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# $CZ$-GROUPS

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Abstract. We describe some aspects of the structure of nonabelian $p$-groups $G$ for which every nonabelian subgroup has a trivial centralizer in $G$, i.e. only it's center. We call such groups $CZ$-groups. The problem of describing the structure of all $CZ$-groups was posted as one of the first research problems in the open problems list in Yakov Berkovich's book 'Groups of prime power order' Vol 1 ([1]). Among other features of such groups, we prove that a minimal $CZ$-group must contain at least $p^5$ elements. The structure of maximal abelian subgroups of these groups is described as well.

## 1. Introduction and definitions

Throughout the entire paper we will think of $G$ as a finite $p$-group. We assume that every nontrivial nonabelian subgroup has a trivial centralizer in $G$, i.e. $C_G(H) = Z(H)$ for every nonabelian $H \leq G$.

We start by a formal definition of our main object to be investigated.

Definition 1.1. *A finite $p$-group $G$ is called a $CZ$-group if for every nonabelian subgroup $M < G$, the centralizer of $M$ in $G$ equals to $Z(M)$.*

We will shortly write $G \in CZ_p$. We start our analysis with one quite trivial observation about the center of any $G \in CZ_p$.

Proposition 1.2. *Let $G \in CZ_p$. Then $Z(G) \leq A$, for every nonabelian $A \leq G$.*

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Proof. Let's assume the opposite. Take some nonabelian group $A < G$. If $g \in Z(G) \setminus A$, then $g \in C_G(A) \neq Z(A)$. Thus, we have a contradiction with $G \in CZ_p$. □

Now, we would like to describe the case when $Z(G)$ is not contained in some abelian subgroup. In that case we prove that such an abelian subgroup can't be a maximal subgroup of some nonabelian subgroup. To be more precise, we have:

Lemma 1.3. *Let $G \in CZ_p$. Let $A \leq G$ be an abelian subgroup such that $Z(G) \nleq A$. If $M > A$ is nonabelian, then $[M : A] \geq p^2$.*

Proof. Take $g \in Z(G) \setminus A$, where $A$ is an abelian subgroup of $G$. Let $M \leq G$ be a nonabelian group such that $[M : A] = p$. Then $A \trianglelefteq M$. Notice that, from $Z(G) \leq M$ we conclude that there is a $g \in Z(G) \leq M$ such that $[\langle g, A\rangle : A] \geq p$ and $\langle g, A\rangle \leq M$, because of which, $M = \langle g, A\rangle$ is abelian. That is a contradiction, hence $[M : A] \geq p^2$. □

In some sense, it is a natural question to ask could CZ structure be inherited from $G$ to some smaller subgroup. The following result provides the answer.

Proposition 1.4. *Let $G \in CZ_p$. If $N < M < G$, where $N$ is nonabelian, then $M \in CZ_p$.*

Proof. Since $N$ is nonabelian, clearly $M' \geq 1$. So $C_G(N) = Z(N)$ and $C_G(M) = Z(M)$. Assume that $C_M(N) > Z(N)$. Then there is some $g \in C_M(N) \setminus Z(N)$. Since $C_M(N) \leq C_G(N)$, we have $g \in C_G(N) \setminus Z(N)$. Thus, $C_G(N) \neq Z(N)$ which is a clear contradiction. Therefore, $C_M(N) = Z(N)$, thus $M \in CZ_p$. □

## 2. Center of a nonabelian subgroup of a $CZ$-group

Next topic that we cover is the question of the center of a $CZ$-group $G$. To be more specific, we will provide properties of the center of a maximal nonabelian subgroup of $G$ and compare the center of $G$ with centers of some of its subgroups.

Lemma 2.1. *Let $G \in CZ_p$ and $M \leq G$ is nonabelian. Then $Z(G) \leq Z(M)$.*

Proof. Assume that $g \in Z(G) \setminus Z(M)$. Then $g \notin M$, otherwise $g \in Z(M)$. Hence, $g \in C_G(M) \setminus Z(M)$. This is a contradiction with $C_G(M) = Z(M)$. □

Next result deals with the center of a maximal nonabelian subgroup.

Theorem 2.2. *Let $G \in CZ_p$ and let $M < G$ be a maximal nonabelian subgroup. Then one of the following is true:*

1. $Z(M) = Z(G)$,
2. $Z(M) > Z(G)$ *and* $[G : M] = p$.

Proof. Take $M < G$, where $M$ is maximal nonabelian. If $Z(M) > Z(G)$, take $g \in Z(M) \setminus Z(G)$. Then there is some $x \in G \setminus M$ such that $[x, g] \neq 1$ and $g \in M$. Notice that $\langle M, x\rangle$ is nonabelian. Assume that $x^p \notin M$. Then $x^p \notin Z(M) = C_G(M)$, so $\langle M, x^p\rangle$ is also nonabelian. If $\langle M, x^p\rangle = G$, then $x^p$ is a generator. On the other hand $x^p \in \Phi(G)$. Therefore, $x$ is not a generator. Hence, $M < \langle M, x^p\rangle < G$, which is a contradiction with the assumption that $M$ is maximal nonabelian. So, $x^p \in M$ and herewith we have proved $[G : M] = p$. □

Lemma 2.3. *Let* $G \in CZ_p$ *and* $A < G$ *be abelian of index* $p$. *Then for every* $x \in G \setminus A$, *there is a nonabelian* $M < G$, *such that* $x^p \in Z(M)$.

Proof. Take $x \in G \setminus A$. We know that $G/A = \langle xA\rangle$. Take $y \in A$, such that $[x, y] \neq 1$ (there is always such an $y$, otherwise $G$ would be abelian). Take $M = \langle x, y\rangle$. Then $C_G(M) = Z(M)$. Notice that $x^p \in A$, so $[x^p, y] = 1$. Now, it is clear that $x^p \in Z(M)$. □

## 3. Minimal $CZ$-groups

In this section we deal with $CZ$-groups which don't possess any nontrivial $CZ$-subgroup. We shall name such groups minimal $CZ$-groups.

We start with a definition of a minimal $CZ$-group.

Definition 3.1. *A group* $G \in CZ_p$ *is called a minimal* $CZ$*-group if it doesn't possess a nontrivial* $CZ$*-subgroup.*

By Proposition 1.4, it is straightforward to see that if $G$ is a minimal $CZ$-group, then every proper nonabelian subgroup is a minimal nonabelian group. For a $CZ$-group $G$ which is determined to be minimal in this sense, we shall write $G \in CZm_p$.

For the sake of completeness, we repeat here the known result that classifies minimal nonabelian $p$-groups.

Theorem 3.2. *Let* $G$ *be a minimal nonabelian* $p$*-group. Then* $|G'| = p$ *and* $G/G'$ *is abelian of rank* 2. $G$ *is isomorphic to one of the following groups:*

1. $G = \langle a, b \mid a^{p^m} = b^{p^n} = 1,\ a^b = a^{1+p^{m-1}}\rangle,\ m \geq 2,\ n \geq 1$ *and* $|G| = p^{m+n}$,
2. $G = \langle a, b, \mid a^{p^m} = b^{p^n} = c^p = 1,\ [a, b] = c,\ [a, c] = [b, c] = 1\rangle$, *where* $|G| = p^{m+n+1}$, *and if* $p = 2$, *then* $m + n > 2$ *and* $G'$ *is maximal cyclic normal subgroup,*
3. $G \cong Q_8$.

Our next result answers the question on the number of generators of a given $G \in CZm_p$.

THEOREM 3.3. *If $G \in CZm_p$, then $[G : \Phi(G)] \leq p^3$.*

PROOF. Let $A < G$ be some maximal abelian subgroup (meaning that there is no abelian subgroup $B$ such that $A < B < G$). Then $A < \langle x, A\rangle \leq G$ for some $x \in G \setminus A$. It is clear that there is an $a \in A$ such that $[a, x] \neq 1$ (otherwise $A$ wouldn't be maximal abelian). Take $M = \langle a, x\rangle$. Clearly $M' > 1$. If $M = G$, then $[G : \Phi(G)] = p^2$.

If $[G : M] \geq p^2$, then there is some $N < G$ such that $M < N < G$. Therefore $N \in CZ_p$ which contradicts to $G \in CZm_p$. If $[G : M] = p$, then it is clear that $G = \langle a, x, y\rangle$, for some $y \in G \setminus M$. Hence $[G : \Phi(G)] \leq p^3$. □

Notice that if $G \in CZm_p$, then it is natural to assume $|G| \geq p^4$. Otherwise, any proper subgroup would be of order at most $p^2$, thus abelian. In order to deliver a description of a minimal $CZ$-group, we need to provide some information about automorphisms of minimal nonabelian groups, since such groups, as we have seen above, are main ingredients of minimal $CZ$-groups.

Thus, we will start our analysis with groups of order $p^4$. For that, we need some technical results regarding the modular group of order $p^3$, which may be a minimal nonabelian subgroup of a putative $CZ$-group $G$. Throughout this paper we will denote that modular group and its generators and relations by

$$M_{p^3} = \langle a, b \mid a^{p^2} = b^p = 1,\ a^b = a^{1+p}\rangle.$$

Another option is that a nonabelian subgroup of order $p^3$ is given by

$$N = \langle a, b, \mid a^p = b^p = c^p = 1,\ [a, b] = c,\ [a, c] = [b, c] = 1\rangle$$

and this notation of the group $N$ will be kept throughout the paper as well.

It is easy to see that for $M_{p^3}$ the following holds: $b^j a^i = a^{i(1-p)^j} b^j$, and

$$(a^i b^j)^k = a^{i[1+(1-p)^j+(1-p)^{2j}+\cdots+(1-p)^{(k-1)j}]} b^{kj}$$

for $k \geq 2$. On the other hand, $o(a^i b^j) = p^2$ for any $i \neq 0, p$ and $i \in [p^2 - 1] \setminus \{0, p\}$, while $o(a^p b^j) = p$. Finally, it is also easy to see that $(a^i b^j)^b = (a^i b^j)^{p+1}$. Also, because of $\langle a\rangle \trianglelefteq M_{p^3}$ we may assume that any automorphism of $M_{p^3}$ is of the form $a^\alpha b^\beta \to a^{\alpha k}(a^{pi} b^j)^\beta$ for some integers $k$, $i$ and $j$.

The next result gives a description of such automorphisms.

LEMMA 3.4. *Let $M_{p^3} = \langle a, b \mid a^{p^2} = b^p = 1,\ a^b = a^{1+p}\rangle$. Let $\varphi_{ij} : M_{p^3} \to M_{p^3}$ be maps defined by $\varphi_{ij}(a^\alpha b^\beta) = a^\alpha (a^{pi} b^j)^\beta$. Then*

1. $\varphi_{ij}(a^\alpha b^\beta a^\gamma b^\delta) = a^{\alpha+\gamma(1-p)^\beta+pi\frac{(1-p)^{(\beta+\delta)j}-1}{(1-p)^j-1}} b^{j(\beta+\delta)}$,
2. $\varphi_{ij}(a^\alpha b^\beta)\varphi_{ij}(a^\gamma b^\delta) = a^{\alpha+pi\frac{(1-p)^{\beta j}-1}{(1-p)^j-1}+\gamma(1-p)^{j\beta}+pi(1-p)^{\beta j}\frac{(1-p)^{\delta j}-1}{(1-p)^j-1}} b^{j(\beta+\delta)}$.

PROOF. Notice that $a^\alpha b^\beta a^\gamma b^\delta = a^\alpha (b^\beta a^\gamma) b^\delta = a^\alpha a^{\gamma(1-p)^\beta} b^\beta b^\delta$. Therefore, we have $\varphi_{ij}(a^\alpha b^\beta a^\gamma b^\delta) = \varphi_{ij}(a^{\alpha+\gamma(1-p)^\beta} b^{\beta+\gamma}) = a^{\alpha+\gamma(1-p)^\beta}(a^{pi} b^j)^{\beta+\delta}$.

But, on the other hand

$$(a^{pi}b^j)^{\beta+\delta} = a^{pi[1+(1-p)^j+(1-p)^{2j}+\cdots+(1-p)^{j(\beta+\delta-1)j}]}b^{j(\beta+\delta)},$$

hence $\varphi_{ij}(a^\alpha b^\beta a^\gamma b^\delta) = a^{\alpha+\gamma(1-p)^\beta+pi\frac{(1-p)^{(\beta+\delta)j}-1}{(1-p)^j-1}}b^{j(\beta+\delta)}$. Furthermore, we have

$$\begin{aligned}\varphi_{ij}(a^\alpha b^\beta)\varphi_{ij}(a^\gamma b^\delta) &= a^\alpha(a^{pi}b^j)^\beta a^\gamma(a^{pi}b^j)^\delta \\ &= a^\alpha a^{pi[1+(1-p)^j+(1-p)^{2j}+\cdots+(1-p)^{j(\beta-1)}]}b^{j\beta} \\ &\quad\cdot a^\gamma a^{pi[1+(1-p)^j+(1-p)^{2j}+\cdots+(1-p)^{j(\delta-1)}]}b^{j\delta}.\end{aligned}$$

Let us introduce shortcuts

$$A = 1+(1-p)^j+(1-p)^{2j}+\cdots+(1-p)^{j(\beta-1)} = \frac{(1-p)^{\beta j}-1}{(1-p)^j-1},$$

$$B = 1+(1-p)^j+(1-p)^{2j}+\cdots+(1-p)^{j(\delta-1)} = \frac{(1-p)^{\delta j}-1}{(1-p)^j-1}.$$

We get

$$\begin{aligned}\varphi_{ij}(a^\alpha b^\beta)\varphi_{ij}(a^\gamma b^\delta) &= a^\alpha a^{piA}b^{j\beta}a^\gamma a^{piB}b^{j\delta} \\ &= a^{\alpha+piA}(b^{j\beta}a^{\gamma+piB})b^{j\delta} = \{\text{since } b^j a^i = a^{i(1-p)^j}b^j\} \\ &= a^{\alpha+piA}a^{(\gamma+piB)(1-p)^{j\beta}}b^{j\beta}b^{j\delta} \\ &= a^{\alpha+pi\frac{(1-p)^{\beta j}-1}{(1-p)^j-1}+\gamma(1-p)^{j\beta}+pi(1-p)^{\beta j}\frac{(1-p)^{\delta j}-1}{(1-p)^j-1}}b^{j(\beta+\delta)}.\end{aligned}$$

□

Throughout the coming results we will deal with the assumption that $M_{p^3}$ is a normal subgroup of $G$.

Proposition 3.5. *Let $G$ be a $p$-group and $M_{p^3} = \langle a,b\rangle \trianglelefteq G$. Let $d \in G\backslash M$ be such that $a^d = a$ and $b^d \in \langle b\rangle$. Then $[b,d] = 1$.*

Proof. Since $M_{p^3} \trianglelefteq G$, the action via conjugation is an inner automorphism of $M_{p^3}$. Let us use the notation $(a^\alpha b^\beta)^d = \varphi_{ij}(a^\alpha b^\beta) = a^\alpha(a^{pi}b^j)^\beta$. Then we would have $\varphi_{ij}(a^\alpha) = a^\alpha = (a^\alpha)^d$, $\varphi_{ij}(b^\beta) = (a^{pi}b^j)^\beta$. Let us assume that $b^d = b^j$. Then $\varphi_{ij}(b) = a^{pi}b^j = b^d$, so $pi \equiv 0 \mod p^2$, hence $i \in \{0,p\}$. We proved that $\varphi_{ij}(a^\alpha b^\beta a^\gamma b^\delta) = a^{\alpha+\gamma(1-p)^\beta}b^{j(\beta+\delta)}$. On the other hand

$$\begin{aligned}\varphi_{ij}(a^\alpha b^\beta)\varphi_{ij}(a^\gamma b^\delta) &= a^\alpha a^0 b^{j\beta}a^\gamma a^0 b^{j\delta} = a^\alpha(a^{j\beta}a^\gamma)b^{j\delta} \\ &= a^\alpha a^{\gamma(1-p)^{j\beta}}b^{j\beta}b^{j\delta} = a^{\alpha+\gamma(1-p)^{j\beta}}b^{j(\beta+\delta)}.\end{aligned}$$

Therefore, it is necessary that $a^{\gamma(1-p)^\beta} = a^{\gamma(1-p)^{j\beta}}$, so $\gamma(1-p)^\beta \equiv \gamma(1-p)^{j\beta} \mod p^2$. Now, we must have $\gamma\left[(1-p)^{j\beta}-(1-p)^\beta\right] \equiv 0 \mod p^2$. From here we get $\beta(1-j) \equiv 0 \mod p$. Since this must be true for any $\beta$, we conclude that $1-j \equiv 0 \mod p$, so $b^d = b$. ∎

PROPOSITION 3.6. *Let $G$ be a $p$-group and $M_{p^3} \trianglelefteq G$. Let $g \in G \setminus M_{p^3}$ such that $a^d = a$ and $b^d \in a^p\langle b\rangle$. Then $b^d = a^p b$.*

PROOF. Notice that $(a^\alpha b^\beta)^d = a^\alpha (a^p b^j)^\beta = \varphi_{1j}(a^\alpha b^\beta)$. It's necessary that $\varphi_{1j}(a^\alpha b^\beta)\varphi_{1j}(a^\gamma b^\delta) = \varphi_{1j}(a^\alpha b^\beta a^\gamma b^\delta)$. Using Lema 3.4 we get

$$\varphi_{1j}(a^\alpha b^\beta a^\gamma b^\delta) = a^{\alpha+\gamma(1-p)^\beta + p\frac{(1-p)^{(\beta+\delta)j}-1}{(1-p)^j-1}} b^{j(\beta+\delta)}.$$

On the other hand

$$\varphi_{1j}(a^\alpha b^\beta)\varphi_{1j}(a^\gamma b^\delta) = a^{\alpha+p\frac{(1-p)^{\beta j}-1}{(1-p)^j-1}+\gamma(1-p)^{\beta j}+p(1-p)^{\beta j}\frac{(1-p)^{\delta j}-1}{(1-p)^j-1}} b^{j(\beta+\delta)}.$$

Let us use abbreviations

$$\Lambda = \alpha + \gamma(1-p)^\beta + p\frac{(1-p)^{(\beta+\delta)j}-1}{(1-p)^j-1},$$
$$\Pi = \alpha + p\frac{(1-p)^{\beta j}-1}{(1-p)^j-1} + \gamma(1-p)^{\beta j} + p(1-p)^{\beta j}\frac{(1-p)^{\delta j}-1}{(1-p)^j-1}.$$

So, it is necessary that $\Lambda \equiv \Pi \mod p^2$. We see that

$$p\left[1+(1-p)^j+(1-p)^{2j}+\cdots+(1-p)^{(\beta-1)j}\right] \equiv p\beta \mod p^2.$$

Similarly, we get

$$p\frac{(1-p)^{(\beta+\delta)j}-1}{(1-p)^j-1} \equiv (\beta+\delta)p \mod p^2$$

and

$$p\frac{(1-p)^{\delta j}-1}{(1-p)^j-1} \equiv \delta p \mod p^2.$$

Hence,

$$a^\Lambda = a^{\alpha+\gamma(1-p)^\beta+(\beta+\gamma)p} = a^\Pi = a^{\alpha+\beta p+\delta p+\gamma(1-p)^{j\beta}}.$$

Then, $\gamma(1-p)^\beta \equiv \gamma(1-p)^{\beta j} \mod p^2$. Thus we get only one possibility: $j \equiv 1 \mod p$. Therefore, $b^d = a^p b$. ∎

Now we will focus our analysis to groups of order $p^4$. The main goal is to determine all minimal $CZ$-groups of order $p^4$.

PROPOSITION 3.7. *Let $G = \langle a, b, c\rangle$ be a group of order $p^4$, $p \geq 2$, where $M_{p^3} = \langle a, b \mid a^{p^2} = b^p = 1,\ a^b = a^{1+p}\rangle$. If $[a,c] = 1$ and $b^c \in \langle b\rangle$, then also $[b,c] = 1$.*

PROOF. If $[a,c]=1$, then $a^c=a$. If $b^c\in\langle b\rangle$, then $b^c=b^j$ for some $j\in[p]$. Notice that $(a^\alpha b^\beta a^\gamma b^\delta)^c=a^{\alpha+\gamma(1-p)^\beta}b^{(\beta+\delta)j}$, while on the other hand we have $(a^\alpha b^\beta)^c(a^\gamma b^\delta)^c=a^\alpha b^{j\beta}a^\gamma b^{j\delta}=a^{\alpha+\gamma(1-p)^{j\beta}}b^{j\beta+j\delta}$. This gives us $\alpha+\gamma(1-p)^{j\beta}\equiv\alpha+\gamma(1-p)^\beta \mod p^2$. Since $\alpha,\gamma\in[p^2]$ and $\beta\in[p]$, we get $\gamma(1-p)^{j\beta}\equiv\gamma(1-p)^\beta \mod p$. Now, it is easy to see that $j\equiv 1 \mod p$, thus $b^c=b$. ∎

PROPOSITION 3.8. *Let $G=\langle a,b,c\rangle$ be a group of order $p^4$, $p\geq 2$, where $M_{p^3}=\langle a,b\mid a^{p^2}=b^p=1,\ a^b=a^{1+p}\rangle$. If $a^c=a$ and $b^c\in a^p\langle b\rangle$, then $b^c=a^pb$.*

PROOF. Put $b^c=a^pb^j$. We use an idea that is similar to the previous proof. Firstly, notice that because of $b^ja^i=a^{i(1-p)^j}b^j$ we get $(a^\alpha b^\beta a^\gamma b^\delta)^c=a^{\alpha+p\beta+p\delta}\cdot a^{\gamma(1-p)^{\beta j}}b^{\beta j+\delta j}$. On the other hand, after we use the automorphism property, we get $(a^\alpha b^\beta a^\gamma b^\delta)^c=a^{\alpha+p\beta+p\delta}\cdot a^{\gamma(1-p)^\beta}b^{\beta j+\delta j}$. This gives us $j\equiv 1 \mod p$. ∎

Notice that if $G=\langle a,b,c\rangle$ is of order $p^4$ and $p>2$, where $M_{p^3}=\langle a,b\rangle$, then from the assumption $G\in CZ$ we get $o(c)\leq p^2$. Otherwise, $o(c)=p^3$, implying $\langle c\rangle\trianglelefteq G$ to be maximal abelian. Then $G$ would be $M_{p^4}$, hence $G$ is not a $CZ$-group. A contradiction.

It can be shown that if $G=\langle a,b,c\rangle\in CZ_p$ and $|G|=p^4$, $p>2$ and $M_{p^3}=\langle a,b\rangle$, then the assumption $[a,c]=1$ yields $b^c=a^pb$ and $o(c)=p^2$, with an additional property $a^p=c^p$. The alternative possibility is $o(c)=p$. But the next result shows that this is not possible.

THEOREM 3.9. *Let $G=\langle M_{p^3},c\rangle$ be of order $p^4$, where $M_{p^3}=\langle a,b\mid a^{p^2}=b^p=1,\ a^b=a^{1+p}\rangle$. If $a^c=a$, $b^c=a^pb$ and $o(c)=p$, then $G\notin CZ_p$.*

PROOF. Let $A=\langle a,c\rangle\cong C_{p^2}\times C_p$. Then $A$ is a maximal abelian subgroup. Take $\phi: A\to A$ where $\phi(x)=[x,b]$. We know that $\phi$ is a homomorphism and $Im(\phi)=G'=\langle a^p\rangle$. Since $a^{ac}=a$, $c^{ac}=c$, $b^{ac}=b$, then $ac\in Z(G)$. Notice that $o(ac)=p^2$ and $|G|=p^4=p\cdot|Z(G)|\cdot|G'|$. Thus $|Z(G)|=p^2$, hence $Z(G)=\langle ac\rangle$. Clearly $ac\notin M_{p^3}$. Therefore $ac\in C_G(M_{p^3})\setminus Z(M_{p^3})$. So, $M_{p^3}$ doesn't have a trivial centralizer, hence $G\notin CZ_p$. ∎

LEMMA 3.10. *Let $G=\langle M_{p^3},c\rangle\in CZ_p$ be of order $p^4$, where $M_{p^3}=\langle a,b\mid a^{p^2}=b^p=1,\ a^b=a^{1+p}\rangle$. If $a^c\in\langle a\rangle$ and $b^c\in\langle b\rangle$, then $[b,c]=1$.*

PROOF. Take $a^c=a^i$, $b^c=b^j$ where $i\in[p^2-1]$, $j\in[p-1]$. Then $(a^\alpha b^\beta a^\gamma b^\delta)^c=(a^\alpha a^{\gamma(1-p)^\beta}b^{\beta+\delta})^c=a^{\alpha i+\gamma(1-p)^\beta i}b^{(\beta+\delta)j}$. On the other hand we have $(a^\alpha b^\beta a^\gamma b^\delta)^c=a^{\alpha i}b^{\beta j}a^{\gamma i}b^{\delta j}=a^{\alpha+\gamma i(1-p)^{\beta j}}b^{(\beta+\delta)j}$. This leads us to $\gamma i[(1-p)^{\beta j}-(1-p)^\beta]\equiv 0 \mod p^2$. Since $\gamma$ is any integer, we get $p\mid(1-p)^{\beta j}-(1-p)^\beta$. Hence $j=1$. ∎

Theorem 3.11. *Let* $G = \langle M_{p^3}, c\rangle$ *be a group of order* $p^4$, $p > 2$ *where* $M_{p^3} = \langle a, b \mid a^{p^2} = b^p = 1,\ a^b = a^{1+p}\rangle$. *If* $o(c) = p^2$, *then* $a^c \in M_{p^3} \setminus \langle a\rangle$ *or* $\langle a\rangle \cap \langle c\rangle > 1$.

Proof. Assume $a^c = a^i$ and $\langle a\rangle \cap \langle c\rangle = 1$. Then $G = \langle a, c\rangle$ and $c^p \in M \setminus \langle a\rangle$. Therefore $c^p = a^{pi}b^j$ where $i \in \{1, p\}$ and $j \in [p-1]$. Notice that because of $Z(M_{p^3}) = \langle a^p\rangle$ we have $(c^p)^k = (a^{pi})^k b^{jk}$. Furthermore, $a^{c^p} = a^{a^{pi}b^j} = a^{b^j} = a^{(1+p)^j} = a^{1+pj}$. For every $k \in \mathbb{N}$ we have $a^{(a^{pi}b^j)^k} = a^{b^{jk}} = a^{(1+pj)^k}$. If we assume that $a^{(1+pj)^k} = a$, then $(1+pj)^k - 1 \equiv 0 \mod p^2$. Therefore, $p(kj-1) \equiv 0 \mod p^2$ and $kj \equiv 1 \mod p$. Notice that such $k$ always exists (and is not divisible by $p$) since $C_p$ is a field. Therefore, without losing generality we can take $c^p = b$. Take $\varphi \in Aut(\langle a\rangle) \cong Aut(C_{p^2}) \cong C_{p(p-1)}$ (here we need the assumption $p > 2$). Put $\varphi(a) = a^i$. Then $\varphi^p(a) = a^{i^p} = a^{c^p} = a^b = a^{1+p}$. On the other hand $\varphi^{p(p-1)}(a) = a$, hence $(p+1)^{p-1} - 1 \equiv 0 \mod p^2$. This gives us $-p \equiv 0 \mod p^2$, which is an obvious contradiction. Therefore, $a^c \in M \setminus \langle a\rangle$ or $\langle a\rangle \cap \langle c\rangle > 1$. ∎

Proposition 3.12. *Let* $G = \langle M_{p^3}, c\rangle$ *be a* $CZ$*-group of order* $p^4$, $p > 2$ *where* $M_{p^3} = \langle a, b \mid a^{p^2} = b^p = 1,\ a^b = a^{1+p}\rangle$. *If* $o(c) = p^2$ *and* $a^c \in \langle a\rangle$, *then* $[b, c] \neq 1$.

Proof. Assume that the claim is not true. That means $[b, c] = 1$. Notice that $c \notin C_G(M_{p^3}) \setminus M_{p^3}$, otherwise $G$ wouldn't be a $CZ$-group. Since $a^c \in \langle a\rangle$, then by the previous theorem $\langle a\rangle \cap \langle c\rangle > 1$. Notice that $\langle a\rangle \cap \langle c\rangle = \langle a^p\rangle$. We can write $a^p = c^p$. Take $a^c = a^{1+i}$. Then $a \mapsto a^{1+i}$ is an automorphism of order $p$, thus $a^{c^p} = a^{(1+i)^p} = a^{1+pi} = a$, hence $pi \equiv 0 \mod p^2$. So without losing generality we may write $a^c = a^{1+p}$. Now, look at the element $cb^{p-1}$. For it we have $[c, cb^{p-1}] = [b, cb^{p-1}] = 1$. On the other hand $a^{cb^{p-1}} = (a^{1+p})^{b^{p-1}} = (a^{(p+1)^{p-1}})^{p+1} = a^{(p+1)^p} = a^{1+p^2} = a$. Hence, $cb^{p-1} \in C_G(M_{p^3}) \setminus M_{p^3}$, which is a contradiction. Thus, $[b, c] \neq 1$. ∎

Lemma 3.13. *Let* $G = \langle M_{p^3}, c\rangle$ *be a* $CZ$*-group of order* $p^4$, $p > 2$ *where* $M_{p^3} = \langle a, b \mid a^{p^2} = b^p = 1,\ a^b = a^{1+p}\rangle$. *If* $a^c = a^{1+p}$, *then* $b^c = a^p b^j$ *where* $j \neq 0$.

Proof. Since $M_{p^3} \trianglelefteq G$, then $b^c \in M_{p^3}$. If $b^c \in \langle a\rangle$, then $o(b) = p$ implies that we can write $b^c = a^p$. Therefore $b^c \in \Phi(M_{p^3}) char\ M_{p^3}$, where $\Phi(M_{p^3})$ stands for the Frattini subgroup, which is characteristic. Thus $b \in \Phi(M_{p^3})$ and so not a generator of $M_{p^3}$, contradiction. Therefore $b^c = b^j$ or $b^c = a^p b^j$, where $j \neq 0$. If $b^c = b^j$, then $b^c \mapsto b^j$ is an automorphism of order $p-1$ (since $Aut(C_p) \cong C_{p-1}$). Therefore, $b^{c^{p-1}} = b$, so $[b, c^{p-1}] = 1$. On the other hand $a^{c^{p-1}} \in \langle a\rangle$ and $c^{p-1}$ is clearly a generator for $G$. But this contradicts Proposition 3.12. ∎

THEOREM 3.14. *Let $G = \langle M_{p^3}, c\rangle$ be a $CZ$-group of order $p^4$, $p > 2$ where $M_{p^3} = \langle a, b \mid a^{p^2} = b^p = 1,\ a^b = a^{1+p}\rangle$. If $o(c) = p^2$ and $a^c = a^{1+p}$, then $b^c = a^p b$.*

PROOF. Because of Theorem 3.11 we have $\langle a\rangle \cap \langle c\rangle > 1$, thus without losing generality we may write $a^p = c^p$. Because of $M_{p^3} \trianglelefteq G$, we have $b^c \in M_{p^3}$. As we have proved in the previous Lemma, we have $b^c = a^p b^j \neq a^p$. From Theorem 2.2 we have $Z(M_{p^3}) = \langle a^p\rangle \geq Z(G) > 1$. Thus $Z(G) = \langle a^p\rangle$. Put $z = a^p$. Then $c^{-1}bc = zb^j$. From here we get $bc = czb^j$ and $b^{-1}cb = zb^{-2}cb^{j+1} = zb^{p-2}cb^{j+1}$. Using this, we get

$$c^b = zb^{p-3}(bc)b^{j+1} = zb^{p-3}(czb^j)b^{j+1} = z^2b^{p-3}cb^{2j+1} = \cdots = z^k b^{p-k-1}cb^{kj+1}.$$

Put $k = p-1$. Then $c^b = z^{p-1}cb^{j(p-1)+1} = z^{p-1}cb^{1-j}$.

Now, let us take a group $N = \langle c, b\rangle$. Since $\langle c\rangle \trianglelefteq N$, we get $c^b \in \langle c\rangle$. From here we get $b^{1-j} \in \langle c\rangle$. If $b^{1-j} \neq b$, then $\langle b^{1-j}\rangle = \langle b\rangle \leq \langle c\rangle$, so $G = \langle a, c\rangle$ is of order $p^3$. Thus, the only option is $b^{1-j} = 1$, hence $b^j = b$. ∎

PROPOSITION 3.15. *Let $M_{p^3} = \langle a, b \mid a^{p^2} = b^p = 1,\ a^b = a^{1+p}\rangle$. Let $G = \langle M_{p^3}, c\rangle$ be a $CZ$-group of order $p^4$ where $o(c) = p^2$. Then $\langle a\rangle \trianglelefteq G$.*

PROOF. Let us assume the opposite. Since $M_{p^3} \trianglelefteq G$, then $a^c = a^i b^j$, where $o(a^i) = p^2$ and $b^j \neq 1$. Take $N = \langle a, c\rangle$. Because $|\langle a\rangle \cap \langle c\rangle| \leq p$ we have $|N| = p^3$, therefore $N \trianglelefteq G$. So $a^c \in N$. This gives us $a^c = a^i b^j \in N$, hence $b^j \in N$. If $b^j \neq 1$, then $b \in N$ and $N = G$, which gives us a contradiction. So, the only case is $b^j = 1$ and $a^c \in \langle a\rangle$, hence $\langle a\rangle \trianglelefteq G$. ∎

Now, we have only one candidate $G = \langle a, b, c \mid a^{p^2} = c^{p^2} = b^p = 1,\ a^b = a^c = az,\ b^c = bz,\ z = a^p\rangle$ for a $CZ$-group of order $p^4$ that contains $M_{p^3}$. The next result will provide an answer regarding the status of such group.

THEOREM 3.16. *The group $G = \langle a, b, c \mid a^{p^2} = c^{p^2} = b^p = 1,\ a^b = a^c = az,\ b^c = bz,\ z = a^p\rangle$ is not a $CZ$-group.*

PROOF. Notice that $M_{p^3} = \langle a, b \mid a^{p^2} = b^p = 1,\ a^b = a^{1+p}\rangle \trianglelefteq G$. Let us assume that $G$ is a $CZ$-group. Then, by previous results we have $Z(G) = Z(M_{p^3}) = \langle a^p\rangle = \langle z\rangle$. Take $x = ab^{p-1}c$. Then $x \notin M_{p^3}$, otherwise $c \in M_{p^3}$ and $|G| \neq p^4$. Then

$$\begin{aligned} a^x &= a^{ab^{p-1}c} = a^{b^{p-1}c} = (a^b)^{b^{p-2}c} = (az)^{b^{p-2}c} \\ &= za^{b^{p-2}c} = z(a^b)^{b^{p-3}c} = z^2a^{b^{p-2}c} = \cdots = z^{p-1}a^c = z^{p-1}za = a, \end{aligned}$$

which gives us $x \in C_G(a)$. On the other hand it is clear that from $b^{-1}ab = az$ we get $ab = zba$, thus $ba = z^{-1}ab$. Using this, we get $b^a = a^{-1}ba = a^{-1}(z^{-1}ab) = z^{-1}b$. Therefore

$$b^x = b^{ab^{p-1}c} = (z^{-1}b)^{b^{p-1}c} = (z^{-1}b)^c = z^{-1}b^c = z^{-1}bz = b,$$

thus $x \in C_G(b)$. So, $x \in C_G(M_{p^3}) \setminus Z(M_{p^3})$, which is a contradiction with the assumption that $G$ is a $CZ$-group. ∎

In other words, we have proved the following result:

THEOREM 3.17. *If $G \in CZm_p$ is of order $p^4$, $p > 2$, then $M_{p^3} \not\leq G$.*

It is easy to check the properties of an exponent of a minimal $CZ$-group of order $p^4$.

LEMMA 3.18. *Let $G \in CZm_p$ and $|G| = p^4$. Then* $\exp(G) \leq p^2$.

PROOF. Let $\exp(G) > p^2$. If $\exp(G) = p^4$ then $G \cong C_{p^4} \notin CZ$. If $\exp(G) = p^3$, then there is some $d \in G$ such that $\langle d \rangle \trianglelefteq G$. Hence $G \cong M_{p^4}$, which is not a $CZ$-group. ∎

Theorem 3.2 motivates us to deal with the possible minimal $CZ$-group that contains minimal nonabelian subgroup different than modular.

PROPOSITION 3.19. *Let $G \in CZm_p$ be of order $p^4$. Let $N = \langle a, b, c \mid a^p = b^p = c^p = 1,\ a^b = ac,\ [a, c] = [b, c] = 1\rangle \leq G$. Then there is a $T \leq N$, such that $T \cong C_p \times C_p$ and $T \trianglelefteq G$.*

PROOF. We know that $|N| = p^3$ and $N$ is minimal nonabelian group. Also $N' = \langle c \rangle$ is maximal cyclic normal subgroup in $N$. Thus every maximal subgroup in $N$ is isomorphic to $C_p \times C_p$. Take $\Gamma_1 = \{T \leq N \mid [N : T] = p\}$. As we've seen, $T \cong C_p \times C_p$ for every $T \in \Gamma_1$. We also know that $|\Gamma_1| \equiv 1 \mod p$. Take $d \in G \setminus N$ such that $G = \langle N, d\rangle$ (such $d$ always exists). Since $d^p \in N$, we know that if $d$ acts on $\Gamma_1$ nontrivially (via conjugation), then the orbits are of order $p$ or 1. Therefore, there is some $T \in \Gamma_1$ which is fixed by conjugation with $d$. Hence $T^d = T$, therefore $T \trianglelefteq G$. ∎

Now, we will use the previous result to describe any minimal $CZ$-group of order $p^4$ that contains a subgroup of order $p^3$ isomorphic to $N$.

PROPOSITION 3.20. *Let $G \in CZm_p$ be of order $p^4$ and $N = \langle a_1, b_1, c_1 \mid a_1^p = b_1^p = c_1^p = 1,\ a_1^{b_1} = a_1c_1,\ [a_1, c_1] = [b_1, c_1] = 1\rangle \trianglelefteq G$. Then there is a $T \leq N$ such that $T = \langle a, b\rangle \cong C_p \times C_p$ and $T \trianglelefteq G$. Additionally, $N = \langle T, b\rangle$ and $[a, b] = c \in Z(N)$.*

PROOF. We know that $Z(N) = \langle c_1\rangle$. Then also $C_p \cong Z(N) \geq Z(G) > 1$. Thus, $Z(N) = Z(G)$. Also, by previous result, we know that there is some $T = \langle a_2, c_2\rangle \cong C_p \times C_p$ that is normal in $G$. Take some $b_2 \in N \setminus T$. Since $N = \langle T, b_2\rangle$, we must have $a_2^{b_2} \neq a_2$. Otherwise, $N$ would be abelian. Since $T \trianglelefteq G$, then $a_2^{b_2} = a_2^i c_2^j$ for some $i, j$. Notice that if $c_2^j \neq 1$, then $\langle a_2, c_2\rangle = \langle a_2, c_2^j\rangle$. Therefore, we can write $a_2^{b_2} = a_2^i c_2$. Another option is $a_2^{b_2} = a_2^i$. Since $T \trianglelefteq G$ we have $T \cap Z(G) > 1$. If $a_2^{b_2} = a_2^i$, then we can write $\langle c_2\rangle = T \cap Z(N)$. Then $N \cong E_{p^3}$ (elementary abelian group) which is a contradiction. So, the only option is $a_2^{b_2} = a_2^i c_2$. Thus, $[a_2, b_2] = a_2^{i-1} c_2$, thus $\langle a_2^{i-1} c_2 \rangle = \langle c_2 \rangle = Z(N) = N'$. Therefore $i = 1$. Now, identify $a_2 = a,\ b_2 = b,\ c_2 = c$. □

THEOREM 3.21. *Let $G \in CZm_p$ be of order $p^4$. Then $G$ has no subgroup isomorphic to the minimal nonabelian group $N = \langle a, b, c \mid a^p = b^p = c^p = 1,\ a^b = ac,\ [a, c] = [b, c] = 1 \rangle$.*

PROOF. Let us assume the opposite. Let $N \trianglelefteq G$, where $N = \langle a, b, c \mid a^p = b^p = c^p = 1,\ a^b = ac,\ [a, c] = [b, c] = 1 \rangle$. By our previous result, without losing generality, we can write $T = \langle a, c \rangle \trianglelefteq G$. Since $|G/T| = p^2$, it is abelian, hence $T \le G'$. Thus $|G'| \ge p^2$. Since $G' \le \Phi(G)$ then also $|\Phi(G)| \ge p^2$. If $|\Phi(G)| = p^3$, then $dim(G) = 1$ and $G \cong C_{p^4}$, which is clearly a contradiction. Hence, the only option is $|\Phi(G)| = p^2$ and $\Phi(G) = G' = T$. Therefore, $G$ has 2 generators. Put $G = \langle x, y \rangle$. Then it is clear that $G' = \langle [x, y] \rangle \cong C_{p^2}$, which is a clear contradiction with $G' = T \cong C_p \times C_p$. □

Using Theorems 3.17 and 3.21 we have reached one of the main results of this paper. We establish now the lower bound for the order of minimal $CZ$-groups.

THEOREM 3.22. *Let $G \in CZm_p$. Then $|G| \ge p^5$.*

PROOF. If $G \in CZm_p \subseteq CZ$, then $G$ has some nonabelian subgroup $S < G$ such that $C_G(S) = Z(S)$. It is clear that $|S| > p^3$, thus $|G| \ge p^4$. If $|G| = p^4$, then there is some minimal nonabelian $S < G$. So $S \cong M_{p^3}$ or $N$. Both cases were eliminated by Theorems 3.17 and 3.21. □

## 4. MAXIMAL NORMAL ABELIAN SUBGROUP OF $G \in CZ_p$

This section collects another type of results, it deals with $CZ$-groups and repercussions on its maximal abelian subgroups. First, we provide a slightly different proof of Lemma 57.1. from Berkovich's and Janko's book *Groups of Prime Power Order*, Vol. 2 ([2]).

We will use notation $A \le_p B$ if $A$ is a subgroup of $B$ whose index is $p$. Similarly, we will write $A \trianglelefteq_p B$ if $A \trianglelefteq B$ and $[B : A] = p$.

LEMMA 4.1. *Let $G$ be a $p$-group and $A \trianglelefteq G$ its maximal abelian subgroup. Then for any $x \in G \backslash A$ there is some $a \in A$ such that $[x, a] \ne 1$ and $[x, a]^p = 1$. Furthermore, $[a, x, x] = 1$, thus $\langle x, a \rangle$ is minimal nonabelian, i.e. every $p$-group is generated by minimal nonabelian subgroups.*

PROOF. Take $C_A(x)$ for some $x \in G \setminus A$. Clearly $C_A(x) < A$, since otherwise $\langle x, A \rangle$ would be abelian and would contain $A$, which is a contradiction. Take $\langle x \rangle C_A(x)$. It is a group because of $\langle x \rangle C_A(x) = C_A(x) \langle x \rangle$. One can see that $\langle x \rangle C_A(x) < \langle x \rangle A$. Take $B \le \langle x \rangle A$ such that $\langle x \rangle C_A(x) \triangleleft_p B$. Notice that $\langle x \rangle C_A(x) \cap A \triangleleft_p A$ . Clearly $\langle x \rangle C_A(x) \cap A = C_A(x)$. Therefore $C_A(x) \triangleleft_p A \cap B$.

On the other hand, take $b = x^i a \in B$ (where $a \in A$.) Take $g \in C_A(x)$. Then (because $g \in A$) we have $b^g = (x^i a)^g = (x^g)^i a^g = x^i a = b$. Thus $C_A(x) \leq Z(B)$.

Now take $a \in (A \cap B) \setminus C_A(x)$ such that $a^p \in C_A(x)$. Clearly $[a, x] \neq 1$. Since $[x, a] \in B' \leq \Phi(B)$, $[x, a] \in C_A(x)\langle x \rangle \lhd_p B$.

Let us assume that $[x, a] \notin C_A(x)$. Then $[x, a] = x^i a_1$ for some $a_1 \in C_A(x)$. Then $x^{-1}a^{-1}xa = x^i a_1$ and $x^a = x^{i+1}a_1$. On the other hand,

$$x^{x^a} = x^{x^{i+1}a_1} = x^{a_1} = x,$$

thus $x^a \in C_A(x)$. Then $(x^a)^{a^{-1}} \in C_A(x)^{a^{-1}} = C_A(x)$, so $x \in C_A(x)$ – a contradiction! Therefore, $[x, a] \in C_A(x)$, so $B' \leq C_A(x)$.

Knowing that for any finite group it holds $[x, yz] = [x, z][x, y]^z$, we have $1 = [x, a^p] = [x, a^{p-1}a] = [x, a][x, a^{p-1}]^a$. Because of $[x, a^{p-1}] \in B' \leq C_A(x)$, we get $[x, a^p] = [x, a][x, a^{p-1}]$. This gives us $[x, a]^p = 1$. Finally, because of $[x, a] \in C_A(x)$, we get $[a, x, x] = 1$. ∎

Now we want to characterize the centralizer of a maximal abelian subgroup of a $CZ$-group because it certainly could be used for further describing the structure of a $CZ$-group.

Lemma 4.2. *Let $G \in CZ_p$ and let $A \trianglelefteq G$ be a maximal abelian subgroup. Let $x \in G \setminus A$. Then $C_A(x) \leq Z(N)$ for every nonabelian $N \leq \langle x \rangle A$.*

Proof. Take $x \in G \setminus A$. If $C_A(x) = A$, then $\langle x, A \rangle > A$ is abelian contradicting to the assumption for $A$ being maximal. Thus, $C_A(x) < A$. Take $N \leq \langle x \rangle A$, where $N$ is nonabelian. Then, $G \in CZ_p$ implies $C_G(N) = Z(N)$. Take $g \in C_A(x)$ and $y \in N$. Then $y = x^j a_1$, $a_1 \in A$. Hence, $(x^j a_1)^g = (x^g)^j a_1^g = x^j a_1$, because $[g, x] = 1$. Also, $g \in A$, so $[g, a_1] = 1$. Therefore, $g \in C_G(N) = Z(N)$, hence $C_A(x) \leq Z(N)$. ∎

Corollary 4.3. *Let $G \in CZ_p$ and let $A \trianglelefteq G$ be a maximal abelian subgroup. Let $x \in G \setminus A$. Then for every nonabelian $N \leq \langle x \rangle A$, it holds $Z(N) \setminus C_A(x) \subseteq \langle x \rangle C_A(x)$.*

Proof. We know that $C_A(x) \leq Z(N) < N \leq \langle x \rangle A$. Take $g = x^j a_1 \in Z(N) \setminus C_A(x)$. Then $[g, x] = 1$ and $x^g = x^{x^j a_1} = x^{a_1} = x$ which yields $a_1 \in C_A(x)$. Thus, $Z(N) \setminus C_A(x) \subseteq \langle x \rangle C_A(x)$. ∎

Corollary 4.4. *Let $G \in CZ_p$ and let $A \trianglelefteq G$ be maximal abelian subgroup. Let $x \in G \setminus A$. Then for every nonabelian $N \leq \langle x \rangle A$, it holds $C_A(x) \leq Z(N) \leq \langle x \rangle C_A(x)$.*

Proof. Since $C_A(x) \leq Z(N)$ and $Z(N) \setminus C_A(x) \subseteq \langle x \rangle C_A(x)$ we have

$$[Z(N) \setminus C_A(x)] \cup C_A(x) \subseteq \langle x \rangle C_A(x) \cup C_A(x).$$

Because $C_A(x) \subseteq \langle x \rangle C_A(x)$, we have $Z(N) \subseteq \langle x \rangle C_A(x)$. Hence $Z(N) \leq \langle x \rangle C_A(x)$. ∎

THEOREM 4.5. *Let $G \in CZ_p$. and let $A \trianglelefteq G$ be a maximal abelian subgroup. Let $x \in G \setminus A$ such that $x^p \in A$. Then $Z(\langle x \rangle A) = C_A(x) = C_G(\langle x \rangle A)$.*

PROOF. Take $x \in G \setminus A$ and $x^p \in A$. Then, by previous result, we have $C_A(x) \leq Z(\langle x \rangle A) \leq \langle x \rangle A C_A(x)$. Clearly $[\langle x \rangle C_A(x) : C_A(x)] = p$. If $Z(\langle x \rangle A) = \langle x \rangle C_A(x)$, then $x \in Z(\langle x \rangle A)$, hence $[x, A] = 1$. A contradiction with maximality of $A$. Therefore $Z(\langle x \rangle A) = C_A(x) = C_G(\langle x \rangle A)$. Last equality is true due to $G \in CZ_p$. ◻

COROLLARY 4.6. *Let $G \in CZ_p$ and let $A \trianglelefteq G$ be a maximal abelian subgroup. Then $C_A(T \setminus A) \leq Z(T) < A$ for any $T \leq G$ such that $A < T$.*

THEOREM 4.7. *Let $G \in CZ_p$ and let $A \trianglelefteq G$ be a maximal abelian subgroup. Then for any $x \in G \setminus A$ there is some $a \in A$ such that $C = \langle a, x \rangle$ is minimal nonabelian and $C_A(x) = Z(C) \cap A \leq C$.*

PROOF. We already know that for any $x \in G \setminus A$, there is some $a \in A$ such that $C = \langle a, x \rangle$ is minimal nonabelian. Take $t \in C_A(x) \setminus C$. Then $t \in A$ and $[t, a] = [t, x] = 1$. Hence $t \in C_G(C) = Z(C)$ (due to $G \in CZ$), which is an obvious contradiction. Therefore, $t \in C$ and $C_A(x) \leq C$. Now, take $g \in C_A(x)$. Then $[g, x] = [g, a] = 1$, so $g \in Z(C) \cap A$. So far we have $C_A(x) \leq Z(C) \cap A$. Now, take $s \in Z(C) \cap A$ but $s \notin C_A(x)$. Then $[s, x] = 1$, so $s \in C_A(x)$. Again a contradiction. This gives us $Z(C) \cap A \leq C_A(x)$. ◻

Finally, we present our second main result, the full description of the centralizer of a generator that lies outside of a maximal abelian subgroup.

THEOREM 4.8. *Let $G \in CZ_p$ and let $A \trianglelefteq G$ be a maximal abelian subgroup. If $x \in G \setminus A$ such that $x^p \in A$, then $Z(B) = C_A(x)$ where $B = \langle x, A \rangle$ . Furthermore, there is a minimal nonabelian group $M = \langle x, a \rangle$, where $a \in A$ such that $Z(M) = C_A(x) \leq A$ and $M \cap A \lhd_p M$.*

PROOF. Take $x \in G \setminus A$ such that $x^p \in A$. Put $B = \langle x, A \rangle$. Clearly $A \lhd_p B$. Take some $g \in C_A(x)$. Then $g \in Z(B)$ since $[g, x] = [g, a_1] = 1$ for any $a_1 \in A$. Hence $C_A(x) \leq Z(B)$.

Now, take $h \in Z(B) \setminus C_A(x)$. If $h \in A$, then $h \in C_A(x)$. A contradiction. If $h \notin A$, then $h \in B \setminus A$ and $[h, A] = 1$. Therefore $\langle A, h \rangle > A$ is abelian, contradiction with the choice of $A$. Thus, $Z(B) \leq C_A(x)$.

We know that there is some $a \in A$ such that $M = \langle x, a \rangle$ is minimal nonabelian. We already know that $Z(M) \cap A = C_A(x)$. Additionally, we have $B = MA$. Then $|B| = \dfrac{|M||A|}{|M \cap A|}$. This gives us $[B : A] = [M : M \cap A] = p$. Thus $M \cap A \lhd_p M$.

Since $M$ is minimal nonabelian, we have $Z(M) \lhd_p M \cap A \lhd_p M$. ◻

## References

[1] Y. Berkovich Groups of Prime Power Order Vol. 1., Walter de Gruyter GmbH & Co. KG, Berlin, 2008.
[2] Y. Berkovich and Z. Janko, Groups of Prime Power Order Vol. 2., Walter de Gruyter GmbH & Co. KG, Berlin, 2008.
[3] Y. Berkovich and Z. Janko, Groups of Prime Power Order Vol. 3., Walter de Gruyter GmbH & Co. KG, Berlin, 2011.

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