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PROPERTIES OF THE DISTRIBUTIONAL FINITE FOURIER TRANSFORM

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This paper is dedicated to the memory of Professor Dr. Dragiša Mitrović of the University of Zagreb, Croatia.

Abstract. The analytic functions in tubes which obtain the distributional finite Fourier transform as boundary value are shown to have a strong boundedness property and to be recoverable as a Fourier-Laplace transform, a distributional finite Fourier transform, and as a Cauchy integral of a distribution associated with the boundary value.

1. Introduction

Results concerning distributional and ultradistributional boundary values of analytic functions in tubes in n-dimensional complex space $\mathbb{C}^n$ are of importance in quantum field theory in which the boundary values are vacuum expectation values in a field theory; see, for example, [7] and [8]. The main focus of our paper [1] was to define a distributional finite Fourier transform and to study analytic functions in tubes which have this transform of distributions as boundary value. As this distributional finite Fourier transform has not been extensively studied, we desire now to produce further analysis of the analytic functions which have this type of boundary value and obtain more general and new results which may have importance in applications. Our results here generalize and extend results in [1] concerning the analytic functions; and we obtain new results here which is a motivation for this paper. Theorem 1 of Section 4 in this paper generalizes the analytic functions of [1, Theorem 10] in that the growth of the analytic functions here holds for $Im(z)$

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arbitrarily bounded away from the origin $\overline{0} = (0, 0, \dots, 0)$ in $\mathbb{R}^n$, but the proof can be obtained by the techniques of [1, Theorem 10]. Theorem 2 extends [1, Theorem 13] for our more general growth here on the analytic functions and yields new results. In addition to representing the analytic functions as a Fourier-Laplace type integral and showing that the analytic functions obtain the distributional finite Fourier transform as distributional boundary value, we obtain a representation of the analytic functions in terms of the finite Fourier transform involving a distribution which yields the boundary value. We further prove a boundedness property of the analytic functions in a distribution space. The lemmas proved in Section 3 below are needed to obtain our results in Section 4 which are the main results of the paper.

In many distributional boundary value results the boundary value is obtained as a transform of a distribution or ultradistribution whose support is in the dual cone C^* of a cone C. This dual cone is used to define the Cauchy and Poisson kernels corresponding to tubes defined by the cone C, and analytic functions having distributional or ultradistributional boundary values can be shown to be recoverable as Cauchy, and sometimes Poisson, integrals of the boundary value. Results of this type are in [3, Chapter 5]. In this paper the support of the constructed distributions in the theorems of representation and boundary values of analytic functions is a set similar to the dual cone of a cone but which has the dual cone as a special case. Corresponding to this new set we will define Cauchy and Poisson type kernels in Section 5 and state some of their important properties. We will construct Cauchy and Poisson integrals of ultradistributions corresponding to these new kernels and note some of their properties. We then prove that analytic functions which we consider in Section 4 of this paper can be represented as a Cauchy integral involving this new Cauchy kernel.

The representation of distributions and ultradistributions as boundary values of analytic functions has been studied for many years and has origins in 1952 and 1953 in work of G. Köthe. The recent book [3] contains an extensive bibliography of papers on this topic including recovery of the analytic functions by Fourier-Laplace, Cauchy, and Poisson integrals of functionals associated with the boundary value; the noted work of Köthe is included in this bibliography. The results obtained in this paper are unique in the correlation between analytic functions of n complex variables in tubes and the distributional finite Fourier transform and in the definition of new Cauchy and Poisson kernels and integrals of ultradistributions with the Cauchy kernel being used to obtain a Cauchy integral representation of the analytic functions studied here.

2. Notation and definitions

Notation in this paper will be the same as in [1] which we restate here. $t = (t_1, \ldots, t_n) \in \mathbb{R}^n$ and $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$. For $A = (a_1, \ldots, a_n) \in \mathbb{R}^n$, $\overline{t/A}$ will denote $(t_1/a_1, \ldots, t_n/a_n)$. Also $t^A = t_1^{a_1} \cdots t_n^{a_n}$ and $z^A = z_1^{a_1} \cdots z_n^{a_n}$. For α being an n-tuple of nonnegative integers, $D^\alpha = D_1^{\alpha_1} \cdots D_n^{\alpha_n}$ where $D_j = (1/2\pi i)(\partial/\partial t_j), j = 1, \ldots, n$. We define $< t, y >= t_1 y_1 + \cdots + t_n y_n, t \in \mathbb{R}^n, y \in \mathbb{R}^n$, with a similar definition for $< t, z >, t \in \mathbb{R}^n, z \in \mathbb{C}^n$. By $|z|, z \in \mathbb{C}^n$, we mean $|z| = (|z_1|^2 + \cdots + |z_n|^2)^{1/2}$. Throughout the paper $\overline{0} = (0, \ldots, 0)$ will denote the origin in $\mathbb{R}^n$, and $N(\overline{0}, m)$ will denote a closed ball of the origin $\overline{0}$ of radius $m > 0$.

A set $C \subseteq \mathbb{R}^n$ is a cone (with vertex at zero) if $y \in C$ implies $\lambda y \in C$ for all $\lambda > 0$. The intersection of a cone with the unit sphere $\{y \in \mathbb{R}^n : |y| = 1\}$ is called the projection of the cone C and is denoted $pr(C)$. If C' and C are cones such that $pr(\overline{C'}) \subset pr(C)$, C' will be called a compact subcone of C. An open convex cone C such that $\overline{C}$ does not contain any entire straight line will be called a regular cone. For a cone C, $T^C = \mathbb{R}^n + iC \subseteq \mathbb{C}^n$ is a tube in $\mathbb{C}^n$. The set $C^* = \{t \in \mathbb{R}^n :< t, y > \geq 0 \; for \; all \; y \in C\}$ is the dual cone of the cone C. The function $u_C(t) = \sup_{y \in pr(C)}(- < t, y >)$ is the indicatrix of the cone C, and we have $C^* = \{t \in \mathbb{R}^n : u_C(t) \leq 0\}$. For further information concerning cones and examples we refer to [3, 4] and [9].

We introduce the test function spaces, spaces of distributions, and transforms to be considered in this paper. We assume familiarity on the part of the reader with the test space $\mathcal{D}$ and space of distributions $\mathcal{D}'$ in n-dimensions given by Schwartz ([6]). Let $A = (a_1, \ldots, a_n)$ be a fixed n-tuple of real numbers such that $a_j > 0$, $j = 1, \ldots, n$; $\mathcal{K}(A)$ is the space of all infinitely differentiable complex valued functions which have support in $S_A = \{t \in \mathbb{R}^n : |t_j| \leq a_j, j = 1, \ldots, n\}$. A topology may be defined on $\mathcal{K}(A)$ by the countable set of norms

$$\|\phi\|_m = \sup_{\substack{x \\ |\alpha| \leq m}} |D^\alpha \phi(x)|, \; m = 0, 1, 2, \ldots,$$

under which $\mathcal{K}(A)$ becomes a locally convex topological vector space. With this topology we say that a sequence $\{\phi_v\}, \phi_v \in \mathcal{K}(A)$, converges to zero in $\mathcal{K}(A)$ as $v \to v_o$ if $\{D^\alpha \phi_v\}$ converges to zero uniformly on S_A as $v \to v_o$ for any n-tuple α of nonnegative integers. $\mathcal{K}(A)$ is a Montel space ([5, p. 510]).

For $R = (r_1, \ldots, r_n)$ being an n-tuple of positive real numbers, $\mathcal{Z}(R)$ is the space of all infinitely differentiable complex valued functions $\psi(x), x \in \mathbb{R}^n$, which can be extended to be entire analytic functions $\psi(z), z \in \mathbb{C}^n$, such that for every n-tuple α of nonnegative integers there exists a constant $M(\alpha)$ depending on α for which

$$|z^\alpha \psi(z)| \leq M(\alpha) \exp(r_1|y_1| + \cdots + r_n|y_n|), \; y = Im(z), \; z \in \mathbb{C}^n.$$

We define a topology on $\mathcal{Z}(R)$ by using the countable set of norms

$$\|\psi\|_m^* \;=\; \sup_{\substack{z \\ |\alpha|\le m}} |z^\alpha \psi(z)| \exp(-r_1|y_1| - \cdots - r_n|y_n|),\; y = Im(z),$$

$m = 0, 1, 2, \ldots$, under which $\mathcal{Z}(R)$ becomes a locally convex topological vector space. With this topology a sequence $\{\psi_v\}, \psi_v \in \mathcal{Z}(R)$, converges to zero in $\mathcal{Z}(R)$ as $v \to v_o$ if for any n-tuple α of nonnegative integers there is a constant $M(\alpha)$ depending only on α such that

$$|z^\alpha \psi_v(z)| \le M(\alpha) \exp(r_1|y_1| + \cdots + r_n|y_n|)$$

for all $z \in \mathbb{C}^n$ and all v and if the sequence $\{\psi_v(x)\}$ converges uniformly to zero as $v \to v_o$ on every bounded set in $\mathbb{R}^n$.

$\mathcal{K}'(A)$ and $\mathcal{Z}'(R)$ will denote the spaces of continuous linear functionals on $\mathcal{K}(A)$ and $\mathcal{Z}(R)$, respectively.

We now define the transforms. The finite Fourier transform of an element $\phi \in \mathcal{K}(A)$, denoted $\mathcal{F}_A[\phi(t); x]$, is

$$\mathcal{F}_A[\phi(t); x] \;=\; \prod_{j=1}^{n} (a_j)^{-1} \int_{S_A} \phi(t) e^{2\pi i <\overline{t/A}, x>} dt.$$

We have proved in [1, section 3] that this finite Fourier transform is a linear, one-one, and onto mapping from $\mathcal{K}(A)$ to $\mathcal{Z}(\overline{2\pi})$, where $\overline{2\pi} = (2\pi, \ldots, 2\pi)$, and in fact is a topological vector space isomorphism of $\mathcal{K}(A)$ onto $\mathcal{Z}(\overline{2\pi})$. Given $\psi \in \mathcal{Z}(\overline{2\pi})$ we proved in [1, section 3] that the element ϕ defined from the given ψ by

$$\phi(t) = \int_{\mathbb{R}^n} \psi(x) e^{-2\pi i <\overline{t/A}, x>} dx$$

is that element in $\mathcal{K}(A)$ such that $\psi(x) = \mathcal{F}_A[\phi(t); x]$.

Now for $U \in \mathcal{K}'(A)$ we define the distributional finite Fourier transform of U, denoted $\mathcal{F}_A[U]$, to be that element $V \in \mathcal{Z}'(\overline{2\pi})$ such that

$$\langle V, \psi\rangle \;=\; \prod_{j=1}^{n} (a_j)^{-1} \langle U, \phi\rangle,\; \phi \in \mathcal{K}(A),\; \psi(x) = \mathcal{F}_A[\phi(t); x] \in \mathcal{Z}(\overline{2\pi}),$$

for all $\psi \in \mathcal{Z}(\overline{2\pi})$. We recall from [1, section 3] that this distributional finite Fourier transform is a continuous, linear, one-one mapping of $\mathcal{K}'(A)$ onto $\mathcal{Z}'(\overline{2\pi})$.

We have noted above that $\mathcal{K}(A)$ is a Montel space ([5, p. 510]). Further, any weakly bounded subset of $\mathcal{K}'(A)$ is strongly bounded, and a weakly convergent sequence in $\mathcal{K}'(A)$ is strongly convergent ([5, p. 510]). Additionally, the distributional finite Fourier transform is a strongly continuous mapping of $\mathcal{K}'(A)$ onto $\mathcal{Z}'(\overline{2\pi})$.

We recall the Fourier transform of $U \in \mathcal{D}'$, denoted $\mathcal{F}[U]$, which is that element $V = \mathcal{F}[U] \in \mathcal{Z}'$ for which

$$\langle V, \psi \rangle \ = \ \langle U, \phi \rangle, \ \phi \in \mathcal{D}, \ \psi(x) = \mathcal{F}[\phi(t); x] \ \in \mathcal{Z}.$$

Here

$$\mathcal{F}[\phi(t); x] \ = \ \int_{\mathbb{R}^n} \phi(t) e^{2\pi i <x,t>} dt, \ x \in \mathbb{R}^n,$$

is the L^1 Fourier transform which is a linear, continuous, one-one, mapping of $\mathcal{D}$ onto $\mathcal{Z}$ with the same being true of the distributional Fourier transform from $\mathcal{D}'$ to $\mathcal{Z}'$ defined above.

3. Preliminary results

We obtain several lemmas which will be used in the main results in Section 4. We first correlate a set that will be a support set in Section 4 with the method of definition of a dual cone of a cone; the set defined in Lemma 1 will be used in the definition of the Cauchy and Poisson kernels in Section 5.

Lemma 1. *Let C be an open convex cone in $\mathbb{R}^n$. We have* $\{t \in \mathbb{R}^n : u_C(\overline{t/A}) \leq 0\} = \{t \in \mathbb{R}^n :< \overline{t/A}, y > \geq \ 0, \ for \ all \ y \in C\}$.

Proof. We call $C^*_A = \{t \in \mathbb{R}^n :< \overline{t/A}, y > \geq \ 0, \ for \ all \ y \in C\}$. By definition of the indicatrix function

$$u_C(\overline{t/A}) = \sup_{y \in pr(C)} (- < \overline{t/A}, y >).$$

If $t \in \{t \in \mathbb{R}^n : u_C(\overline{t/A}) \leq 0\}$, then $< \overline{t/A}, y' > \geq \ 0$ for every $y' \in pr(C)$ which implies $< \overline{t/A}, y > \geq \ 0$ for every $y \in C$ since any point of C is a positive scalar multiple of a point on $pr(C)$. Thus $t \in C^*_A$. Conversely, if $t \in C^*_A$, $< \overline{t/A}, y > \geq \ 0$ for every $y \in C$ which implies $- < \overline{t/A}, y' > \leq \ 0$ for every $y' \in pr(C)$; hence $t \in \{t \in \mathbb{R}^n : u_C(\overline{t/A}) \leq 0\}$. The proof is complete. $\square$

Lemmas 2 and 3 will be used in the proof of Theorem 2 below.

Lemma 2. *Let C be an open convex cone. Let $C' \subset C$ be an arbitrary compact subcone of C, and let $m > 0$ be an arbitrary real number. Let $g(t), t \in \mathbb{R}^n$, be a continuous function with support in C^*_A. Let $\sigma > 0$ be an arbitrary real number. Let*

$$|g(t)| \leq M(C', m) \exp(2\pi < \overline{t/A}, B > + \sigma|B|), \ t \in \mathbb{R}^n, \qquad (1)$$

with (1) being independent of $B \in (C' \backslash (C' \cap N(\overline{0}, m)))$ for all $C' \subset C$ and all $m > 0$ and $M(C', m)$ being a constant depending on $C' \subset C$ and $m > 0$; that is, (1) holds for all $B \in (C' \backslash (C' \cap N(\overline{0}, m))), C' \subset C, m > 0$. Let y be any point of C. We have $exp(-2\pi < \overline{t/A}, y >) g(t) \in L^p, 1 \leq p < \infty$, as a function of $t \in \mathbb{R}^n$.

PROOF. Let $y \in C$; there is a compact subcone $C' \subset C$ and a $m > 0$ such that $y \in (C' \backslash (C' \cap N(\overline{0}, m)))$. For $y \in (C' \backslash (C' \cap N(\overline{0}, m)))$ now, choose $\mu > 0$ such that $0 < m/|y| < \mu < 1$. Then $\mu y \in (C' \backslash (C' \cap (N(\overline{0}, m)))$, and put $B = \mu y$ in (1). From the proof of [4, Lemma 4.3.2, p. 155] we have a fixed real number $\delta > 0$ such that

$$< \overline{t/A}, y > \geq \delta |y| |\overline{t/A}|, \ t \in C_A^*, \ y \in C' \subset C, \tag{2}$$

and δ depends on $C' \subset C$ and not on $y \in C'$. Combining (1) and (2) we have for $t \in C_A^*$ and $y \in (C' \backslash (C' \cap N(\overline{0}, m)))$

$$\begin{aligned} |e^{-2\pi < \overline{t/A}, y>} g(t)| &\leq M(C', m) e^{2\pi\sigma\mu|y|} e^{-2\pi(1-\mu) < \overline{t/A}, y>} \\ &\leq M(C', m) e^{2\pi\sigma\mu|y|} \exp(-2\pi(1-\mu)\delta|y||\overline{t/A}|). \end{aligned} \tag{3}$$

Let $1 \leq p < \infty$; recall $\operatorname{supp}(g) \subseteq C_A^*$. From (3)

$$\begin{aligned} \int_{\mathbb{R}^n} |e^{-2\pi < \overline{t/A}, y>} g(t)|^p dt &= \int_{C_A^*} |e^{-2\pi < \overline{t/A}, y>} g(t)|^p dt \\ &\leq (M(C', m))^p e^{2\pi\sigma\mu p|y|} \int_{C_A^*} exp(-2\pi(1-\mu)\delta pa|y||t|) dt \\ &\leq (M(C', m))^p Z_n e^{2\pi\sigma\mu p|y|} \int_0^\infty s^{n-1} exp(-2\pi(1-\mu)\delta pa|y|s) ds \\ &= (M(C', m))^p Z_n e^{2\pi\sigma\mu p|y|} (n-1)! (2\pi(1-\mu)\delta pa|y|)^{-n}, \end{aligned} \tag{4}$$

where $(a_j)^{-2} \geq a^2 > 0$, $j = 1, \ldots, n$, for a chosen fixed $a > 0$ and Z_n is the surface area of the unit sphere in $\mathbb{R}^n$. (4) holds for all $\sigma > 0$ and for the arbitrary point $y \in C$ chosen at the beginning of this proof. Recall $0 < \mu < 1$. We thus conclude from (4) that $exp(-2\pi < \overline{t/A}, y >) g(t) \in L^p, 1 \leq p < \infty$, as a function of $t \in \mathbb{R}^n$ for each $y \in C$. □

LEMMA 3. *Let C be an open convex cone, and let C' be an arbitrary compact subcone of C. Let α be an arbitrary n-tuple of nonnegative integers. Let $U_t = D_t^\alpha g(t)$ be the distributional derivative of $g(t)$ where $g(t)$ is a continuous function on $\mathbb{R}^n$ which satisfies (1). Let $\operatorname{supp}(g) \subseteq C_A^*$. We have $f(z) = \langle U_t, e^{2\pi i < \overline{t/A}, z>} \rangle$ is an analytic function of z for $z \in T^C = \mathbb{R}^n + iC$; and for any $m > 0$ and any $\sigma > 0$*

$$|f(z)| \leq K(C', m)(1 + |z|)^N e^{2\pi\sigma|y|}, \tag{5}$$

for $z = x + iy \in T(C', m) = \{z = x + iy \in \mathbb{C}^n : x \in \mathbb{R}^n, y \in (C' \backslash (C' \cap N(\overline{0}, m)))\}$, where $K(C', m)$ is a constant depending on $C' \subset C$ and on m and $N > 0$ is a fixed real number.

PROOF. Let $z \in T^C$. There is a compact subcone $C' \subset C$ and a $m > 0$ such that $z \in T(C', m)$. Thus to prove $f(z)$ is analytic in T^C it suffices to prove that $f(z)$ is analytic in $T(C', m)$ for all compact subcones $C' \subset C$ and any $m > 0$. C_A^* is a regular set ([6, pp. 98–99]); thus $\text{supp}(U) = \text{supp}(g) \subseteq C_A^*$. Consider

$$f(z) = \langle U_t, e^{2\pi i <\overline{t/A}, z>}\rangle, z \in T(C', m), \tag{6}$$

for any $C' \subset C$ and any $m > 0$ (with $U_t = D_t^\alpha g(t)$ we formally use distributional differentiation in (6); by the analysis below, this formalism is valid). Consider

$$h(z) = \int_{C_A^*} g(t) e^{2\pi i <\overline{t/A}, z>} dt, z \in T(C', m). \tag{7}$$

Let z_o be an arbitrary but fixed point of $T(C', m)$ and let $R(z_o, r) \subset T(C', m)$ be an arbitrary but fixed open neighborhood of z_o with radius r whose closure is in $T(C', m)$. Let $z \in R(z_o, r)$ and let β be an arbitrary n-tuple of nonnegative integers. Since $R(z_o, r)$ is fixed and has closure in $T(C', m)$ there exist two balls of the origin in $\mathbb{R}^n$ of radius b and c, respectively, such that $0 < m < b < |y| < c$ for all $y = Im(z), z = x+iy \in R(z_o, r)$. Now let $\mu = m/b$; then $0 < \mu = m/b < 1$. Further $\mu y \in C'$ for $y = Im(z), z = x + iy \in R(z_o, r)$, since C' is a cone; and

$$|\mu y| = \mu |y| = (m/b)|y| > (m/b)b = m$$

which yields $\mu y \in (C' \backslash (C' \cap N(\overline{0}, m)))$. We now choose $B = \mu y, \mu = m/b, y = Im(z), z = x + iy \in R(z_o, r)$ in (1). From (1), (2), the fact that $|y| < c$ for all $y = Im(z), z = x + iy \in R(z_o, r)$, we proceed as in (3) and (4) to obtain

$$\begin{aligned} &|\int_{C_A^*} g(t) t^\beta e^{2\pi i <\overline{z/A}, t>} dt| \leq \int_{C_A^*} |t^\beta||g(t)| e^{-2\pi <\overline{t/A}, y>} dt \\ &\leq M(C', m) \int_{C_A^*} |t^\beta| e^{2\pi <\overline{t/A}, \mu y>} e^{2\pi\sigma|\mu y|} e^{-2\pi <\overline{t/A}, y>} dt \\ &\leq M(C', m) e^{2\pi\sigma\mu c} \int_{C_A^*} |t^\beta| \exp(-2\pi(1-\mu)\delta |y| |\overline{t/A}|) dt \\ &\leq M(C', m) Z_n e^{2\pi\sigma\mu c} \int_0^\infty s^{|\beta|+n-1} \exp(-2\pi(1-\mu)\delta bas) ds \ < \ \infty, \end{aligned} \tag{8}$$

where $(a_j)^{-2} \geq a^2 > 0, \ j = 1, \ldots, n$, and Z_n is the surface area of the unit sphere in $\mathbb{R}^n$. From (8) we conclude that the integral defining $h(z)$ in (7) and any derivative $D_z^\beta h(z)$ of it converges absolutely and uniformly for $z \in R(z_o, r)$ (recall $\mu = m/b, \sigma > 0, b$, and c are all independent of $z \in R(z_o, r)$). Since z_o is an arbitrary point in $T(C', m), h(z)$ is analytic for $z \in T(C', m)$ and hence for $z \in T^C$. Thus $f(z)$ is analytic in T^C with

$$f(z) = \langle U_t, e^{2\pi i <\overline{z/A}, t>}\rangle = (-1)^{|\alpha|} (z/A)^\alpha \int_{C_A^*} g(t) e^{2\pi i <\overline{z/A}, t>} dt. \tag{9}$$

We now obtain (5). Let $C' \subset C, m > 0$, and $z = x + iy \in T(C', m)$. For the given $C' \subset C$ and $m > 0$ choose $b > 0$ such that $0 < b < m$, and put $\mu = b/m$ so that $0 < \mu = b/m < 1$. Here μ is independent of $y \in C' \backslash (C' \cap N(\overline{0}, m))$. Put $B = \mu y$, and $B \in C' \subset C$ such that

$$|B| = \mu |y| = (b/m)|y| > (b/m)m = b.$$

Thus $B \in C' \backslash (C' \cap N(\overline{0}, b))$ and (1) holds for this B. From (9) and the analysis of (4) and (8) we have for $z = x + iy \in T(C', m)$

$$|f(z)| \leq M(C', m) Z_n |(z/A)^\alpha| e^{2\pi\sigma|y|} (n-1)! (2\pi(1-\mu)\delta ab)^{-n}$$

where $\delta > 0$ is from (2). (5) now follows. ∎

We now present three lemmas concerning bounded sets in $\mathcal{K}(A)$ which will be used in the proof of Theorem 2.

LEMMA 4. *Let $S \subseteq \mathcal{E}(\mathbb{R}^n)$ be a set of functions such that for each n-tuple ρ of nonnegative integers $\{D_t^\rho \phi(t) : \phi \in S\}$ is uniformly bounded. Let T be a bounded set in $\mathcal{K}(A)$. We have $ST = \{\phi\psi : \phi \in S, \psi \in T\}$ is a bounded set in $\mathcal{K}(A)$.*

PROOF. We have $ST \subset \mathcal{E}(\mathbb{R}^n)$. Let $m = 0, 1, 2, 3, \ldots$ be arbitrary but fixed. For the fixed m let ρ satisfy $|\rho| \leq m$. By the generalized Leibnitz rule

$$D_t^\rho(\phi(t)\psi(t)) \;=\; \sum_{\beta+\gamma=\rho} \frac{\rho!}{\beta!\gamma!} D_t^\beta \phi(t) D_t^\gamma \psi(t).$$

Letting N_β denote the uniform bound on $\{D_t^\beta \phi : \phi \in S\}$ and recalling $\|\psi\|_m$ from the definition of $\mathcal{K}(A)$ in Section 2,

$$|D_t^\rho(\phi(t)\psi(t))| \leq \|\psi\|_m \sum_{\beta+\gamma=\rho} \frac{\rho!}{\beta!\gamma!} N_\beta$$

which holds for each $\rho, |\rho| \leq m$, and all $t \in \mathbb{R}^n$. Thus

$$\|\phi\psi\|_m \leq \|\psi\|_m \sup_{|\rho| \leq m} \Big(\sum_{\beta+\gamma=\rho} \frac{\rho!}{\beta!\gamma!} N_\beta \Big)$$

which proves $ST \subset \mathcal{K}(A)$ and is a bounded set in $\mathcal{K}(A)$. ∎

Let C be an open convex cone in $\mathbb{R}^n$. Let $q(u) \in \mathcal{E}(\mathbb{R}^1), u \in \mathbb{R}^1$, such that $q(u) = 1, u \geq 0, q(u) = 0, u \leq -\epsilon, \epsilon > 0$ and fixed, and $0 \leq q(u) \leq 1$. Put

$$r_y(t) = q(<\overline{t/A}, y>), \; y \in C, \; t \in \mathbb{R}^n. \tag{10}$$

We have $r_y(t) \in \mathcal{E}(\mathbb{R}^n), t \in \mathbb{R}^n$, for any $y \in C$.

LEMMA 5. *Let C be an open convex cone in $\mathbb{R}^n$. Put*

$$\phi_y(t) = r_y(t) e^{-2\pi <\overline{t/A}, y>}, \; y \in C, \; t \in \mathbb{R}^n,$$

where we recall $A = (a_1, a_2, \ldots, a_n)$ is a fixed n-tuple of positive real numbers. Let $y \in C$ be arbitrary but fixed. We have $D_t^\rho(\phi_y(t)), t \in \mathbb{R}^n, y \in C$ fixed, is uniformly bounded in $t \in \mathbb{R}^n$ for each n-tuple ρ of nonnegative integers.

PROOF. From the definition of $r_y(t)$ in (10), $D_t^\rho(r_y(t)) = 0$ for $t \in \mathbb{R}^n$ such that $< \overline{t/A}, y > \leq -\epsilon$, $\epsilon > 0$ and fixed. Hence

$$D_t^\rho(\phi_y(t)) = \sum_{\beta+\gamma=\rho} \frac{\rho!}{\beta!\gamma!} D_t^\beta(r_y(t)) D_t^\gamma(e^{-2\pi<\overline{t/A},y>})$$
$$= \sum_{\beta+\gamma=\rho} \frac{\rho!}{\beta!\gamma!} D_t^\beta(r_y(t))(-1)^{|\gamma|} i^{-|\gamma|}(y/A)^\gamma e^{-2\pi<\overline{t/A},y>}$$

and

$$|D_t^\rho(\phi_y(t))| \leq \sum_{\beta+\gamma=\rho} \frac{\rho!}{\beta!\gamma!} K_{\beta,y}|(y/A)^{\beta+\gamma}|e^{2\pi\epsilon}.$$

Thus for each ρ, $D_t^\rho(\phi_y(t))$ is bounded independently of $t \in \mathbb{R}^n$ for $y \in C$ fixed. ∎

LEMMA 6. *Let C be an open convex cone in $\mathbb{R}^n$. Let T be a bounded set in $\mathcal{K}(A)$ and $r_y(t)$ be the function defined in* (10). *Put $S = \{r_y(t)e^{-2\pi<\overline{t/A},y>} : t \in \mathbb{R}^n, y \in C$ fixed$\}$. We have ST is a bounded set in $\mathcal{K}(A)$.*

PROOF. The proof follows by combining Lemmas 4 and 5. ∎

4. Analytic functions

We obtain extensions and new results in this section related to the results in [1, section 6]. Let C be an open convex cone and $C' \subset C$ be an arbitrary compact subcone of C. For arbitrary $m > 0$ and $C' \subset C$ recall $T(C', m) = \mathbb{R}^n + i(C' \backslash (C' \cap N(\overline{0}, m)))$ as before where $N(\overline{0}, m) = \{y \in \mathbb{R}^n : |y| \leq m\}$.

We consider functions $f(z)$ which are analytic in $T^C = \mathbb{R}^n + iC$ and satisfy

$$(11) \qquad |f(z)| \leq P(C', m)(1 + |z|)^N e^{2\pi(b+\sigma)|y|}, z = x + iy \in T(C', m),$$

where $b \geq 0$ and $N \geq 0$ are fixed real numbers, $\sigma > 0$ and $m > 0$ are arbitrary real numbers which are independent of any entity, $C' \subset C$ is an arbitrary compact subcone of C, and $P(C', m)$ is a constant which depends on the choice of $C' \subset C$ and on $m > 0$.

THEOREM 1. *Let $A = (a_1, a_2, \ldots, a_n)$ be a fixed n-tuple of positive real numbers. Let $f(z)$ be analytic in T^C and satisfy* (11). *There exists an element $U \in \mathcal{K}'(A)$ having support in $\{t \in \mathbb{R}^n : u_C(\overline{t/A}) \leq b\}$ such that $f(z) \to \mathcal{F}_A[U]$ in $\mathcal{Z}'(\overline{2\pi})$ as $y \to \overline{0}, y \in C' \subset C$, for any compact subcone $C' \subset C$.*

PROOF. For any fixed point $y \in C$ or any fixed compact set in the cone C there is a compact subcone $C' \subset C$ and a $m > 0$ such that the point or the compact set is in $(C' \backslash (C' \cap N(\overline{0}, m)))$. The growth (11) holds for $C' \subset C$ being any compact subcone of C and for $m > 0$ and $\sigma > 0$ being arbitrary and not dependent on any entity. Because of these facts, the proof of this theorem can be accomplished by the method of proof of [1, Theorem 10] as follows. We summarize the proof; the details are exactly those in the proof of [1, Theorem 10]. For n being the dimension and $\epsilon > 0$ we may choose a n-tuple $\alpha = (\alpha_1, \ldots, \alpha_n)$ of nonnegative integers such that

$$|z^{-\alpha} f(z)| \leq M(C', m)(1 + |z|)^{-n-\epsilon} e^{2\pi(b+\sigma)|y|}, \ z = x + iy \in T(C', m),$$

for any $C' \subset C$ and any $m > 0$. Put

$$g_y(t) = \int_{\mathbb{R}^n} z^{-\alpha} f(z) e^{-2\pi i <\overline{t/A}, z>} dx, \ z = x + iy \in T^C.$$

Proceeding as in the proof of [1, Theorem 10] we have $g_y(t)$ is independent of $y = Im(z) \in C$; hence we refer to this function as $g(t)$ in the remainder of this paper. Further, applying the proof of [1, Theorem 10] we have that $g(t)$ is a continuous function of $t \in \mathbb{R}^n$ and $\text{supp}(g) \subseteq \{t \in \mathbb{R}^n : u_C(\overline{t/A}) \leq b\}$. By the growth condition on $z^{-\alpha} f(z)$, $z^{-\alpha} f(z) \in L^1 \cap L^2$ as a function of $x \in \mathbb{R}^n$ for $y \in C$. By the Plancherel theory of Fourier transform, $e^{-2\pi <\overline{t/A}, y>} g(t) \in L^2, y \in C$, and

$$z^{-\alpha} f(z) = \mathcal{F}[e^{-2\pi <\overline{t/A}, y>} g(t); \overline{x/A}], \ z = x + iy \in T^C,$$

with this Fourier transform being in the L^2 sense. For $\psi \in \mathcal{Z}(\overline{2\pi})$ and $\phi \in \mathcal{K}(A)$ such that $\psi(x) = \mathcal{F}_A[\phi(t); x]$ we use change of variable and the distributional Fourier transform on $\mathcal{D}'$ to obtain

$$\langle z^{-\alpha} f(z), \psi(x) \rangle = \langle e^{-2\pi <y,t>} g(a_1 t_1, \ldots, a_n t_n), \phi(a_1 t_1, \ldots, a_n t_n) \rangle$$

for $y \in C$. For C' being any compact subcone of C we now have

$$\begin{aligned} \langle z^{-\alpha} f(z), \psi(x) \rangle &\to \langle g(a_1 t_1, \ldots, a_n t_n), \phi(a_1 t_1, \ldots, a_n t_n) \rangle \\ &= \prod_{j=1}^{n} (a_j)^{-1} \langle g(t), \phi(t) \rangle = \langle \mathcal{F}_A[g], \psi \rangle \end{aligned}$$

as $y \to \overline{0}, y \in C' \subset C$ and

$$\langle f(z), \psi(x) \rangle = \langle z^{-\alpha} f(z), z^{\alpha} \psi(x) \rangle \to \langle \mathcal{F}_A[g], x^{\alpha} \psi(x) \rangle = \langle x^{\alpha} \mathcal{F}_A[g], \psi(x) \rangle,$$

$\psi \in \mathcal{Z}(\overline{2\pi})$, as $y \to \overline{0}, y \in C' \subset C$. Put

$$\Delta \ = \ A^{\alpha} (-2\pi i)^{-|\alpha|} \frac{\partial^{|\alpha|}}{\partial t_1^{\alpha_1} \ldots \partial t_n^{\alpha_n}}$$

and $U = \Delta g(t)$. $U \in \mathcal{K}'(A)$ and using Schwartz ([6, pp. 98–99]) $\text{supp}(U) = \text{supp}(g) \subseteq \{t \in \mathbb{R}^n : u_C(\overline{t/A}) \leq b\}$ since this support set is a regular set. We continue the proof now to show that $\mathcal{F}_A[U] = x^{\alpha} \mathcal{F}_A[g]$ in $\mathcal{Z}'(\overline{2\pi})$; and from the above convergence, $f(x + iy) \to \mathcal{F}_A[U]$ in $\mathcal{Z}'(\overline{2\pi})$ as $y \to \overline{0}, y \in C' \subset C$ for C' being any compact subcone of C. The proof is complete. □

We now obtain additional results to those of [1, Theorems 10 and 13] for our more general analytic functions here.

Theorem 2. *Let $A = (a_1, \dots, a_n)$ be a fixed n-tuple of positive real numbers. Let $f(z)$ be analytic in T^C and satisfy* (11) *with $b = 0$. There exists an element $U \in \mathcal{K}'(A)$ having support in $C_A^* = \{t \in \mathbb{R}^n : <\overline{t/A}, y> \geq 0, \text{ for all } y \in C\} = \{t \in \mathbb{R}^n : u_C(\overline{t/A}) \leq 0\}$ such that*

$$e^{-2\pi<\overline{t/A},y>} U_t \in \mathcal{K}'(A),\ y \in C; \tag{12}$$

$$f(z) = \langle U_t, e^{2\pi i<\overline{t/A},z>} \rangle,\ z \in T^C; \tag{13}$$

$$f(x+iy) \ = \ \prod_{j=1}^{n} a_j \ \mathcal{F}_A[e^{-2\pi<\overline{t/A},y>} U_t],\ z = x+iy \in T^C; \tag{14}$$

in $\mathcal{Z}'(\overline{2\pi})$;

$$\lim_{y \to \overline{0},\ y \in C' \subset\ C} f(x+iy) \ = \ \mathcal{F}_A[U] \tag{15}$$

in $\mathcal{Z}'(\overline{2\pi})$ for any compact subcone $C' \subset C$;

$$\{f(x+iy) : y \in C \text{ fixed}\} \tag{16}$$

is a strongly bounded set in $\mathcal{Z}'(\overline{2\pi})$.

Proof. Let $z \in T^C$; then $z \in T(C', m)$ for some $C' \subset C$ and $m > 0$. Here $b = 0$ in (11). For the n-tuple α of nonnegative integers and the function $g(t)$ in the proof of Theorem 1 we have $g(t)$ is independent of $y \in C$, $\text{supp}(g) \subseteq C_A^*$, and $g(t)$ is continuous on $\mathbb{R}^n$. Further for $t \in \mathbb{R}^n$ we have from the construction of $g(t)$ in the proof of Theorem 1 that

$$\begin{aligned} |g(t)| &\leq M(C', m) e^{2\pi<\overline{t/A},y>} e^{2\pi\sigma|y|} \int_{\mathbb{R}^n} (1+|x|)^{-n-\epsilon} dx \\ &\leq Q(C', m) e^{2\pi<\overline{t/A},y>} e^{2\pi\sigma|y|}, \end{aligned} \tag{17}$$

which holds for all $y \in (C' \backslash (C' \cap N(\overline{0}, m)))$, for all $C' \subset C$, all $m > 0$, and all $\sigma > 0$. Put $U = \Delta g(t)$ for the differential operator Δ as defined in the proof of Theorem 1. We have $U \in \mathcal{K}'(A)$; and $e^{-2\pi<\overline{t/A},y>} U_t \in \mathcal{K}'(A), y \in C$, which is (12). (15) follows from the proof of Theorem 1. Also $\text{supp}(U) = \text{supp}(g) \subseteq C_A^*$ since C_A^* is a regular set ([6, pp. 98–99]).

We now obtain (13). Because of the properties on $g(t)$ here, by Lemma 3 $\langle U_t, e^{2\pi i<\overline{t/A},z>} \rangle$ is analytic for $z \in T^C$. For each $z \in T^C, z \in T(C', m)$ for some $C' \subset C$ and some $m > 0$; and hence $y = Im(z) \in (C' \backslash (C' \cap N(\overline{0}, m)))$. Again by the properties of $g(t)$ in this proof including (17) we have by the proof of Lemma 2 that $exp(-2\pi < \overline{t/A}, y >) g(t) \in L^1 \cap L^2$ for any $y \in C$.

Thus for $z \in T^C$ and the construction of $U = \Delta g(t)$ we have

$$\begin{aligned}\langle U_t, e^{2\pi i<\overline{t/A},z>}\rangle &= z^\alpha \int_{C_A^*} g(t)e^{2\pi i<\overline{t/A},z>}dt \\ &= z^\alpha \mathcal{F}[e^{-2\pi<\overline{t/A},y>}g(t);\overline{x/A}]\end{aligned} \tag{18}$$

with this Fourier transform being in both the L^1 and L^2 sense. Recall the definition of the function $g_y(t) = g(t)$ in the proof of Theorem 1; from the construction of the function $g(t)$ from the given function $f(z)$ in the proof of Theorem 1 applied to the present theorem, the Fourier transform theory yields

$$\mathcal{F}[e^{-2\pi<\overline{t/A},y>}g(t);\overline{x/A}] \ = \ z^{-\alpha}f(z), \ z \in T^C, \tag{19}$$

for the α chosen in the proof of Theorem 1; and $z^{-\alpha}f(z) \in L^1 \cap L^2$ as a function of $x \in \mathbb{R}^n$ for each $y \in C$. Combining (18) and (19) we have (13).

To obtain (14) let $\psi \in \mathcal{Z}(\overline{2\pi})$. Put

$$\phi(-t) \ = \ \int_{\mathbb{R}^n} \psi(x)e^{2\pi i<\overline{t/A},x>}dx.$$

By the analysis of [1, pp. 739 - 740, (5) and (6)], $\check{\phi}(t) = \phi(-t) \in \mathcal{K}(A)$ such that $\psi(x) = \mathcal{F}_A[\check{\phi}(t);x]$. Using (13) and the distributional finite Fourier transform, we have for $z = x+iy \in T^C$

$$\begin{aligned}\langle f(x+iy),\psi(x)\rangle &= \langle\langle U_t, e^{2\pi i<\overline{t/A},z>}\rangle,\psi(x)\rangle \\ &= \int_{\mathbb{R}^n} z^\alpha \int_{C_A^*} g(t)e^{2\pi i<\overline{t/A},z>}dt\psi(x)dx \\ &= \int_{C_A^*} g(t) \int_{\mathbb{R}^n} z^\alpha e^{2\pi i<\overline{t/A},z>}\psi(x)dxdt \\ &= \langle e^{-2\pi<\overline{t/A},y>}U_t, \int_{\mathbb{R}^n} \psi(x)e^{2\pi i<\overline{t/A},x>}dx\rangle \\ &= \langle e^{-2\pi<\overline{t/A},y>}U_t, \check{\phi}(t)\rangle \\ &= \prod_{j=1}^n a_j \ \langle \mathcal{F}_A[e^{-2\pi<\overline{t/A},y>}U_t],\psi(x)\rangle\end{aligned}$$

for $z = x+iy \in T^C$ which proves (14).

(16) remains to be proved. Let T be an arbitrary bounded set in $\mathcal{K}(A)$ and $y \in C$ be arbitrary but fixed. Recalling the function $r_y(t)$ of (10) and the set $S = \{r_y(t)e^{-2\pi<\overline{t/A},y>} : t \in \mathbb{R}^n, y \in C \text{ fixed}\}$, we apply Lemma 6 to obtain ST is a bounded set in $\mathcal{K}(A)$. Since $U \in \mathcal{K}'(A)$, U is continuous and

hence bounded on $\mathcal{K}(A)$. Thus

$$\{\langle e^{-2\pi<\overline{t/A},y>}U_t, \check{\phi}(t)\rangle : \phi \in T,\ y \in C \text{ fixed}\}$$
$$= \{\langle U_t, r_y(t)e^{-2\pi<\overline{t/A},y>}\check{\phi}(t)\rangle : \phi \in T,\ y \in C \text{ fixed}\}$$

is a bounded set in the complex plane. Thus $\{e^{-2\pi<\overline{t/A},y>}U_t : y \in C \text{ fixed}\}$ is a strongly bounded set in $\mathcal{K}'(A)$ since T is an arbitrary bounded set in $\mathcal{K}(A)$. But the distributional finite Fourier transform analyzed in [1, pp. 740–741] is a strongly continuous mapping of $\mathcal{K}'(A)$ onto $\mathcal{Z}'(\overline{2\pi})$. Using this fact and (14), we obtain (16). □

5. Cauchy and Poisson kernels

Let C be a regular cone; that is, C is an open convex cone which does not contain an entire straight line. Here we define Cauchy and Poisson kernels corresponding to the support set C_A^* of section 3.

If A is the n-tuple $(1, 1, \ldots, 1)$, $C^*_{(1,1,\ldots,1)} = C^*$, the dual cone of C, from which the Cauchy and Poisson kernels corresponding to C

$$K(z-t) = \int_{C^*} e^{2\pi i<z-t,u>}du,\ z \in T^C,\ t \in \mathbb{R}^n,$$

and

$$Q(z;t) = \frac{K(z-t)\overline{K(z-t)}}{K(2iy)},\ z = x+iy \in T^C,\ t \in \mathbb{R}^n,$$

are defined. Further if in $\mathbb{R}^1, C = (0,\infty)$ or $C = (-\infty, 0)$, we have $C_a^* = C^*$ for any $a > 0$; hence the Cauchy and Poisson kernels defined below corresponding to the set C_A^* are the usual one-dimensional kernels in these cases.

We refer to [3, Chapter 2] where the ultradifferentiable functions $\mathcal{D}(*, L^s)$, $1 \le s \le \infty$, and ultradistributions $\mathcal{D}'(*, L^s), 1 \le s \le \infty$, are defined. Here $*$ represents either (M_p) or $\{M_p\}$ which signifies that the functions and ultradistributions are of Beurling or Roumieu type, respectively. In [2] and [3] we have proved that the Cauchy and Poisson kernels $K(z-t)$ and $Q(z;t)$ are elements of the $\mathcal{D}(*, L^s)$ spaces for $z \in T^C$ and have defined and obtained properties of the Cauchy and Poisson integrals (transforms) of elements in $\mathcal{D}'(*, L^s)$.

In this section we define and state properties of Cauchy and Poisson kernels corresponding to the support set C_A^* of section 3. The Cauchy kernel corresponding to C_A^* is

$$K_A(z-t) \ = \ \int_{C_A^*} e^{2\pi i<z-t,\overline{u/A}>}du,\ z \in T^C,\ t \in \mathbb{R}^n,$$

and the Poisson kernel is

$$Q_A(z;t) \ = \ \frac{K_A(z-t)\overline{K_A(z-t)}}{K_A(2iy)},\ z = x+iy \in T^C,\ t \in \mathbb{R}^n.$$

Using the same proofs as in [3, Chapter 4] we can show $K_A(z-t) \in \mathcal{D}(*, L^s), 1 < s \leq \infty$, and $Q_A(z;t) \in \mathcal{D}(*, L^s), 1 \leq s \leq \infty$, as functions of $t \in \mathbb{R}^n$ for $z \in T^C$. For $U \in \mathcal{D}'(*, L^s)$ we can proceed to define corresponding Cauchy and Poisson integrals as

$$C_A(U;z) \;=\; \langle U_t, K_A(z-t)\rangle,\; z \in T^C,\; 1 < s \leq \infty,$$

and

$$P_A(U;z) \;=\; \langle U_t, Q_A(z;t)\rangle,\; z \in T^C,\; 1 \leq s \leq \infty,$$

and can proceed to obtain results for these functions of $z \in T^C$ like those obtained in [2] and [3] for $C(U;z)$ and $P(U;z)$. (For example $C_A(U;z)$ will be analytic in T^C and will have both pointwise and norm growths.) The proofs of the corresponding results will be equivalent. We note as above that for $A = a$ in one dimension, $C_a^* = C^*$ for $C = (0,\infty)$ or $C = (-\infty, 0)$.

We now add information to Theorem 2 by showing that the analytic function $f(z)$ there can be recovered as a Cauchy integral of the type defined above corresponding to the support set C_A^*.

Corollary 1. *Under the hypotheses of Theorem 2 with C being a regular cone, we have*

$$f(z) \;=\; \langle \mathcal{F}[U], K_A(z - (a_1t_1, \overset{\vee}{\ldots}, a_nt_n))\rangle,\; z \in T^C,$$

for the element $U \in \mathcal{D}' \subset \mathcal{K}'(A)$ in Theorem 2.

Proof. From the form of the constructed U in Theorem 2, $U \in \mathcal{D}' \subset \mathcal{K}'(A)$. From Theorem 2 we have for $z \in T^C$ that

$$f(z) \;=\; \langle U_u, I_{C_A^*}(u) r_y(u) e^{2\pi i < \overline{u/A}, z>}\rangle$$

where $I_{C_A^*}(u)$ is the characteristic function of C_A^*. Using the Fourier transform on $\mathcal{D}'$ and because of the properties of the $U \in \mathcal{D}'$ constructed in this paper we have

$$\begin{aligned} f(z) &= \langle \mathcal{F}[U], \mathcal{F}[I_{C_A^*}(u) r_y(u) e^{2\pi i < \overline{u/A}, z>}; t]\rangle \\ &= \langle \mathcal{F}[U], \int_{C_A^*} \exp(2\pi i < \overline{u/A}, z - (a_1t_1, \overset{\vee}{\ldots}, a_nt_n) >) du\rangle \\ &= \langle \mathcal{F}[U], K_A(z - (a_1t_1, \overset{\vee}{\ldots}, a_nt_n))\rangle. \end{aligned}$$

□

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