THE NAPOLEON-BARLOTTI THEOREM IN PENTAGONAL QUASIGROUPS

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Abstract. Pentagonal quasigroups are IM-quasigroups in which the additional identity $(ab \cdot a)b \cdot a = b$ holds. GS-quasigroups are IM-quasigroups in which the identity $a(ab \cdot c) \cdot c = b$ holds. The relation between these two subclasses of IM-quasigroups is studied. The geometric concepts of GS-trapezoid and affine regular pentagon, previously defined and studied in GS-quasigroups, are now defined in a general pentagonal quasigroup. Along with the concepts of the regular pentagon and the centre of the regular pentagon, previously defined in pentagonal quasigroups, this enables formulations and proofs of some theorems of the Euclidean plane in a general pentagonal quasigroup. Among these theorems is the famous Napoleon-Barlotti theorem in the case $n = 5$.

1. Introduction

A quasigroup $(Q, \cdot)$ is a groupoid in which for given $a, b \in Q$ each of the equations $a \cdot x = b$ and $y \cdot a = b$ has the unique solution. A quasigroup $(Q, \cdot)$ is called *IM-quasigroup* if it satisfies the identities of *idempotency* and *mediality*:

$$(1.1) \qquad\qquad aa = a,$$

$$(1.2) \qquad\qquad ab \cdot cd = ac \cdot bd.$$

The immediate consequences of these identities are the identities known as *elasticity*, *left distributivity* and *right distributivity*:

$$(1.3) \qquad\qquad ab \cdot a = a \cdot ba,$$

$$(1.4) \qquad\qquad a \cdot bc = ab \cdot ac,$$

(1.5) $ab \cdot c = ac \cdot bc.$

DEFINITION 1.1. Pentagonal quasigroup *is an IM-quasigroup* $(Q, \cdot)$ *in which the identity of* pentagonality *holds:*

(1.6) $(ab \cdot a)b \cdot a = b.$

The previous definition can be found in [8]. In pentagonal quasigroups, along with pentagonality and the identities that hold in any IM-quasigroup, some other identities hold. They are stated in the next theorem proved in [8].

THEOREM 1.2. *In every IM-quasigroup* $(Q, \cdot)$ *the identity (1.6) and the identities*

(1.7) $(ab \cdot a)c \cdot a = bc \cdot b,$

(1.8) $(ab \cdot a)a \cdot a = ba \cdot b,$

(1.9) $ab \cdot (ba \cdot a)a = b$

are mutually equivalent and they imply the identity

(1.10) $a(b \cdot (ba \cdot a)a) \cdot a = b$

for every $a, b, c \in Q$.

EXAMPLE 1.3. The basic example of the pentagonal quasigroup is $C(q) = (\mathbb{C}, *)$, where $*$ is defined by

$$a * b = (1 - q)a + qb$$

and q is a solution of the equation

(1.11) $q^4 - 3q^3 + 4q^2 - 2q + 1 = 0.$

This example motivates the introduction of many geometric concepts in pentagonal quasigroups. Actually, some concepts can be defined more generally in medial and IM-quasigroups and the whole idea of geometry of medial quasigroups and its subclasses comes from [10, 12].

We can regard elements of the set $\mathbb{C}$ as points of the Euclidean plane. For any two different points $a, b \in \mathbb{C}$ the equality $a * b = (1 - q)a + qb$ can be written in the form

$$\frac{a * b - a}{b - a} = \frac{q - 0}{1 - 0}.$$

That means that the points a, b and $a * b$ are vertices of a triangle directly similar to the triangle with vertices 0, 1 and q. Each q_i, $i = 1, 2, 3, 4$ gives a certain type of triangle and we get so called *characteristic triangles* for pentagonal quasigroups (see [8]). In $C(q_1)$ the point $a * b$ is the third vertex of the regular pentagon determined by its adjacent vertices a and b. Any identity in the pentagonal quasigroup $C(q) = (\mathbb{C}, *)$ can be interpreted as a theorem of the Euclidean geometry. Some illustrations can be found in [8].

In [9] basic geometry of pentagonal quasigroups was developed. In this paper the concepts of the regular pentagon, the regular decagon and the centre of the regular pentagon, all defined and studied in [9], will be of a particular interest.

DEFINITION 1.4. GS-quasigroup *is an idempotent quasigroup* $(Q, \cdot)$ *in which the identity*

$$(1.12) \qquad\qquad a(ab \cdot c) \cdot c = b$$

holds.

EXAMPLE 1.5. The basic example of the GS-quasigroup is $C(q) = (\mathbb{C}, *)$, where $*$ is defined by

$$a * b = (1 - q)a + qb$$

and q is a solution of the equation

$$(1.13) \qquad\qquad q^2 - q - 1 = 0.$$

GS-quasigroups were defined and studied in [11] where the next theorem was proved.

THEOREM 1.6. *In any GS-quasigroup* $(Q, \cdot)$ *the mediality holds.*

It follows immediately that GS-quasigroups are actually IM-quasigroups in which the identity (1.12) holds.

In already mentioned article some other identities that hold in GS-quasigroups were studied and some basic geometric concepts in GS-quasigroups were defined. Further geometric concepts were defined and studied in [4, 5]. Among these the most important for this paper will be the GS-trapezoid and the affine regular pentagon which we revisit in the third section. In the Euclidean geometry *GS-trapezoid* is a trapezoid whose lengths of parallel sides are in the golden section ratio and *affine regular pentagon* is the affine image of a regular pentagon.

Motivated by the example $C(q)$ of pentagonal quasigrops, where q is a solution of (1.11), we prove that we can "find" a GS-quasigroup "inside" any pentagonal quasigroup. More precisely, in a given pentagonal quasigroup $(Q, \cdot)$ we can define a new binary operation $*$ on the set Q by $a * b = (ba \cdot a)a \cdot b$, for $a, b \in Q$. It turns out that $(Q, *)$ is a GS-quasigroup. Furthermore, using that result in the third section we define the concepts of the GS-trapezoid and the affine regular pentagon in pentagonal quasigroups.

In the last section we bring two theorems of the Euclidean plane stated and proved in a pentagonal quasigroup. First is the famous Napoleon-Barlotti theorem (see [1]) in the case $n = 5$. The claim of the theorem is:

On the sides of an affine regular n-gon regular n-gons (all either of the same orientation as the base n-gon or all opposite to it) are constructed. Then the centres of these regular n-gons form a regular n-gon.

In order to state and prove it the concept of the regular pentagon and its centre and the concept of the affine regular pentagon will be needed.

The other theorem is a problem proposed in [3]. In the statement and the proof of this theorem the concept of the affine regular pentagon and the concept of the regular decagon, along with its centre, will be needed.

2. Relation between pentagonal and GS-quasigroups

Firstly we bring a very useful identity that holds in GS-quasigroups and give a characterization of GS-quasigroups involving that identity.

THEOREM 2.1. *Let $(Q, \cdot)$ be a GS-quasigroup. Then for every $a, b \in Q$ the identity*

$$(2.1) \qquad\qquad\qquad (a \cdot ab)b = a$$

holds.

PROOF. We have

$$(a \cdot ab)b \overset{(1.1)}{=} a(aa \cdot b) \cdot b \overset{(1.12)}{=} a,$$

which proves (2.1). $\square$

THEOREM 2.2. *Let $(Q, \cdot)$ be an IM-quasigroup in which (2.1) holds. Then the identity (1.12) holds in Q.*

PROOF. Since $(Q, \cdot)$ is an IM-quasigroup, besides idempotency (1.1) and mediality (1.2), the elasticity (1.3) holds. Let $a, b, c \in Q$. Then there exists $d \in Q$ such that $a = bd$. We have:

$$a(ab \cdot c) \cdot c = (bd \cdot (bd \cdot b)c)c \overset{(1.2)}{=} (b(bd \cdot b) \cdot dc)c \overset{(1.3)}{=} ((b \cdot bd)b \cdot dc)c$$

$$\overset{(1.2)}{=} (\ (b \cdot bd)d \cdot bc)c \overset{(2.1)}{=} (b \cdot bc)c \overset{(2.1)}{=} b.$$

That proves (1.12). $\square$

The previous two results combined give the next theorem.

THEOREM 2.3. *A quasigroup $(Q, \cdot)$ is a GS-quasigroup if and only if it is an IM-quasigroup in which the identity (2.1) holds.*

This theorem is very useful when showing that a certain quasigroup is actually a GS-quasigroup, since (2.1) has less number of variables and multiplications than (1.12).

In a quasigroup $(Q, \cdot)$ operations of left and right division are defined by

$$a \backslash c = b \Leftrightarrow ab = c \Leftrightarrow c/b = a.$$

Formula is an expression built up from variables using the operations $\cdot$, $\backslash$ and $/$. More precisely:

(1) elements of the set Q (variables) are formulae;

(2) if φ and ψ are formulae, then so are $\varphi \cdot \psi$, $\varphi \backslash \psi$ and φ / ψ.

A formula φ containing at most two variables gives rise to a new binary operation $Q \times Q \to Q$, which will also be denoted by φ.

In [6] the next corollary was proved. We will use it in the proof of the main theorem of the section.

COROLLARY 2.4. *If $(Q, \cdot)$ is a medial quasigroup, then binary operation defined by the formula φ is also medial.*

If we choose the model $C(q_1)$ for pentagonal quasigroups, the elements a, b and ab of a pentagonal quasigroup $(Q, \cdot)$ form an isosceles triangle (a, b, ab) with angle sizes $36°$, $108°$ and $36°$ respectively (so called characteristic triangle). Similarly, if a and b are elements of a GS-quasigroup $(Q, *)$, then the point $a * b$ lies on the line determined by the points a and b and divides the segment $\{a, b\}$ in the golden section ratio.
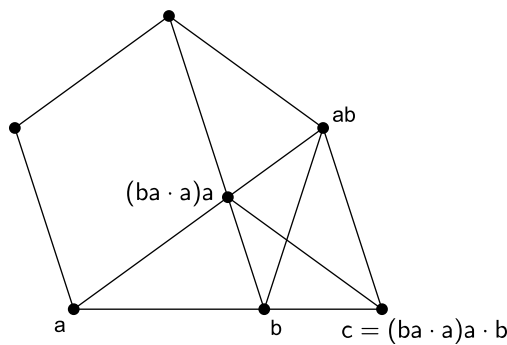

FIGURE 1. Definition of GS-quasigroup "inside" of pentagonal quasigroup

The next statement is justified by consideration of the geometrical interpretation of a binary operation in a pentagonal quasigroup.

THEOREM 2.5. *Let $(Q, \cdot)$ be a pentagonal quasigroup and let $*\colon Q \times Q \to Q$ be a binary operation defined by*

$$a * b = (ba \cdot a)a \cdot b.$$

Then $(Q, *)$ is a GS-quasigroup.

PROOF. By Theorem 2.3 it is enough to prove that $(Q, *)$ is a quasigroup which satisfies identities of idempotency, mediality and (2.1).

Let $a, b \in Q$ and we observe the equation $a * x = b$. We want to prove that there exists a unique $x \in Q$ which satisfies this equation. Rewriting the

expression $a * x$ we get:

$$a * x = (xa \cdot a)a \cdot x \stackrel{(1.9)}{=} (xa \cdot a)a \cdot (ax \cdot (xa \cdot a)a)$$

$$\stackrel{(1.2)}{=} ((xa \cdot a) \cdot ax)(a \cdot (xa \cdot a)a) \stackrel{(1.3),(1.8)}{=} ((xa \cdot a) \cdot ax)(xa \cdot x)$$

$$\stackrel{(1.2)}{=} ((xa \cdot a) \cdot xa)(ax \cdot x) \stackrel{(1.5)}{=} (xa \cdot x)a \cdot (ax \cdot x)$$

$$\stackrel{(1.4),(1.6)}{=} ((xa \cdot x)a \cdot ax)a \stackrel{(1.4),(1.6)}{=} ((xa \cdot x)a \cdot a)a \cdot a$$

$$\stackrel{(1.8)}{=} (((ax \cdot a)a \cdot a)a \cdot a)a \cdot a.$$

We get the equation

$$(((ax \cdot a)a \cdot a)a \cdot a)a \cdot a = b.$$

Since $(Q, \cdot)$ is a quasigroup, dividing by a seven times we get the equivalent equation

$$ax = c,$$

where $c \in Q$ and $c = \varphi(a, b)$ for some formula φ. Now we know that there exists a unique $x \in Q$ such that $ax = c$ holds so we conclude that there exists a unique $x \in Q$ which satisfies

$$a * x = (((ax \cdot a)a \cdot a)a \cdot a)a \cdot a = b.$$

Let us now check that $y = (ab \cdot a)a$ satisfies the equation $y * a = b$. We have:

$$y * a = (ay \cdot y)y \cdot a = (((a \cdot (ab \cdot a)a) \cdot (ab \cdot a)a) \cdot (ab \cdot a)a)a$$

$$\stackrel{(1.2)}{=} ((a(ab \cdot a) \cdot ((ab \cdot a)a \cdot a)) \cdot (ab \cdot a)a)a$$

$$\stackrel{(1.8)}{=} ((a(ab \cdot a) \cdot (ba \cdot b)) \cdot (ab \cdot a)a)a$$

$$\stackrel{(1.2)}{=} (((a \cdot ba) \cdot (ab \cdot a)b) \cdot (ab \cdot a)a)a$$

$$\stackrel{(1.3),(1.2)}{=} ((ab \cdot a)(ab \cdot a) \cdot ((ab \cdot a)b \cdot a))a$$

$$\stackrel{(1.1),(1.6)}{=} (ab \cdot a)b \cdot a \stackrel{(1.6)}{=} b.$$

Let us assume that there exist $y_1, y_2 \in Q$ such that $y_1 * a = y_2 * a = b$. Rewriting that we get

$$(ay_1 \cdot y_1)y_1 \cdot a = (ay_2 \cdot y_2)y_2 \cdot a.$$

Cancelling by a from the right that equation is equivalent to

$$(ay_1 \cdot y_1)y_1 = (ay_2 \cdot y_2)y_2.$$

Multiplying by $y_1 a$ from the left and applying (1.9) we get

$$a = y_1 a \cdot (ay_2 \cdot y_2)y_2.$$

Using (1.9) again it follows

$$y_2 a \cdot (ay_2 \cdot y_2)y_2 = y_1 a \cdot (ay_2 \cdot y_2)y_2.$$

Cancelling by $(ay_2 \cdot y_2)y_2$ from the right we get

$$y_2 a = y_1 a,$$

which canceling from the right by a finally gives $y_2 = y_1$. We conclude that there is a unique $y \in Q$ such that $y * a = b$ holds and

$$(2.2) \qquad\qquad y = (ab \cdot a)a.$$

Hence, $(Q, *)$ is a quasigroup.

We have $a * a = (aa \cdot a)a \cdot a \overset{(1.1)}{=} a$ so idempotency holds in $(Q, *)$. By Corollary 2.4 mediality in $(Q, *)$ follows immediately.

It remains to prove

$$(a * (a * b)) * b = a.$$

Since the solution of the equation $y * b = a$ is unique, by (2.2) it is enough to prove

$$a * (a * b) = (ba \cdot b)b.$$

Since $a * b = (ba \cdot a)a \cdot b$ holds, we get:

$$a * (a * b) = ((a * b)a \cdot a)a \cdot (a * b) = ((a * b)a \cdot a)a \cdot ((ba \cdot a)a \cdot b)$$

$$\overset{(1.2)}{=} (((a * b)a \cdot a) \cdot (ba \cdot a)a) \cdot ab$$

$$= ((((ba \cdot a)a \cdot b)a \cdot a) \cdot (ba \cdot a)a) \cdot ab$$

$$\overset{(1.5),(1.3),(1.8)}{=} (((ba \cdot a)a \cdot b)a \cdot (ba \cdot a)a)(ba \cdot b) \cdot ab$$

$$\overset{(1.5),(1.3),(1.8)}{=} ((((ba \cdot a)a \cdot b) \cdot (ba \cdot a)a)(ba \cdot b) \cdot (ba \cdot b)) \cdot ab$$

$$\overset{(1.2),(1.5)}{=} ((((ba \cdot b)a \cdot a) \cdot ba)(ba \cdot b) \cdot (ba \cdot b)) \cdot ab$$

$$\overset{(1.5)}{=} (((ba \cdot b)a \cdot b)a \cdot (ba \cdot b))(ba \cdot b) \cdot ab$$

$$\overset{(1.6),(1.1)}{=} (a(ba \cdot b) \cdot (ba \cdot b)) \cdot ab$$

$$\overset{(1.2)}{=} (a(ba \cdot b) \cdot a)(ba \cdot b)b \overset{(1.4)}{=} ((a \cdot ba) \cdot ab)a \cdot (ba \cdot b)b$$

$$\overset{(1.3)}{=} ((ab \cdot a) \cdot ab)a \cdot (ba \cdot b)b \overset{(1.4)}{=} ((ab \cdot a)a \cdot (ab \cdot a)b)a \cdot (ba \cdot b)b$$

$$\overset{(1.5),(1.8),(1.6)}{=} (ba \cdot b)b \cdot (ba \cdot b)b \overset{(1.1)}{=} (ba \cdot b)b.$$

Finally, $(Q, *)$ is a GS-quasigroup. $\qquad\qquad\qquad\qquad$ $\square$

In the previous proof we implicitly proved an identity which will be of a great use in the following sections. We bring it as a corollary.

COROLLARY 2.6. *Let* $(Q, \cdot)$ *be a pentagonal quasigroup and* $a, b \in Q$. *Let* $(Q, *)$ *be its associated GS-quasigroup, where* $a * b = (ba \cdot a)a \cdot b$. *Then*

$$a * (a * b) = (ba \cdot b)b$$

holds.

3. AFFINE REGULAR PENTAGONS IN PENTAGONAL QUASIGROUPS

In [4] the concept of GS-trapezoid was defined in a GS-quasigroup. That is, according to the example $C(\frac{1+\sqrt{5}}{2})$, a trapezoid whose lengths of parallel sides are in the golden section ratio. In [4] it was formulated in the following way.

DEFINITION 3.1. *Let* $(Q, \cdot)$ *be a GS-quasigroup and* $a, b, c, d \in Q$. *We say that quadrilateral* (a, b, c, d) *is* GS-trapezoid *and we denote it by* $GST(a, b, c, d)$ *if*

$$a \cdot ab = d \cdot dc$$

holds (Figure 2).

We notice that a GS-trapezoid in a GS-quasigroup is completely determined by its three vertices. It is obvious that $GST(a, b, c, d) \Leftrightarrow GST(d, c, b, a)$ holds.
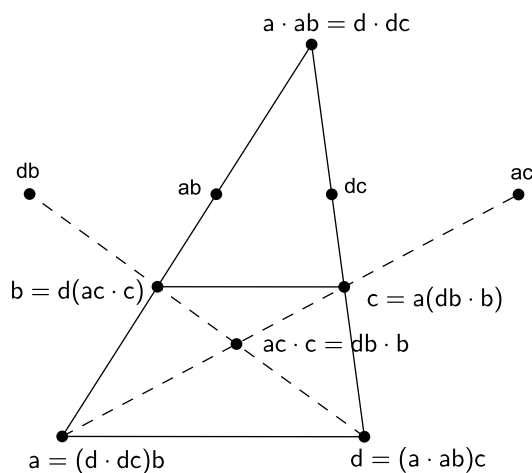

FIGURE 2. GS-trapezoid in GS-quasigroup ([4])

In the already mentioned article the next theorem was proved.

THEOREM 3.2. *Let* $(Q, \cdot)$ *be a GS-quasigroup and* $a, b, c, d \in Q$. *The relation* $GST(a, b, c, d)$ *is equivalent to any of the following four equalities*

$$a = (d \cdot dc)b, \ b = d(ac \cdot c), \ c = a(db \cdot b), \ d = (a \cdot ab)c.$$

In the previous section we proved that "inside" of any pentagonal quasigroup we can define a GS-quasigroup. Now we want to define GS-trapezoid in a pentagonal quasigroup.

THEOREM 3.3. *Let* $(Q, \cdot)$ *be a pentagonal quasigroup and* $a, b, c, d \in Q$. *Let* $(Q, *)$ *be its associated GS-quasigroup, where* $a * b = (ba \cdot a)a \cdot b$. *Then*

$$GST(a, b, c, d) \ in \ (Q, *) \Leftrightarrow d = (ca \cdot b)a \cdot c.$$

PROOF. The relation $GST(a, b, c, d)$ in $(Q, *)$ is by Theorem 3.2 equivalent to the equality $d = (a * (a * b)) * c$. By Corollary 2.6 $a * (a * b) = (ba \cdot b)b$ holds. That implies:

$$\begin{aligned}
d = (a * (a * b)) * c &= (ba \cdot b)b * c \\
&= (((c \cdot (ba \cdot b)b) \cdot (ba \cdot b)b) \cdot (ba \cdot b)b)c \\
&\overset{(1.2),(1.8)}{=} ((c(ba \cdot b) \cdot (ab \cdot a)) \cdot (ba \cdot b)b)c \\
&\overset{(1.2)}{=} (((c \cdot ab) \cdot (ba \cdot b)a) \cdot (ba \cdot b)b)c \\
&\overset{(1.2),(1.6)}{=} ((c \cdot ab)(b \cdot ab) \cdot a)c \\
&\overset{(1.5)}{=} (cb \cdot ab)a \cdot c \overset{(1.5)}{=} (ca \cdot b)a \cdot c.
\end{aligned}$$

$\square$

That justifies the next definition of a GS-trapezoid in a pentagonal quasigroup.

DEFINITION 3.4. *Let* $(Q, \cdot)$ *be a pentagonal quasigroup and* $a, b, c, d \in Q$. *We say that quadrilateral* (a, b, c, d) *is a* GS-trapezoid *and we denote it by* $GST(a, b, c, d)$ *if*

$$d = (ca \cdot b)a \cdot c$$

holds (Figure 3).

We immediately see that a GS-trapezoid in a pentagonal quasigroup is completely determined by its three vertices.

Affine regular pentagon in the Euclidean plane is the affine image of a regular pentagon. In [5] the notion of the affine regular pentagon in GS-quasigroups was defined. In the mentioned paper the next theorem was proved.

THEOREM 3.5. *Let* $(Q, \cdot)$ *be a GS-quasigroup and* $a, b, c, d, e \in Q$. *Any two out of five following relations imply the remaining ones:*

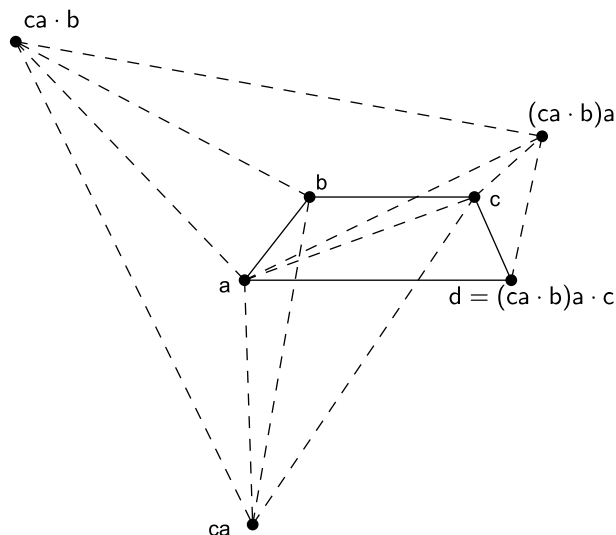$$GST(a, b, c, d), GST(b, c, d, e), GST(c, d, e, a), GST(d, e, a, b), GST(e, a, b, c).$$

FIGURE 3. Definition of GS-trapezoid in pentagonal quasigroup

That enables the next definition.

DEFINITION 3.6. *Let $(Q, \cdot)$ be a GS-quasigroup and $a, b, c, d, e \in Q$. We say that pentagon (a, b, c, d, e) is* affine regular pentagon *and we denote it by $ARP(a, b, c, d, e)$, if any two (and then all five) out of the following five relations*

$$GST(a, b, c, d), GST(b, c, d, e), GST(c, d, e, a), GST(d, e, a, b), GST(e, a, b, c)$$

hold (Figure 4).

It is immediately clear that an affine regular pentagon in GS-quasigroup is uniquely determined by any of its three vertices. We would like to define the affine regular pentagon in the pentagonal quasigroup associated to the GS-quasigroup where that pentagon is already defined.

THEOREM 3.7. *Let $(Q, \cdot)$ be a pentagonal quasigroup and $a, b, c, d, e \in Q$. Let $(Q, *)$ be its associated GS-quasigroup, where $a * b = (ba \cdot a)a \cdot b$. Then*

$$ARP(a, b, c, d, e) \text{ in } (Q, *) \Leftrightarrow d = (ca \cdot b)a \cdot c, \ e = (ac \cdot b)c \cdot a.$$

PROOF. In $(Q, *)$ we have the equivalence

$$ARP(a, b, c, d, e) \Leftrightarrow GST(a, b, c, d), \ GST(e, a, b, c).$$

By Theorem 3.3 $GST(a, b, c, d) \Leftrightarrow d = (ca \cdot b)a \cdot c$. Since $GST(e, a, b, c) \Leftrightarrow GST(c, b, a, e)$ (see [4]) and by Theorem 3.3 we have $GST(c, b, a, e) \Leftrightarrow e =$

$(ac \cdot b)c \cdot a$, we get $GST(e, a, b, c) \Leftrightarrow e = (ac \cdot b)c \cdot a$. Finally we conclude

$$ARP(a, b, c, d, e) \text{ in } (Q, *) \Leftrightarrow d = (ca \cdot b)a \cdot c, \ e = (ac \cdot b)c \cdot a.$$
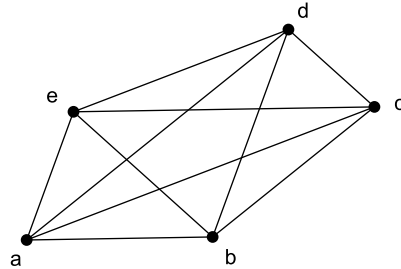
$\square$

FIGURE 4. Definition of affine regular pentagon in GS-quasigroup

That brings us to the next definition.

DEFINITION 3.8. *Let $(Q, \cdot)$ be a pentagonal quasigroup and $a, b, c, d, e \in Q$. We say that pentagon (a, b, c, d, e) is* affine regular pentagon *and we denote it by $ARP(a, b, c, d, e)$ if*

$$d = (ca \cdot b)a \cdot c, \ e = (ac \cdot b)c \cdot a$$

hold (Figure 5).

It is clear that an affine regular pentagon is uniquely determined by its three vertices.

The concept of the regular pentagon in pentagonal quasigroups was defined in [9].

DEFINITION 3.9. *We shall say that the points a, b, c, d, e in a pentagonal quasigroup $(Q, \cdot)$ are the vertices of* regular pentagon *and denote it by $RP(a, b, c, d, e)$ if $ab = c$, $bc = d$ and $cd = e$ hold.*

The next theorem says that any regular pentagon in pentagonal quasigroups is affine regular.

THEOREM 3.10. *Let $(Q, \cdot)$ be a pentagonal quasigroup and $a, b, c, d, e \in Q$. The next implication holds*
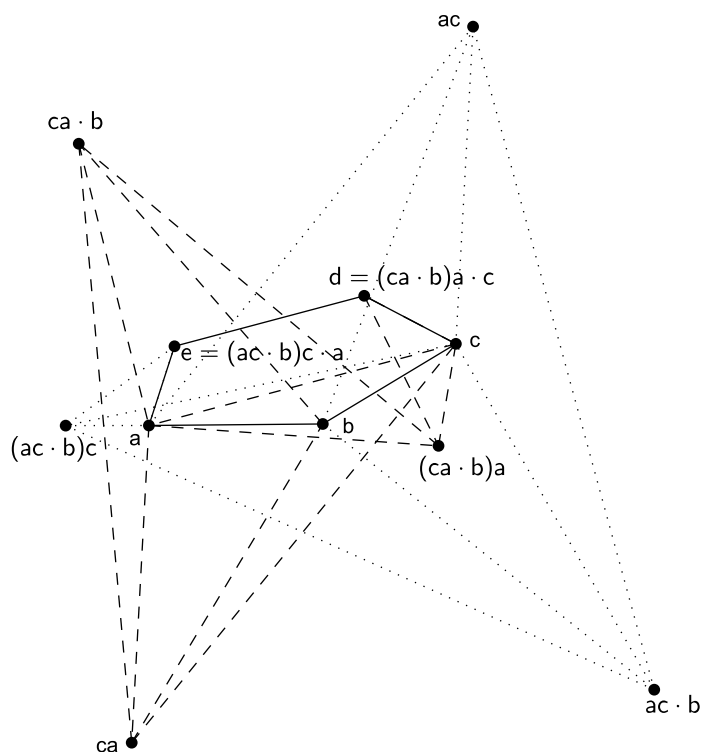
$$RP(a, b, c, d, e) \Rightarrow ARP(a, b, c, d, e).$$

FIGURE 5. Definition of affine regular pentagon in pentagonal quasigroup

PROOF. Because of $RP(a,b,c,d,e)$ we have $c = ab$ and $d = bc = b \cdot ab$ (see [9]). We get

$$(ca \cdot b)a \cdot c = ((ab \cdot a)b \cdot a) \cdot ab \stackrel{(1.6)}{=} b \cdot ab = d.$$

Similarly, because of $RP(a,b,c,d,e)$ we have $e = (ab \cdot a)b$ (see [9]). We get

$$(ac \cdot b)c \cdot a = ((a \cdot ab)b \cdot ab)a \stackrel{(1.5)}{=} ((a \cdot ab)a \cdot b)a$$

$$\stackrel{(1.7)}{=} (ab \cdot b) \cdot ab \stackrel{(1.5)}{=} (ab \cdot a)b = e,$$

which means that $ARP(a,b,c,d,e)$ holds. □

Along with the concepts of GS-trapezoid and affine regular pentagon, Theorem 2.5 enables us to define all other geometric concepts from GS-quasigroups in pentagonal quasigroups. We can say that pentagonal quasigroups "inherit" the entire geometry of GS-quasigroups (see [4,5]) and have their own geometry based on regular pentagons and regular decagons (see [9]).

4. The Napoleon-Barlotti theorem and similar results

In [7] the famous Napoleon-Barlotti theorem in the case $n = 3$ was stated and proved in Napoleon's quasigroups. The Napoleon-Barlotti theorem in the case $n = 5$ says:

On the sides of an affine regular pentagon regular pentagons are constructed (all on the outer or all on the inner side of that pentagon). Then the centres of these regular pentagons form a regular pentagon.

It is enough to treat the case where all regular pentagons are constructed on the outer side of the affine regular pentagon. In the Figure 6 all regular pentagons are constructed on the outer side of the original affine regular pentagon and that figure is made in the model $C(q_1)$ for pentagonal quasigroups. The case where regular pentagons are constructed on the inner side would correspond to the model $C(q_2)$, so once the theorem is proved in a general pentagonal quasigroup, the both cases are covered.

A minor problem occurs in the statement of the Napoleon-Barlotti theorem in pentagonal quasigroups since the centre of a regular pentagon does not have to exists and if it exists, it does not have to be unique. That problem produces a slightly weaker version of the theorem. The concept of the centre of a regular pentagon was defined in [9].

DEFINITION 4.1. *Let $(Q, \cdot)$ be a pentagonal quasigroup and $a, b, c, d, e \in Q$ such that $RP(a, b, c, d, e)$ holds. We say that $o \in Q$ is the* centre *of the regular pentagon (a, b, c, d, e) if $oa \cdot b = o$ holds.*

EXAMPLE 4.2. The smallest non-trivial example of finite pentagonal quasigroup is $(Q_5, \cdot_1)$.

TABLE 1. Pentagonal quasigroup $(Q_5, \cdot_1)$

$\cdot_1$	0	1	2	3	4
0	0	2	4	1	3
1	4	1	3	0	2
2	3	0	2	4	1
3	2	4	1	3	0
4	1	3	0	2	4

In $(Q_5, \cdot_1)$ the statement $RP(0, 1, 2, 3, 4)$ holds. The pentagon $(0, 1, 2, 3, 4)$ is determined by its vertices 0 and 1 and it has no centre, as shown in [9].
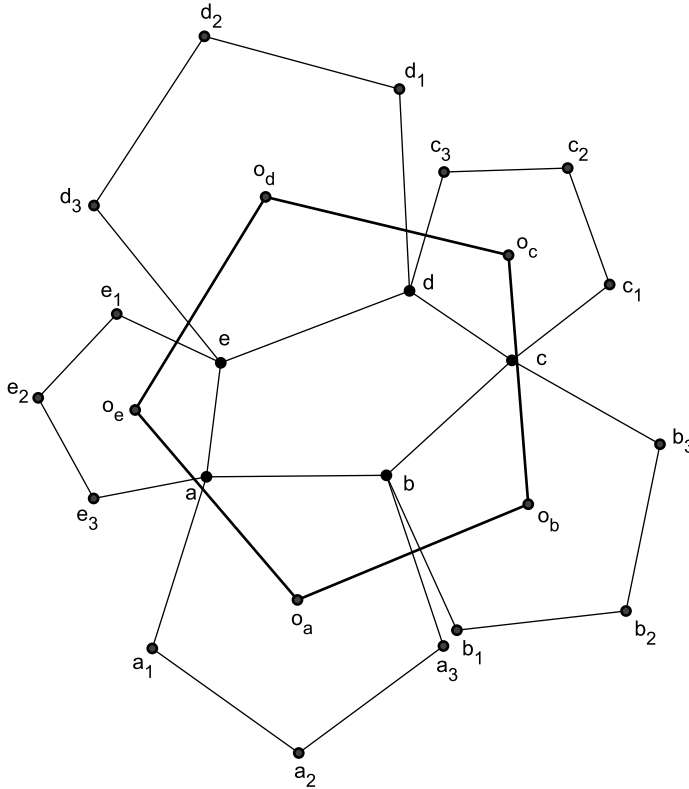
FIGURE 6. The Napoleon-Barlotti theorem

However, the statement $RP(0, 0, 0, 0, 0)$ also holds. The pentagon $(0, 0, 0, 0, 0)$ is determined by its vertices 0 and 0. It is easily checked that every element of Q_5 is its centre:

$$00 \cdot_1 0 = 00 = 0, \ 10 \cdot_1 0 = 40 = 1, \ 20 \cdot_1 0 = 30 = 2,$$

$$30 \cdot_1 0 = 20 = 3, \ 40 \cdot_1 0 = 10 = 4.$$

EXAMPLE 4.3. The next example of pentagonal quasigroup is $(Q_{11}, \cdot_2)$.

In $(Q_{11}, \cdot_2)$ the statement $RP(0, 1, 4, 2, 7)$ holds. The pentagon $(0, 1, 4, 2, 7)$ is determined by its vertices 0 and 1. We have:

$$00 \cdot_2 1 = 4 \neq 0, \ 10 \cdot_2 1 = 2 \neq 1, \ 20 \cdot_2 1 = 0 \neq 2, \ 30 \cdot_2 1 = 9 \neq 3,$$

$$40 \cdot_2 1 = 7 \neq 4, \ 50 \cdot_2 1 = 5, \ 60 \cdot_2 1 = 3 \neq 6, \ 70 \cdot_2 1 = 1 \neq 7,$$

$$80 \cdot_2 1 = 10 \neq 8, \ 90 \cdot_2 1 = 8 \neq 9, \ 100 \cdot_2 1 = 6 \neq 10.$$

That means that 5 is the unique centre of the pentagon $(0, 1, 4, 2, 7)$.

TABLE 2. Pentagonal quasigroup $(Q_{11}, \cdot_2)$

$\cdot_2$	0	1	2	3	4	5	6	7	8	9	10
0	0	4	8	1	5	9	2	6	10	3	7
1	8	1	5	9	2	6	10	3	7	0	4
2	5	9	2	6	10	3	7	0	4	8	1
3	2	6	10	3	7	0	4	8	1	5	9
4	10	3	7	0	4	8	1	5	9	2	6
5	7	0	4	8	1	5	9	2	6	10	3
6	4	8	1	5	9	2	6	10	3	7	0
7	1	5	9	2	6	10	3	7	0	4	8
8	9	2	6	10	3	7	0	4	8	1	5
9	6	10	3	7	0	4	8	1	5	9	2
10	3	7	0	4	8	1	5	9	2	6	10

EXAMPLE 4.4. The direct product of the pentagonal quasigroups $(Q_5, \cdot_1)$ and $(Q_{11}, \cdot_2)$ is the pentagonal quasigroup $(Q_{55}, \cdot)$. For $(a_1, a_2), (b_1, b_2) \in Q_{55}$ we have

$$(a_1, a_2) \cdot (b_1, b_2) = (a_1 \cdot_1 b_1, a_2 \cdot_2 b_2).$$

From the previous two examples we see that $RP((0,0), (0,1), (0,4), (0,2),$ $(0,7))$ holds. The pentagon $((0,0), (0,1), (0,4), (0,2), (0,7))$ is determined by its vertices $(0,0)$ and $(0,1)$. From

$$(0,5)(0,0) \cdot (0,1) = (0,5),$$

$$(1,5)(0,0) \cdot (0,1) = (1,5),$$

$$(2,5)(0,0) \cdot (0,1) = (2,5),$$

$$(3,5)(0,0) \cdot (0,1) = (3,5),$$

$$(4,5)(0,0) \cdot (0,1) = (4,5),$$

we conclude that $(0,5), (1,5), (2,5), (3,5), (4,5)$ are 5 distinct centres of the pentagon $((0,0), (0,1), (0,4), (0,2), (0,7))$.

Here we bring the version of the Napoleon-Barlotti theorem in pentagonal quasigroups.

THEOREM 4.5. Let $(Q, \cdot)$ be a pentagonal quasigroup. Let $ARP(a, b, c, d, e)$, $RP(b, a, a_1, a_2, a_3)$, $RP(c, b, b_1, b_2, b_3)$, $RP(d, c, c_1, c_2, c_3)$, $RP(e, d, d_1, d_2, d_3)$, $RP(a, e, e_1, e_2, e_3)$ hold and let o_a be the centre of (b, a, a_1, a_2, a_3) and o_b be the centre of (c, b, b_1, b_2, b_3). If $RP(o_a, o_b, o_c, o_d, o_e)$ holds, then o_c, o_d and o_e are the centres of the regular pentagons (d, c, c_1, c_2, c_3), (e, d, d_1, d_2, d_3) and (a, e, e_1, e_2, e_3), respectively.

PROOF. Since o_a and o_b are centres of the regular pentagons (b, a, a_1, a_2, a_3) and (c, b, b_1, b_2, b_3), respectively, we have

(4.1) $$o_a b \cdot a = o_a,$$

(4.2) $$o_b c \cdot b = o_b.$$

Multiplying the equalities (4.1) and (4.2) and using (1.2) we get:

$$(o_a b \cdot a)(o_b c \cdot b) = o_a o_b,$$
$$(o_a b \cdot o_b c) \cdot ab = o_a o_b,$$
$$(o_a o_b \cdot bc) \cdot ab = o_a o_b.$$

Because of $RP(o_a, o_b, o_c, o_d, o_e)$, we have $o_c = o_a o_b$, and we want to prove that o_c is the centre of the regular pentagon (d, c, c_1, c_2, c_3). Hence, we have

(4.3) $$o_c = (o_c \cdot bc) \cdot ab,$$

and it is enough to prove

$$o_c d \cdot c = o_c.$$

Since $ARP(a, b, c, d, e)$ holds, by Theorem 3.7 or by Definition 3.8, we have $d = (ca \cdot b)a \cdot c$. Hence, we get successively:

$$o_c d \cdot c = o_c((ca \cdot b)a \cdot c) \cdot c \overset{(4.3)}{=} ((o_c \cdot bc) \cdot ab)((ca \cdot b)a \cdot c) \cdot c$$

$$\overset{(1.2)}{=} ((o_c \cdot bc) \cdot (ca \cdot b)a)(ab \cdot c) \cdot c \overset{(1.2)}{=} (o_c(ca \cdot b) \cdot (bc \cdot a))(ab \cdot c) \cdot c$$

$$\overset{(1.2)}{=} ((o_c(ca \cdot b) \cdot ab) \cdot (bc \cdot a)c)c \overset{(1.2)}{=} ((o_c a \cdot (ca \cdot b)b) \cdot (bc \cdot a)c)c$$

$$\overset{(1.2)}{=} (o_c a \cdot (bc \cdot a))((ca \cdot b)b \cdot c) \cdot c \overset{(1.5)}{=} ((o_c \cdot bc)a \cdot ((ca \cdot b)b \cdot c))c$$

$$\overset{(1.2)}{=} (((o_c \cdot bc) \cdot (ca \cdot b)b) \cdot ac)c \overset{(1.2)}{=} ((o_c(ca \cdot b) \cdot (bc \cdot b)) \cdot ac)c$$

$$\overset{(1.3)}{=} ((o_c(ca \cdot b) \cdot (b \cdot cb)) \cdot ac)c \overset{(1.2),(1.5)}{=} ((o_c b \cdot (ca \cdot c)b) \cdot ac)c$$

$$\overset{(1.2)}{=} (o_c b \cdot a)((ca \cdot c)b \cdot c) \cdot c \overset{(1.7)}{=} (o_c b \cdot a)(ab \cdot a) \cdot c$$

$$\overset{(1.5)}{=} (o_c b \cdot ab)a \cdot c \overset{(1.5)}{=} (o_c a \cdot b)a \cdot c$$

$$\overset{(1.6)}{=} (o_c a \cdot b)a \cdot ((bc \cdot b)c \cdot b) \overset{(1.2)}{=} ((o_c a \cdot b) \cdot (bc \cdot b)c) \cdot ab$$

$$\overset{(1.2)}{=} ((o_c a \cdot (bc \cdot b)) \cdot bc) \cdot ab \overset{(1.2)}{=} (((o_c \cdot bc) \cdot ab) \cdot bc) \cdot ab$$

$$\overset{(4.3)}{=} (o_c \cdot bc) \cdot ab \overset{(4.3)}{=} o_c,$$

exactly what we wanted to show. Hence, o_c is the centre of the regular pentagon (d, c, c_1, c_2, c_3).

It can be shown analogously that o_d is the centre of the regular pentagon (e, d, d_1, d_2, d_3) and that o_e is the centre of the regular pentagon (a, e, e_1, e_2, e_3).   □

At the end we bring another interesting theorem of the Euclidean plane in pentagonal quasigroups. In [3] the next theorem was stated as a problem:

Let $ABCDE$ be an affine regular pentagon and D' and E' vertices of the regular pentagons on the sides $\overline{BA}$ and $\overline{CB}$ opposite to that sides, respectively. Let B'' be the vertex of the regular pentagon on the side $\overline{D'E'}$ opposite to that side. Then $B'' = CD \cap EA$.

In order to state and prove that theorem in pentagonal quasigroups we can firstly notice that the vertex of the regular pentagon on the side $\overline{BA}$ opposite to that side is actually the fourth vertex of that regular pentagon and that is exactly the centre of the regular decagon on $\overline{BA}$. We also have to determine what is the intersection of the mentioned lines CD and EA. Since $AEDC$ is a GS-trapezoid, that intersection can be reached using only golden section ratio on the line EA or on the line CD (see Figure 2 and [4]).
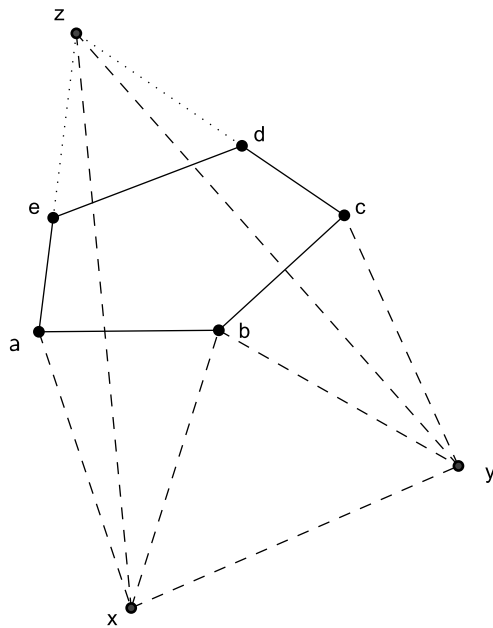

FIGURE 7. The geometrical presentation of Theorem 4.6

THEOREM 4.6. *Let $(Q, \cdot)$ be a pentagonal quasigroup and let $(Q, *)$ be its associated GS-quasigroup. Let $ARP(a, b, c, d, e)$ hold and let x and y be the fourth vertices of the regular pentagons determined by segments $\{b, a\}$ and $\{c, b\}$, respectively. If $z = a * (a * e)$, then z is the fourth vertex of the regular pentagon determined by segment $\{x, y\}$.*

PROOF. Since $GST(a, e, d, c)$ holds, $z = a * (a * e) = c * (c * d)$. By Corollary 2.6 we have $z = a * (a * e) = (ea \cdot e)e$.

Since x is the vertex of the regular pentagon determined by segment $\{b, a\}$, we have $x = ab \cdot a$ (see [9]). Analogously we have $y = bc \cdot b$.

In order to prove the statement of the theorem it is enough to prove $z = yx \cdot y$. Since $ARP(a, b, c, d, e)$ holds, by Theorem 3.7 or by Definition 3.8 we have $e = (ac \cdot b)c \cdot a$. We get successively:

$$z = (ea \cdot e)e = (((ac \cdot b)c \cdot a)a \cdot ((ac \cdot b)c \cdot a))e$$

$$\overset{(1.5)}{=} (((ac \cdot b)c \cdot a) \cdot (ac \cdot b)c)a \cdot e \overset{(1.2)}{=} (((ac \cdot b)c \cdot (ac \cdot b)) \cdot ac)a \cdot e$$

$$\overset{(1.2)}{=} ((((ac \cdot b) \cdot ac) \cdot cb) \cdot ac)a \cdot e \overset{(1.7)}{=} ((b \cdot cb)b \cdot a) \cdot ((ac \cdot b)c \cdot a)$$

$$\overset{(1.3),(1.5)}{=} ((bc \cdot b)b \cdot (ac \cdot b)c)a \overset{(1.2)}{=} ((bc \cdot b)(ac \cdot b) \cdot bc)a$$

$$\overset{(1.5),(1.5)}{=} ((ba \cdot c)b \cdot bc)a \overset{(1.5)}{=} (((ba \cdot b) \cdot cb) \cdot bc)a$$

$$\overset{(1.2)}{=} ((ba \cdot b)b \cdot (cb \cdot c))a \overset{(1.8),(1.6)}{=} (((ab \cdot a)a \cdot a)((ab \cdot a)b \cdot a) \cdot (cb \cdot c))a$$

$$\overset{(1.5)}{=} (((ab \cdot a)a \cdot (ab \cdot a)b)a \cdot (cb \cdot c))a \overset{(1.4)}{=} (((ab \cdot a) \cdot ab)a \cdot (cb \cdot c))a$$

$$\overset{(1.5)}{=} (((ab \cdot a)a \cdot (ab \cdot a)) \cdot (cb \cdot c))a \overset{(1.2)}{=} (((ab \cdot a)a \cdot cb) \cdot (ab \cdot a)c)a$$

$$\overset{(1.2)}{=} (((ab \cdot a)c \cdot ab) \cdot (ab \cdot a)c)a \overset{(1.5)}{=} ((ab \cdot a)c \cdot ab)a \cdot ((ab \cdot a)c \cdot a)$$

$$\overset{(1.5)}{=} ((ab \cdot a)c \cdot a)(ab \cdot a) \cdot ((ab \cdot a)c \cdot a) \overset{(1.7),(1.7)}{=} (bc \cdot b)(ab \cdot a) \cdot (bc \cdot b)$$

$$= yx \cdot y,$$

which is exactly what we needed to prove.   □

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