Q}(f_k(\alpha))$ and that $|f_k(\alpha^{(i)})| \leq 1$ for all $i = 1, \ldots, k$ and $k \geq 2$, we obtain that $h(\alpha) = (\log\alpha)/k$ and $h(f_k(\alpha)) = (\log a_0)/k$, where a_0 is the leading coefficient of minimal primitive polynomial over the integers of $f_k(\alpha)$. Put

$$g_k(x) = \prod_{i=1}^{k}\left(x - f_k(\alpha^{(i)})\right) \in \mathbb{Q}[x] \quad \text{and} \quad \mathcal{N} = \mathrm{N}_{\mathbb{K}/\mathbb{Q}}(2 + (k+1)(\alpha - 2)) \in \mathbb{Z}.$$

We conclude that $\mathcal{N}g_k(x) \in \mathbb{Z}[x]$ vanishes at $f_k(\alpha)$. Thus, a_0 divides $|\mathcal{N}|$. But, for $k \geq 2$

$$\begin{aligned} |\mathcal{N}| = \left|\prod_{i=1}^{k}\left(2 + (k+1)(\alpha^{(i)} - 2)\right)\right| &= (k+1)^k \left|\prod_{i=1}^{k}\left(2 - \frac{2}{k+1} - \alpha^{(i)}\right)\right| \\ &= (k+1)^k \left|\Psi_k\left(2 - \frac{2}{k+1}\right)\right| \\ &= \frac{2^{k+1}k^k - (k+1)^{k+1}}{k-1} < 2^k k^k. \end{aligned}$$

Hence, we will use the following inequalities

$$h(\alpha) < 0.7/k \quad \text{and} \quad h(f_k(\alpha)) < 2\log k, \qquad \text{for all } k \geq 2. \tag{2.1}$$

3. Proof of Theorem 1.2

Assume first that we have a nontrivial solution (n, k, m) of equation (1.3); hence, $n \geq k+3$ and so $n \geq 5$. To begin with, by (1.3) and inequalities (1.1) and (1.7), we have

$$\alpha^{n-2} \leq F_n^{(k)} = 2^m - 1 < 2^{n-2},$$

getting

$$(n-2)\frac{\log\alpha}{\log 2} < m \leq n-2, \tag{3.1}$$

which is a relation between n and m. We shall have some use for it later. Using now (1.3) once again and (1.6) we get that

$$\left|f_k(\alpha)\alpha^{n-1} - 2^m\right| < \frac{1}{2} + 1 = \frac{3}{2},$$

giving

$$\left|1 - 2^m\alpha^{-(n-1)}(f_k(\alpha))^{-1}\right| < \frac{3}{\alpha^{n-1}}, \tag{3.2}$$

where we used the fact that $f_k(\alpha) > 1/2$ as has already been mentioned (see (1.5)). In order to use the result of Matveev Theorem 2.1, we take $t := 3$ and

$$\gamma_1 := 2, \qquad \gamma_2 := \alpha, \qquad \gamma_3 := f_k(\alpha).$$

We also take $b_1 := m$, $b_2 := -(n-1)$ and $b_3 := -1$. We begin by noticing that the three numbers $\gamma_1, \gamma_2, \gamma_3$ are positive real numbers and belong to $\mathbb{K} = \mathbb{Q}(\alpha)$, so we can take $D := [\mathbb{K} : \mathbb{Q}] = k$. The left–hand size of (3.2) is not zero. Indeed, if this were zero, we would then get that $f_k(\alpha) = 2^m \cdot \alpha^{-(n-1)}$ and so $f_k(\alpha)$ would be an algebraic integer, contradicting something previously mentioned.

Since $h(\gamma_1) = \log 2$, it follows that we can take $A_1 := k \log 2$. Further, in view of (2.1), we can take $A_2 = 0.7$ and $A_3 := 2k \log k$. Finally, by recalling that $m \leq n-2$, we can take $B := n-1$. Then, Matveev's theorem together with a straightforward calculation gives

$$\left|1 - 2^m\alpha^{-(n-1)}(f_k(\alpha))^{-1}\right| > \exp(-8.34 \times 10^{11} k^4 \log^2 k \log(n-1)), \tag{3.3}$$

where we used that $1 + \log k \leq 3 \log k$ for all $k \geq 2$ and $1 + \log(n-1) \leq 2\log(n-1)$ for all $n \geq 4$. Comparing (3.2) and (3.3), taking logarithms and then performing the respective calculations, we get that

$$\frac{n-1}{\log(n-1)} < 1.76 \times 10^{12} k^4 \log^2 k. \tag{3.4}$$

We next use the fact that the inequality $x/\log x < A$ implies $x < 2A \log A$ whenever $A \geq 3$ in order to get an upper bound for n depending on k. Indeed, taking $x := n-1$ and $A := 1.76 \times 10^{12} k^4 \log^2 k$, and performing the respective calculations, inequality (3.4) yields $n < 1.6 \times 10^{14} k^4 \log^3 k$. We record what we have proved so far as a lemma.

Lemma 3.1. *If (n, m, k) is a nontrivial solution in positive integers of equation* (1.3)*, then $n \geq k+3$ and*

$$m + 2 \leq n < 1.6 \times 10^{14} k^4 \log^3 k.$$

3.1. *The case $k > 180$.* In this case the following inequalities hold

$$m + 2 \le n < 1.6 \times 10^{14} k^4 \log^3 k < 2^{0.49k}. \tag{3.5}$$

By recalling that $\alpha > 2(1 - 2^{-k})$ and using the inequality $\log(1 - x) > -2x$, which holds for all $x \in (0, 1/2)$, we get

$$\frac{\log \alpha}{\log 2} > 1 + \frac{\log(1 - 2^{-k})}{\log 2} > 1 - \frac{2}{2^k \log 2}.$$

Thus, from (3.1) and (3.5) we obtain

$$\begin{aligned} n - 2 \ge m > (n-2)\frac{\log \alpha}{\log 2} &> n - 2 - \frac{2}{\log 2} \cdot \frac{n-2}{2^k} \\ &> n - 2 - \frac{2}{\log 2} \cdot 2^{-0.51k} \\ &> n - 2 - 10^{-27}, \end{aligned}$$

where we used that $k > 180$ in the last inequality. That is, $n - 2 - 10^{-27} < m \le n - 2$ implying that $m = n - 2$. Therefore, if $k > 180$, then all is reduced in finding solutions of the equation

$$F_n^{(k)} = 2^{n-2} - 1 \tag{3.6}$$

in positive integers n and k with $n \ge k + 3$. We shall distinguish two cases, namely, the cases when $k + 3 \le n \le 2k + 2$ or $n > 2k + 2$.

Case 1. $k + 3 \le n \le 2k + 2$. In this case, in view of (1.2), equation (3.6) becomes

$$2^{n-2} - (n-k) \cdot 2^{n-k-3} = 2^{n-2} - 1, \quad \text{or equivalently,} \quad (n-k) \cdot 2^{n-k-3} = 1.$$

But the last equation clearly has no solutions because its left–hand side is always ≥ 3 since $n - k \ge 3$.

To deal with the case $n > 2k + 2$, we will use the following result:

Lemma 3.2. *If $r < 2^k$, then the following estimate holds:*

$$F_r^{(k)} = 2^{r-2}\left(1 + \frac{k-r}{2^{k+1}} + \zeta(k, r)\right),$$

where $\zeta = \zeta(k, r)$ is a real number such that $|\zeta| < 4r^2/2^{2k+2}$.

Proof. From the Cooper and Howard's formula Lemma 1.1, we get that

$$\begin{aligned} |\zeta| \le \sum_{j=2}^{\lfloor \frac{r+k}{k+1} \rfloor - 1} \frac{|C_{r,j}|}{2^{(k+1)j}} &< \sum_{j \ge 2} \frac{2r^j}{2^{(k+1)j}(j-2)!} \\ &< \frac{2r^2}{2^{2k+2}} \sum_{j \ge 2} \frac{(r/2^{k+1})^{j-2}}{(j-2)!} < \frac{2r^2}{2^{2k+2}} e^{r/2^{k+1}}. \end{aligned}$$

Further, since $r < 2^k$ we have that $e^{r/2^{k+1}} < e^{1/2} < 2$. Thus, $|\zeta| < 4r^2/2^{2k+2}$. ∎

Case 2. $n > 2k+2$. Here, by Lemma 3.2, we see that equation (3.6) is transformed into the expression

$$(k-n)2^{n-k-3} + \zeta \cdot 2^{n-2} = -1,$$

where ζ is a real number satisfying $|\zeta| < 4n^2/2^{2k+2}$. From this, it follows that

$$\left|(n-k)2^{n-k-3} - 1\right| < 2^n n^2/2^{2k+2},$$

and dividing it across by $(n-k)2^{n-k-3}$ and using the fact that $n < 2^{k/2}$, we get

$$\left|1 - \frac{1}{(n-k)2^{n-k-3}}\right| < \frac{2}{n-k}.$$

However, the above relation is not possible since its left–hand side exceeds $1/2$, because $(n-k)2^{n-k-3} > (k+2)\cdot 2^{k-1} \geq 183 \cdot 2^{180}$, while its right–hand side is smaller than $2/(k+2) \leq 2/183 < 1/2$.

In conclusion, the Diophantine equation (1.3) has no solution with $n \geq k+3 \geq 184$.

3.2. *The case* $2 \leq k \leq 180$. For these values of k, we will use the following lemma, which is an immediate variation of the result due to Dujella and Pethő from [8], and will be the key tool used in this paper to reduce the upper bounds on the variables of the Diophantine equation (1.3).

Lemma 3.3. *Let A, B, γ, μ be positive real numbers and M a positive integer. Suppose that p/q is a convergent of the continued fraction expansion of the irrational γ such that $q > 6M$. Put $\epsilon = ||\mu q|| - M||\gamma q||$, where $||\cdot||$ denotes the distance from the nearest integer. If $\epsilon > 0$, then there is no positive integer solution (u, v, w) to the inequality*

$$0 < |u\gamma - v + \mu| < AB^{-w},$$

subject to the restrictions that

$$u \leq M \quad \text{and} \quad w \geq \frac{\log(Aq/\epsilon)}{\log B}.$$

In order to apply this result, we let $z := m\log 2 - (n-1)\log\alpha - \log f_k(\alpha)$ and we observe that (3.2) can be rewritten as

$$|e^z - 1| < \frac{3}{\alpha^{n-1}}. \tag{3.7}$$

Note that $z \neq 0$; thus, we distinguish the following cases. If $z > 0$, then $e^z - 1 > 0$, so from (3.7) we obtain

$$0 < z < \frac{3}{\alpha^{n-1}}.$$

Suppose now that $z < 0$. Since the dominant zeros of $F^{(k)}$ are strictly increasing as k increases, we deduce that $3/\alpha^{n-1} \leq 3/(\alpha(2))^{n-1} < 1/2$ for all $n \geq 5$.

Here, $\alpha(2)$ denotes the golden section as mentioned before. Then, from (3.7), we have that $|e^z - 1| < 1/2$ and therefore $e^{|z|} < 2$. Since $z < 0$, we have

$$0 < |z| \leq e^{|z|} - 1 = e^{|z|}|e^z - 1| < \frac{6}{\alpha^{n-1}}.$$

In any case, we have that the inequality

$$0 < |z| < \frac{6}{\alpha^{n-1}}$$

holds for all $k \geq 2$ and $n \geq 5$. Replacing z in the above inequality by its formula and dividing it across by $\log \alpha$, we conclude that

$$0 < \left| m\left(\frac{\log 2}{\log \alpha}\right) - n + \left(1 - \frac{\log f_k(\alpha)}{\log \alpha}\right)\right| < 13 \cdot \alpha^{-(n-1)}, \tag{3.8}$$

where we have used the fact that $1/\log \alpha < 2.1$. We put

$$\hat{\gamma} := \hat{\gamma}(k) = \frac{\log 2}{\log \alpha}, \quad \hat{\mu} := \hat{\mu}(k) = 1 - \frac{\log f_k(\alpha)}{\log \alpha}, \quad A := 13 \quad \text{and} \quad B := \alpha.$$

We also put $M_k := \lfloor 1.6 \times 10^{14} k^4 \log^3 k \rfloor$, which is an upper bound on m by Lemma 3.1. The fact that α is a unit in $\mathcal{O}_{\mathbb{K}}$, the ring of integers of $\mathbb{K}$, ensures that $\hat{\gamma}$ is an irrational number. Even more, $\hat{\gamma}$ is transcendental by the Gelfond-Schneider Theorem. Then, the above inequality (3.8) yields

$$0 < |m\hat{\gamma} - n + \hat{\mu}| < A \cdot B^{-(n-1)}. \tag{3.9}$$

It then follows from Lemma 3.3, applied to inequality (3.9), that

$$n - 1 < \frac{\log(Aq/\epsilon)}{\log B},$$

where $q = q(k) > 6M_k$ is a denominator of a convergent of the continued fraction of $\hat{\gamma}$ such that $\epsilon = \epsilon(k) = ||\hat{\mu}q|| - M_k||\hat{\gamma}q|| > 0$. A computer search with *Mathematica* revealed that if $k \in [2, 180]$, then the maximum value of $\log(Aq/\epsilon)/\log B$ is < 357. Hence, we deduce that the possible solutions (n, k, m) of the equation (1.3) for which k is in the range $[2, 180]$ all have $n < 360$.

Finally, a brute force search with *Mathematica* in the range

$$2 \leq k \leq 180 \quad \text{and} \quad k + 3 \leq n < 360$$

gives no solutions for the equation (1.3). This completes the analysis in the case $k \in [2, 180]$ and therefore the proof of Theorem 1.2.

4. Related equations

As has already been mentioned, F. Luca ([11]) proved that $F_{10} = 55$ is the largest repdigit in base 10 in the Fibonacci sequence. D. Marques ([13]) looked for repdigits in base 10 in the Tribonacci sequence and proved that $T_8 = 44$ is the largest such. Moreover, Bravo and Luca showed that there are no repdigits in base 10, with at least two digits, belonging to $F^{(k)}$, for $k > 3$, proving a conjecture proposed by D. Marques ([13]).

Let $b \geq 2$ an integer. A repdigit in base b (or $b-$repdigit) is an integer whose digits in its base $b-$representation are all equal to d. In particular, such number has the form $d(b^m - 1)/(b - 1)$ for some $m \geq 1$ and $1 \leq d \leq b - 1$. One obvious question that arises is whether there are finitely or infinitely many $b-$repdigits in $F^{(k)}$; that is, what can we say about solutions of the Diophantine equation

$$F_n^{(k)} = d \cdot \frac{b^m - 1}{b - 1} \tag{4.1}$$

in positive integers n, k, d, b, m with $k, b \geq 2$ and $1 \leq d \leq b - 1$?

To make this question meaningful we want to assume that $m \geq 2$ to avoid trivial cases in which $F_n^{(k)}$ could be a digit. Let us consider some cases.

Case 1. $2 \leq n \leq k+1$. In this case we already know that $F_n^{(k)}$ is a power of 2 and therefore equation (4.1) is transformed into the equation

$$2^{n-2} = d \cdot \frac{b^m - 1}{b - 1} \tag{4.2}$$

to be solved in positive integers n, d, b, m with $b \geq 2$ and $1 \leq d \leq b - 1$. However, it is not difficult to see that equation (4.2) has no solutions. In fact, since $(b^m - 1)/(b - 1)$ is a power of 2 we deduce that b is odd and therefore m must be even, say $m = 2s$ for some $s \in \mathbb{Z}^+$. Consequently, $(b^{2s} - 1)/(b - 1) = (b^s + 1)(b^s - 1)/(b - 1)$ is a power of 2 implying that $b^s + 1$ is a power of 2 which is not possible. Indeed, if s is even then $b^s + 1 = (b^{s/2})^2 + 1 \equiv 2 \pmod 4$, while if s is odd then $(b^s + 1)/(b + 1) = b^{s-1} - b^{s-2} + \cdots + 1$ is odd and larger than 1 so in either case $b^s + 1$ is not a power of 2.

Case 2. $n \geq k+2$. In this case we do not know much about the solutions of the equation (4.1). By using similar arguments to those given in this paper, one can prove that, if b is polynomially bounded in terms of k, then there exist effectively computable constants c_0 and k_0 such that, if $k > k_0$, then

$$n < c_0 k^4 \log^4 k < 2^{k/2}.$$

In this case, we use Lemma 3.2 to get

$$|F_n^{(k)} - 2^{n-2}| \leq \frac{2^{n-2}}{2^{k+1}}(n - k) + 2^{n-2}|\zeta|,$$

where ζ is a real number satisfying $|\zeta| < 4n^2/2^{2k+2}$. From the above, (4.1) and some elementary algebra, we obtain the inequality

$$\left|\left(\frac{d}{b-1}\right)\cdot b^m\cdot 2^{-(n-2)}-1\right| < \frac{7}{2^{k/2}}, \tag{4.3}$$

which holds for all $k > k_0$.

If the left–hand side of (4.3) is zero, then $db^m = (b-1)\cdot 2^{n-2}$ and so $b = 2^t$ for some integer t such that $1 \le t \le n-2$. Thus,

$$d\cdot 2^{mt} = (2^t-1)\cdot 2^{n-2} \quad \text{with} \quad 1 \le d \le 2^t - 1.$$

But the above relation is possible only when $d = 2^t - 1$ and therefore $mt = n-2$. Consequently, our equation (4.1) now looks like

$$F_n^{(k)} = 2^{mt}-1 \tag{4.4}$$

to be solved in positive integers n, k, m, t with $n \ge k+2$, $m \ge 2$ and $1 \le t \le n-2$. However, as it has been discussed in previous sections of this paper, all solutions of (4.4) have $n = k+2$, so that $mt = k$.

We now conclude from above discussion that if $n > k+2$ or b is not a power of 2, then the left–hand side of (4.3) is not zero. In this case, we apply the Matveev's theorem to the left–hand side of inequality (4.3), in order to find a lower bound of such expression, and after comparing this lower bound with the upper bound of (4.3), we get an upper bound of k. If $f(x)$ is a polynomial function such that $b < f(k)$, then it follows that there exists an effectively computable constant c, depending only on $f(x)$ and k_0, such that

$$\max\{n, k, d, b, m\} < c.$$

Suppose now that $n = k+2$ and b is a power of 2, say $b = 2^t$ for some $t \in \mathbb{Z}^+$. In this case, from (4.1) we have that

$$2^k - 1 = d\cdot\frac{2^{mt}-1}{2^t-1} \quad \text{with} \quad m \ge 2 \quad \text{and} \quad 1 \le d \le 2^t-1. \tag{4.5}$$

Since

$$d\cdot\frac{2^{mt}-1}{2^t-1} \ge \frac{2^{2t}-1}{2^t-1} = 2^t+1$$

we deduce that $k > t$. In addition, since

$$(2^k-1)(2^t-1) = d(2^{mt}-1)$$

and $k > t$, it follows that $d+1 \equiv 0 \pmod{2^t}$. But this is only possible when $d = 2^t - 1$ because $1 \le d \le 2^t - 1$. Combining this with (4.5), we conclude that $k = mt$ and therefore $n = mt+2$.

Finally, we see that the tuples

$$(n, k, d, b, m) = (mt+2, mt, 2^t-1, 2^t, m) \tag{4.6}$$

are all integer solutions of equation (4.1) for all $m \ge 2$ and t such that $mt = k$.

Thus, we have the following theorem:

Theorem 4.1. *Consider the diophantine equation* (4.1) *with* $m \geq 2$.

(*a*) *If* $2 \leq n \leq k+1$, *then* (4.1) *has no solutions.*
(*b*) *If* $n = k+2$ *and* b *is a power of 2, then the solutions of* (4.1) *are given by* (4.6).
(*c*) *Let* $f(x)$ *be a polynomial function and suppose that* $2 \leq b < f(k)$. *If* $n > k+2$ *or* b *is not a power of 2, then* (4.1) *has only finitely many solutions.*

Finally, it is important to point out that, if we remove the restriction $b < f(k)$ in Theorem 4.1(c), then equation (4.1) could have infinitely many solutions. Indeed, for $m = 2$ and $k = 2$ equation (4.1) becomes $F_n = d(b+1)$. It is well known that the largest square in the Fibonacci sequence is $F_{12} = 144$. McDaniel in [15] showed that 2 is the largest Fibonnaci number of the form $x(x+1)$. So, if $n > 12$ and n not prime, then F_n is composite and neither of the form x^2 nor of the form $x(x+1)$. Thus, considering any factorization of it of the form $F_n = a(b+1)$ with $a \leq b+1$, we actually have $a \leq b-1$, so $F_n = aa_{(b)}$. Certainly, similar things could happen for larger $k's$.

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J. J. Bravo
Departamento de Matemáticas
Universidad del Cauca
Calle 5 No 4–70, Popayán
Colombia
E-mail: jbravo@unicauca.edu.co

C. A. Gómez
Departamento de Matemáticas
Universidad del Valle
Calle 13 No 100–00, Cali
Colombia
E-mail: carlos.a.gomez@correounivalle.edu.co

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