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CERTAIN WEAKLY GENERATED NONCOMPACT, PSEUDO-COMPACT TOPOLOGIES ON TYCHONOFF CUBES

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Abstract. Given an uncountable cardinal $\aleph$, the product space $I^{\aleph}$, $I = [0,1]$, is called a Tychonoff cube. A collection of closed subsets of a subspace Y of a Tychonoff cube $I^{\aleph}$ that covers Y determines a weak topology for Y. The collection of compact subsets of $I^{\aleph}$ that have a countable dense subset covers $I^{\aleph}$. It is known from work of the author and I. Ivanšić that the weak topology generated by this collection is pseudo-compact. We are going to show that it is not compact. The author and I. Ivanšić have also considered weak topologies on some other "naturally occurring" subspaces of such $I^{\aleph}$. The new information herein along with the previous examples will lead to the existence of vast naturally occurring classes of pseudo-compacta any set of which has a pseudo-compact product. Some of the classes consist of Tychonoff spaces, so the product spaces from subsets of these are also Tychonoff spaces.

1. Introduction

For each uncountable cardinal $\aleph$, we are going to consider the Tychonoff cube $I^{\aleph}$ and the weak topology on it determined by the collection of closed subsets of $I^{\aleph}$ that have a countable dense subset. This space was denoted $X_2(\aleph)$ in [5, Definitions 3.4 and 3.13]. As a set, $X_2(\aleph) = I^{\aleph}$ ([5, Lemma 3.14(3)]), and the set-wise identity function $\lambda_2(\aleph) : X_2(\aleph) \to I^{\aleph}$ turns out to be a bijective map meaning that the topology of $X_2(\aleph)$ contains that of $I^{\aleph}$. Moreover, by [5, Lemmas 3.15 and 3.16 and Theorem 3.30], $X_2(\aleph)$ is an arcwise connected Hausdorff space that is pseudo-compact, that is, every map of it to $\mathbb{R}$ has compact image. According to [5, Proposition 3.27], if the

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cardinal $\aleph$ is "sufficiently large," then the topology of $X_2(\aleph)$ is larger than that of $I^\aleph$. This implies that the bijection $\lambda_2(\aleph)$ is not a homeomorphism and that $X_2(\aleph)$ is not compact.

We shall show in Theorem 2.5, as one of our two main results, that for *every* uncountable cardinal $\aleph$, the topology of $X_2(\aleph)$ is larger than that of $I^\aleph$ and hence that $\lambda_2(\aleph) : X_2(\aleph) \to I^\aleph$ is never a homeomorphism. Our other main result, Theorem 3.6, is a product theorem that is parallel to but stronger than [5, Theorem 3.30] which concerns the pseudo-compactness of products of certain sets of the examples that were described in the latter. A reader who would like to see a historical perspective about such product theorems can find a good source of information in [6]. The publications [2–4] can also be used as sources for this topic.

2. Main Theorem

When we say that a sequence (s_n) of elements of a set is non-repeating, we mean that if $1 \le m < n$, then $s_m \neq s_n$.

Lemma 2.1. *Let Z be a Hausdorff space, S, $C \subset Z$, and $S \cap C = \emptyset$. Suppose that:*

1. *S is countably infinite, and*
2. *$C \subset \mathrm{cl}_Z(S)$.*

Then for each $z \in C$, there exists a non-repeating sequence in S that converges to z in Z.

Proof. Select a directed set A and a net $(s_\alpha)_{\alpha\in A}$ in S converging to z in Z. For each $s \in S$, let $A_s = \{\alpha \in A \mid s_\alpha = s\}$. Such an A_s cannot be cofinal in A because this would imply the existence of a constant net in S that converges to z. This is impossible since Z is Hausdorff, $z \in C$, and $S \cap C = \emptyset$. Hence for each $s \in S$, there exists $\alpha_s \in A$ such that for all $\alpha \in A_s$, $\alpha < \alpha_s$. One sees that the countable set $\{\alpha_s \mid s \in S\}$ is cofinal in A since if $\alpha \in A$, then $\alpha \in A_s$ for $s = s_\alpha$, and $\alpha < \alpha_s$.

The existence of a countable cofinal subset of A implies the existence of a cofinal sequence in A. This produces a sequence in S that converges to z in Z. From this, the fact that $S \cap C = \emptyset$, and the fact that Z is Hausdorff, one could extract a non-repeating subsequence to obtain a sequence of the type requested. □

Let Ω denote the first uncountable ordinal, $[0,\Omega)$ the first uncountable ordinal space, and $[0,\Omega]$ its one-point compact extension. As usual, each of these well-ordered spaces is given the order topology. Hence $[0,\Omega]$ is a compact Hausdorff space, and $[0,\Omega)$ is a dense subspace of it.

Lemma 2.2. *The weight of $[0,\Omega]$ is Ω. Hence for each uncountable cardinal $\aleph$, $[0,\Omega]$ embeds in $I^\aleph$.*

Proof. For each element $\alpha \in [0,\Omega)$, choose a countable local base B_α at α consisting of open neighborhoods of α in $[0,\Omega]$. Surely the collection B_Ω of intervals $(\beta,\Omega]$, $0 \le \beta < \Omega$, is a local base at Ω consisting of open neighborhoods of Ω in $[0,\Omega]$. The cardinality of B_Ω is Ω. Hence $\bigcup\{B_\alpha \mid \alpha \in [0,\Omega]\}$ is a base for the topology of $[0,\Omega]$ whose cardinality is Ω. By [1, Theorem 3.2.5], $[0,\Omega]$ embeds in $I^{\aleph}$. ∎

Next is a lemma that will lead to the proofs of Theorems 2.5 and 4.1.

Lemma 2.3. *Let $\aleph$ be an uncountable cardinal, $\kappa : [0,\Omega] \to I^{\aleph}$ an embedding, and Z a compact subset of $I^{\aleph}$ having a countable dense subset. Then $Z \cap \kappa([0,\Omega))$ is closed in Z.*

Proof. Let S be a countable dense subset of Z. To save on notation, let us assume that $[0,\Omega] \subset I^{\aleph}$. We must prove that $A = Z \cap [0,\Omega)$ is closed in Z. If there exists $0 \le \alpha < \Omega$ such that $A \subset [0,\alpha]$, then $A = Z \cap [0,\alpha]$ which is surely closed in Z. So to arrive at a contradiction we assume that there is no such α. This implies that A is an uncountable subset of $[0,\Omega)$. Since S is countable, choose $0 \le \beta < \Omega$ such that $(Z \cap [\beta,\Omega)) \cap S = \emptyset$, and put $B = Z \cap [\beta,\Omega)$.

It is not difficult to see that there is an uncountable subset C of B such that in the relative topology of $[0,\Omega)$, C is discrete. For each $c \in C$, let

$$J_c = [c+1,\Omega] \cap Z.$$

Each such J_c is a closed subset of Z that is disjoint from S. We are now going to perform a transfinite construction on the well-ordered set C. Prior to the start of this, let us agree that when we are given a sequence $\sigma = (s_n)$ in S, we shall denote $\tilde{\sigma} = \{s_n \mid n \in \mathbb{N}\} \subset S$. The sequences we shall choose below will be non-repeating. Therefore each such $\tilde{\sigma}$ will be a countably infinite set.

Denote by c_0 the first element of the well-ordered set C. Then of course $c_0 \notin J_{c_0}$. There is a closed neighborhood U_{c_0} of c_0 in Z such that $U_{c_0} \cap J_{c_0} = \emptyset$. Notice that $c_0 \in \mathrm{cl}_Z(S \cap U_{c_0})$. Since $c_0 \notin S$, S is countable, and Z is Hausdorff, then $S \cap U_{c_0}$ is countably infinite. Applying Lemma 2.1 with S replaced by $S \cap U_{c_0}$ and $C = \{c_0\}$, select a non-repeating sequence σ_{c_0} in $S \cap U_{c_0}$ that converges to c_0 in Z. Thus $\tilde{\sigma}_{c_0} \subset U_{c_0}$, and it follows that $\mathrm{cl}_Z(\tilde{\sigma}_{c_0}) \cap J_{c_0} = \emptyset$.

Now we make an inductive assumption. Suppose that $d \in C$ and for each $c \in C$ with $c \le d$ we have selected a non-repeating sequence σ_c in S. We require that:

($\dagger_1$) σ_c converges to c in Z,

($\dagger_2$) if $a \in [0,d] \cap C$ and $a \ne c$, then $\tilde{\sigma}_a \cap \tilde{\sigma}_c = \emptyset$, and

($\dagger_3$) $\mathrm{cl}_Z(\bigcup\{\tilde{\sigma}_a \mid a \in [0,d] \cap C\}) \cap J_d = \emptyset$.

We have just satisfied these inductive assumptions for the case that $d = c_0$. Let b be the first element of C that is greater than d. Then $b \in J_d$. This and an application of ($\dagger_3$) show that $b \in Z \setminus \mathrm{cl}_Z(\bigcup\{\tilde{\sigma}_a \mid a \in [0,d] \cap C\})$. Of course $b \in Z \setminus J_b$, so there is a closed neighborhood U_b of b in Z such that

$U_b \cap \mathrm{cl}_Z(\bigcup\{\tilde\sigma_a \mid a \in [0,d] \cap C\}) = \emptyset = U_b \cap J_b$. Applying Lemma 2.1 once again, choose a non-repeating sequence σ_b in $S \cap U_b$ that converges to b in Z. This gives us $(\dagger_1)$ for $c = b$. It follows that $\tilde\sigma_b \subset U_b = \mathrm{cl}_Z(U_b)$, so $\tilde\sigma_b \cap (\bigcup\{\tilde\sigma_a \mid a \in [0,d] \cap C\}) = \emptyset = \mathrm{cl}_Z(\tilde\sigma_b) \cap J_b$. The inductive assumption $(\dagger_2)$ and the first of these null intersections give us $(\dagger_2)$ for b in place of d. Since $J_b \subset J_d$, then the second null intersection and the inductive assumption $(\dagger_3)$ yield $(\dagger_3)$ for b in place of d.

By virtue of $(\dagger_2)$ of this construction, one achieves the contradiction that the countable set S is uncountable. ∎

Lemma 2.3 leads to the following corollary.

Corollary 2.4. *For each uncountable cardinal $\aleph$, every embedding $\kappa : [0,\Omega] \to I^\aleph$ has the property that $\kappa([0,\Omega))$ is closed in $X_2(\aleph)$.*

Theorem 2.5. *For each uncountable cardinal $\aleph$, the bijective map $\lambda_2(\aleph) : X_2(\aleph) \to I^\aleph$ is not a homeomorphism; hence the topology of $X_2(\aleph)$ is larger than that of $I^\aleph$.*

Proof. Using Lemma 2.2, choose an embedding $\kappa : [0,\Omega] \to I^\aleph$. By Lemma 2.3, $\kappa([0,\Omega))$ is closed in $X_2(\aleph)$, but of course it is not closed in $I^\aleph$. ∎

Corollary 2.6. *For each uncountable cardinal $\aleph$, $X_2(\aleph)$ is not compact.*

3. Product Theorem

In [5], for each uncountable cardinal $\aleph$, we identified 4 different topological spaces, $X_i(\aleph)$, $i \in \{0,1,2,3\}$. We have described and discussed $X_2(\aleph)$ in Section 2. Let us now review the other 3 and make some observations that were not mentioned in [5].

(*i*) $X_0(\aleph)$ as a set is the same as $I_0^\aleph$, which is the set of points in $I^\aleph$ having at most countably many coordinates different from 0. The topology of $X_0(\aleph)$ is weakly generated by the set of compacta X in $I^\aleph$ that are the closure in $I^\aleph$ of a countable subset of $I_0^\aleph$. Such X is metrizable, lies in $I_0^\aleph$, and $I_0^\aleph$ is a dense, but not closed subset of $I^\aleph$.

(*ii*) $X_1(\aleph)$ as a set is $I^\aleph$; its topology is weakly generated by the set of metrizable compacta in $I^\aleph$.

(*iii*) $X_2(\aleph)$ was discussed in Section 2.

Before we describe the next class of spaces, let us recall [5, Definition 3.6].

Definition 3.1. *Let* $\aleph^* = \max\{\mathrm{card}(\beta(\mathbb{N})), \mathrm{card}(\mathbb{R})\}$.

(*iv*) $X_3(\aleph)$ as a set is $I_3^\aleph$; this consists of those $x \in I^\aleph$ having at most $\aleph^*$ coordinates different from 0. The topology of $X_3(\aleph)$ is determined weakly by the set of compacta X in $I^\aleph$ such that X is the closure in $I^\aleph$ of a subset of

$I_3^{\aleph}$ of cardinality $\leq \aleph^*$. By [5, Lemma 3.26], if $\aleph^* < \aleph$, then $I_3^{\aleph}$ is a dense but not closed subspace of $I^{\aleph}$.

[5, Lemma 3.15 of] goes as follows.

LEMMA 3.2. *For each uncountable cardinal $\aleph$ and for each $i \in \{0, 1, 2, 3\}$, $X_i(\aleph)$ is arcwise connected.*

Taking into account [5, Propositions 3.23, 3.22, and 3.28] and Corollary 2.6, we get the next set of information.

PROPOSITION 3.3. *For each uncountable cardinal $\aleph$, the spaces $X_0(\aleph)$, $X_1(\aleph)$, and $X_2(\aleph)$ are not compact, and if $\aleph^* < \aleph$, then $X_3(\aleph)$ is not compact.*

Since pseudo-compactness is preserved by surjective maps, then we can obtain facts that were not revealed in [5]. First, we state [5, Theorem 3.30].

THEOREM 3.4. *Let $\{\aleph_i \,|\, i \in \Gamma\}$ be a nonempty indexed set of uncountable cardinals and for each $i \in \Gamma$ let Y_i be a space. Suppose that for each $i \in \Gamma$, there is $j(i) \in \{0, 2, 3\}$ such that $Y_i = X_{j(i)}(\aleph_i)$ and if $j(i) = 3$, then $\aleph^* < \aleph_i$. Then $\prod\{Y_i \,|\, i \in \Gamma\}$ is Hausdorff, pseudo-compact, and noncompact.*

From Theorem 3.4 we deduce that always $X_0(\aleph)$ and $X_2(\aleph)$ are pseudo-compact and that $X_3(\aleph)$ is pseudo-compact if $\aleph^* < \aleph$. Since $X_0(\aleph)$ maps bijectively to the subspace $I_0^{\aleph}$ of $I^{\aleph}$ and $X_3(\aleph)$ maps bijectively to the subspace $I_3^{\aleph}$ of $I^{\aleph}$, then we obtain the next result.

COROLLARY 3.5. *For each uncountable cardinal $\aleph$,*

1. *$I_0^{\aleph}$ is a non compact, pseudo-compact, arcwise connected subspace of $I^{\aleph}$, and*
2. *if $\aleph^* < \aleph$, then $I_3^{\aleph}$ is a non compact, pseudo-compact, arcwise connected subspace of $I^{\aleph}$.*

Of course the subspaces $I_0^{\aleph}$ and $I_3^{\aleph}$ of $I^{\aleph}$ mentioned in Corollary 3.5 are Tychonoff spaces. Hence we have uncovered significant naturally occurring classes of non compact, pseudo-compact, arcwise connected Tychonoff spaces. Now we can strengthen Theorem 3.4.

THEOREM 3.6. *Let $\{\aleph_i \,|\, i \in \Gamma\}$ be a nonempty indexed set of uncountable cardinals and for each $i \in \Gamma$ let Y_i be a space. Suppose that for each $i \in \Gamma$, there is $j(i) \in \{0, 2, 3\}$ such that,*

1. *if $j(i) = 0$, then $Y_i \in \{X_0(\aleph_i), I_0^{\aleph}\}$,*
2. *if $j(i) = 2$, then $Y_i = X_2(\aleph_i)$, and*
3. *if $j(i) = 3$, then $\aleph^* < \aleph_i$ and $Y_i \in \{X_3(\aleph_i), I_3^{\aleph}\}$.*

Then $Y = \prod\{Y_i \,|\, i \in \Gamma\}$ is Hausdorff, pseudo-compact, arcwise connected, and noncompact. In case $j(i) \in \{0, 3\}$ for all i, $Y_i = I_0^{\aleph}$ when $j(i) = 0$, and $Y_i = I_3^{\aleph}$ when $j(i) = 3$, then in addition Y is a Tychonoff space.

4. The Example $X_0(I^{\aleph})$

In [5], we used $\lambda_0(\aleph) : X_0(\aleph) \to I_0^{\aleph}$ to denote the bijective map that is the identity on sets (see (i) of Section 3). We were not able to detect there whether or not $\lambda_0(\aleph)$ is a homeomorphism. Our next result provides a criterion by which one might determine that it is not.

Theorem 4.1. *Let* $\aleph$ *be an uncountable cardinal, and suppose that* $[0,\Omega]$ *embeds in* $I_0^{\aleph}$. *Then the bijective map* $\lambda_0(\aleph) : X_0(\aleph) \to I_0^{\aleph}$, *which is the identity on sets, is not a homeomorphism.*

Proof. Let $\kappa : [0,\Omega] \to I_0^{\aleph} \subset I^{\aleph}$ be an embedding. The topology of $X_0(\aleph)$ is weakly generated by compact subsets Z of $I_0^{\aleph} \subset I^{\aleph}$ that have countable dense subsets. For each such Z, apply Lemma 2.3 to see that $Z \cap \kappa([0,\Omega))$ is closed in Z. This shows that $\kappa([0,\Omega))$ is closed in $X_0(\aleph)$; but it is not closed in $I_0^{\aleph}$. ∎

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