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DYNAMIC PROPERTIES FOR THE INDUCED MAPS ON n-FOLD SYMMETRIC PRODUCT SUSPENSIONS

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Abstract. Let X be a continuum. For any positive integer n we consider the hyperspace $\mathcal{F}_n(X)$ and if n is greater than or equal to two, we consider the quotient space $\mathcal{SF}_n(X)$ defined in [3]. For a given map $f : X \to X$, we consider the induced maps $\mathcal{F}_n(f) : \mathcal{F}_n(X) \to \mathcal{F}_n(X)$ and $\mathcal{SF}_n(f) : \mathcal{SF}_n(X) \to \mathcal{SF}_n(X)$ defined in [4]. Let $\mathcal{M}$ be one of the following classes of maps: exact, mixing, weakly mixing, transitive, totally transitive, strongly transitive, chaotic, minimal, irreducible, feebly open and turbulent. In this paper we study the relationships between the following statements: $f \in \mathcal{M}$, $\mathcal{F}_n(f) \in \mathcal{M}$ and $\mathcal{SF}_n(f) \in \mathcal{M}$.

1. Introduction

A continuum is a nonempty compact connected metric space. Given a continuum X and a positive integer n, we consider the hyperspaces 2^X, $\mathcal{C}_n(X)$ and $\mathcal{F}_n(X)$ of X, topologized with the Hausdorff metric. We recall that 2^X consists of all nonempty and closed subsets of X, $\mathcal{C}_n(X)$ consists of all elements of 2^X with at most n components and $\mathcal{F}_n(X)$ consists of all elements of 2^X with at most n points. If n is an integer greater than or equal to two, by $\mathcal{SF}_n(X)$ we mean the quotient space $\mathcal{F}_n(X)/\mathcal{F}_1(X)$. The space $\mathcal{SF}_n(X)$ is called the n-fold symmetric product suspension of the continuum X. Some topological properties of $\mathcal{SF}_n(X)$ are studied in [3] and [5].

A map $f\colon X \to X$, where X is a continuum, induces a map on the hyperspace 2^X denoted by $2^f\colon 2^X \to 2^X$ and defined by $2^f(A) = f(A)$, for each $A \in 2^X$. The induced map to the other hyperspaces mentioned are

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simply the restriction of 2^f to each of such hyperspaces, denoted by $\mathcal{C}_n(f)$ and $\mathcal{F}_n(f)$, respectively, for each positive integer n. If n is an integer greater than or equal to two, we consider the induced map $\mathcal{SF}_n(f) : \mathcal{SF}_n(X) \to \mathcal{SF}_n(X)$, which is called induced map of f on the n-fold symmetric product suspension of X. Some topological properties of $\mathcal{SF}_n(f)$ are studied in [4] and [6].

A dynamical system is a pair (X, f), where X is a continuum and $f : X \to X$ is a map. The dynamical system (X, f) induce the dynamical systems $(2^X, 2^f)$, $(\mathcal{C}_n(X), \mathcal{C}_n(f))$ and $(\mathcal{F}_n(X), \mathcal{F}_n(f))$. Because dynamics is obtained by iterating the map, it is important to study the dynamical properties of the map. Hence, in recent times, a natural problem has been to study connections between dynamical properties of f (individual dynamics) and dynamical properties of induced maps (collective dynamics). Some dynamical properties of the induced maps 2^f, $\mathcal{C}_n(f)$ and $\mathcal{F}_n(f)$, and others set-valued maps are studied, for instance in [1, 2, 7, 8, 10–12, 16, 20, 21, 23].

In this paper, we introduce the dynamical system $(\mathcal{SF}_n(X), \mathcal{SF}_n(f))$ and we investigate connections between dynamical properties of f and the dynamical properties of the induced maps $\mathcal{F}_n(f)$ and $\mathcal{SF}_n(f)$. Specifically, if $\mathcal{M}$ is one of the following classes of maps: exact, mixing, weakly mixing, transitive, totally transitive, strongly transitive, chaotic, minimal, irreducible, feebly open and turbulent, we study the relationships between the following statements:

(1) $f \in \mathcal{M}$;
(2) $\mathcal{F}_n(f) \in \mathcal{M}$;
(3) $\mathcal{SF}_n(f) \in \mathcal{M}$.

This paper is organized as follows: In Section 2, we recall basic definitions and introduce some notation. In Section 3, we present some preliminary results needed for the rest of the paper. In particular, we prove results respect to quotient spaces. Section 4 is devoted to study the problem posed if $\mathcal{M}$ is one of the following classes of maps: exact, mixing, weakly mixing, transitive, totally transitive, strongly transitive, chaotic and minimal. Finally, in Section 5, we investigate our problem when $\mathcal{M}$ is one of the following classes of maps: irreducible, feebly open and turbulent.

2. Definitions and notations

A *continuum* is a nonempty compact connected metric space. A continuum is said to be *nondegenerate* if it has more than one point. A *subcontinuum* is a continuum contained in a topological space. Given a continuum X, a point $a \in X$ and $\epsilon > 0$, $V_\epsilon(a)$ denotes the open ball with center a and radius ϵ. A *map* is a continuous function. We denote by Id_X the identity map on a continuum X. A *dynamical system* is a pair (X, f), where X is a continuum and $f : X \to X$ is a map.

The symbol $\mathbb{N}$ denote the set of positive integers. Given a dynamical system (X, f), define $f^0 = Id_X$ and for each $k \in \mathbb{N}$, let $f^k = f \circ f^{k-1}$. A point $p \in X$ is a *periodic point* in (X, f) provided that there exists $k \in \mathbb{N}$ such that $f^k(p) = p$. The set of periodic points of (X, f) is denoted by $\mathrm{per}(f)$. Given $x \in X$, the *orbit of x under f* is the set $\mathrm{orb}(x, f) = \{f^k(x) \mid k \in \mathbb{N} \cup \{0\}\}$. A subset K of X is said to be *invariant under f* if $f(K) \subseteq K$ and *strongly invariant under f* if $f(K) = K$.

Let X be a continuum with metric d and let $f : X \to X$ be a map. We say that f is:

- *exact* if for each nonempty open subset U of X, there exists $k \in \mathbb{N}$ such that $f^k(U) = X$;
- *mixing* if for every pair of nonempty open subsets U and V of X, there exists $N \in \mathbb{N}$ such that $f^k(U) \cap V \neq \emptyset$, for every $k \geq N$;
- *weakly mixing* if for all nonempty open subsets U_1, U_2, V_1 and V_2 of X, there exists $k \in \mathbb{N}$ such that $f^k(U_i) \cap V_i \neq \emptyset$, for each $i \in \{1, 2\}$;
- *transitive* if for every pair of nonempty open subsets U and V of X, there exists $k \in \mathbb{N}$ such that $f^k(U) \cap V \neq \emptyset$;
- *totally transitive* if f^s is transitive, for all $s \in \mathbb{N}$;
- *strongly transitive* if for each nonempty open subset U of X, there exists $s \in \mathbb{N}$ such that $X = \bigcup_{k=0}^{s} f^k(U)$;
- *chaotic* if it is transitive and $\mathrm{per}(f)$ is dense in X;
- *minimal* if there is no proper subset $M \subseteq X$ which is nonempty, closed and M is invariant under f; equivalently, if the orbit of every point of X is dense in X;
- *irreducible* if the only closed subset $A \subseteq X$ for which $f(A) = X$ is $A = X$;
- *feebly open* if for every nonempty open subset U of X, there is a nonempty open subset V of X such that $V \subseteq f(U)$;
- *turbulent* if there are compact nondegenerate subsets C and K of X such that $C \cap K$ has at most a point and $K \cup C \subseteq f(K) \cap f(C)$;
- *isometry* if $d(x, y) = d(f(x), f(y))$, for each $x, y \in X$.

Diagram 1 shows inclusions between some classes of maps, which are considered here, an arrow means inclusion; i.e., the class of maps above is contained in the class of maps below. For some of this inclusions see, for instance, [14] and [15].

Given a continuum X and a positive integer n, we consider the following hyperspaces of X:

$$2^X = \{A \subseteq X \mid A \text{ is closed and nonempty}\};$$
$$\mathcal{C}_n(X) = \{A \in 2^X \mid A \text{ has at most } n \text{ components}\};$$
$$\mathcal{F}_n(X) = \{A \in 2^X \mid A \text{ has at most } n \text{ points}\}.$$

We topologize these sets with the Hausdorff metric ([18, (0.1)]). The hyperspace $\mathcal{C}_n(X)$ is the *n-fold hyperspace of* X and the hyperspace $\mathcal{F}_n(X)$ is the *n-fold symmetric product* of X.

$$\begin{tikzcd}
 & \text{Exact} \arrow[d] & & \\
 & \text{Mixing} \arrow[dl] \arrow[d] & \text{Minimal} \arrow[d] \arrow[dr] & \\
\text{Totally transitive} \arrow[dr] & \text{Weakly mixing} \arrow[d] & \text{Strongly transitive} \arrow[dl] & \text{Irreducible} \arrow[d] \arrow[ddll] \\
\text{Chaotic} \arrow[r] & \text{Transitive} \arrow[d] & & \text{Feebly open} \\
 & \text{Surjective} & &
\end{tikzcd}$$

DIAGRAM 1

Given a finite collection $U_1, U_2, \ldots, U_m$ of nonempty subsets of X, we denote by $\langle U_1, U_2, \ldots, U_m \rangle$ the following subset of 2^X:

$$\left\{ A \in 2^X \mid A \subseteq \bigcup_{i=1}^{m} U_i \text{ and } A \cap U_i \neq \emptyset, \text{ for each } i \in \{1, 2, \ldots, m\} \right\}.$$

The family:

$$\{\langle U_1, U_2, \ldots, U_l \rangle \mid l \in \mathbb{N} \text{ and } U_1, U_2, \ldots, U_l \text{ are open subsets of } X\}$$

forms a base for a topology on 2^X called the *Vietoris topology* ([18, (0.11)]). It is well known that the Vietoris topology and the topology induced by the Hausdorff metric coincide ([18, (0.13)]). For those who are interested in learning more about this topics can see [13, 17] and [19].

NOTATION 2.1. *Let X be a continuum, let n be a positive integer, and let $U_1, U_2, \ldots, U_m$ be a finite family of open subsets of X. Then $\langle U_1, U_2, \ldots, U_m \rangle_n$ denotes the set $\langle U_1, U_2, \ldots, U_m \rangle \cap \mathcal{F}_n(X)$.*

Let n be an integer greater than or equal to two. Then the *n-fold symmetric product suspension* ([3]) of a continuum X, denoted by $\mathcal{SF}_n(X)$, is the quotient space $\mathcal{F}_n(X)/\mathcal{F}_1(X)$, with the quotient topology. Here, we denote the quotient map by $q : \mathcal{F}_n(X) \to \mathcal{SF}_n(X)$ and $q(\mathcal{F}_1(X))$ by F_X. Thus,

$$\mathcal{SF}_n(X) = \{\{A\} \mid A \in \mathcal{F}_n(X) \setminus \mathcal{F}_1(X)\} \cup \{F_X\}.$$

REMARK 2.2. The space $\mathcal{SF}_n(X) \setminus \{F_X\}$ is homeomorphic to $\mathcal{F}_n(X) \setminus \mathcal{F}_1(X)$, using the appropriate restriction of q.

Let n be a positive integer and let X be a continuum. If $f : X \to X$ is a map, we consider the function $\mathcal{F}_n(f) : \mathcal{F}_n(X) \to \mathcal{F}_n(X)$ defined by $\mathcal{F}_n(f)(A) = f(A)$, for all $A \in \mathcal{F}_n(X)$; it is called the *induced map* of f on the n-fold symmetric product of X. Note that $\mathcal{F}_n(f)$ is continuous ([17, 1.8.23]). Also, if n is greater than or equal to two, we consider the function $\mathcal{SF}_n(f) : \mathcal{SF}_n(X) \to \mathcal{SF}_n(X)$ given by

$$\mathcal{SF}_n(f)(\chi) = \begin{cases} q(\mathcal{F}_n(f)(q^{-1}(\chi))), & \text{if } \chi \neq F_X; \\ F_X, & \text{if } \chi = F_X. \end{cases}$$

Note that, by [9, 4.3, p. 126], $\mathcal{SF}_n(f)$ is continuous, it is called *induced map* of f on n-fold symmetric product suspension (see [4] and [6]). In addition, the diagram:

$$\begin{tikzcd} \mathcal{F}_n(X) \arrow[r, "\mathcal{F}_n(f)"] \arrow[d, "q"'] & \mathcal{F}_n(X) \arrow[d, "q"] \\ \mathcal{SF}_n(X) \arrow[r, "\mathcal{SF}_n(f)"'] & \mathcal{SF}_n(X) \end{tikzcd} \qquad (*)$$

is commutative, that is $q \circ \mathcal{F}_n(f) = \mathcal{SF}_n(f) \circ q$.

As a consequence of Diagram 1 and [4, Theorem 3.2], we obtain:

Lemma 2.3. *Let X be a continuum, let n be an integer greater than or equal to two, and let $f : X \to X$ be a map. Let $\mathcal{M}$ be one of the following classes of maps: exact, mixing, weakly mixing, transitive, totally transitive, strongly transitive, chaotic, minimal and irreducible. If $f \in \mathcal{M}$, then $f, \mathcal{F}_n(f)$ and $\mathcal{SF}_n(f)$ are surjective.*

3. Preliminary results

Let X be a continuum, let $f : X \to X$ be a surjective map, and let K be a subcontinuum of X such that K is strongly invariant under f. Consider the quotient space X/K and let $q_X : X \to X/K$ be the quotient map. We denote $q_X(K)$ by K_X. Note that f induces a function $f_\star : X/K \to X/K$ ([9, 7.7, p. 17]) given by

$$f_\star(\chi) = \begin{cases} q_X(f((q_X)^{-1}(\chi))), & \text{if } \chi \neq K_X; \\ K_X, & \text{if } \chi = K_X. \end{cases}$$

The continuity of $f_\star$ follows from [9, 4.3, p. 126]. Observe that the diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "q_X"'] & X \arrow[d, "q_X"] \\ X/K \arrow[r, "f_\star"'] & X/K \end{tikzcd} \qquad (**)$$

is commutative, that is $q_X \circ f = f_\star \circ q_X$.

REMARK 3.1. Let X be a continuum, let $f : X \to X$ be a surjective map, and let K be a subcontinuum of X such that K is strongly invariant under f. It follows that the dynamical system (X, f) induces the dynamical system $(X/K, f_\star)$.

As an easy consequence from the definition of f^k and from commutativity of $(**)$, we have the following:

PROPOSITION 3.2. *Let X be a continuum, let $f : X \to X$ be a surjective map, let K be a subcontinuum of X such that K is strongly invariant under f, and let $k, s \in \mathbb{N}$. Then the following holds:*

(a) $q_X \circ f^k = (f_\star)^k \circ q_X$,
(b) $q_X \circ f^k = (f^k)_\star \circ q_X$,
(c) $f^k \circ f^s = f^{k+s}$,
(d) $(f^s)^k = f^{sk}$,
(e) $q_X \circ (f^s)^k = ((f_\star)^s)^k \circ q_X$,
(f) $q_X \circ (f^s)^k = ((f^s)_\star)^k \circ q_X$.

LEMMA 3.3. *Let X be a continuum, let $f : X \to X$ be a surjective map, and let K be a subcontinuum of X such that K is strongly invariant under f. If per(f) is dense in X, then per$(f_\star)$ is dense in X/K.*

PROOF. Let $\mathcal{U}$ be a nonempty open subset of X/K. Since q_X is continuous, $q_X^{-1}(\mathcal{U})$ is a nonempty open subset of X. Since $\mathrm{per}(f)$ is dense in X, we have that $q_X^{-1}(\mathcal{U}) \cap \mathrm{per}(f) \neq \emptyset$. Hence, there exists $a \in q_X^{-1}(\mathcal{U})$ and there exists $k \in \mathbb{N}$ such that $f^k(a) = a$. Thus,

$$q_X(a) \in \mathcal{U} \quad \text{and} \quad q_X(f^k(a)) = q_X(a).$$

By Proposition 3.2 (a), it follows that

$$q_X(a) \in \mathcal{U} \quad \text{and} \quad (f_\star)^k(q_X(a)) = q_X(a).$$

This implies that $\mathcal{U} \cap \mathrm{per}(f_\star) \neq \emptyset$. Thus, we conclude that $\mathrm{per}(f_\star)$ is dense in X/K. ∎

THEOREM 3.4. *Let X be a continuum, let $f : X \to X$ be a map, and let K be a subcontinuum of X such that K is strongly invariant under f. Let $\mathcal{M}$ be one of the following classes of maps: exact, mixing, weakly mixing, transitive, totally transitive, strongly transitive, chaotic and minimal. If $f \in \mathcal{M}$, then $f_\star \in \mathcal{M}$.*

PROOF. Suppose that f is exact, we see that $f_\star$ is exact. For this, let $\mathcal{U}$ be a nonempty open subset of X/K. Since q_X is continuous, we have that $q_X^{-1}(\mathcal{U})$ is a nonempty open subset of X. Since f is exact, there exists $k \in \mathbb{N}$ such that $f^k(q_X^{-1}(\mathcal{U})) = X$. Thus, since q_X is surjective, we have that $q_X(f^k(q_X^{-1}(\mathcal{U}))) = X/K$. Hence, by Proposition 3.2 (a), we obtain that $(f_\star)^k(q_X((q_X^{-1}(\mathcal{U}))) = X/K$. This implies that $(f_\star)^k(\mathcal{U}) = X/K$. Therefore, $f_\star$ is exact.

Assume that f is mixing, we prove that $f_\star$ is mixing. Let $\mathcal{U}$ and $\mathcal{V}$ be nonempty open subsets of X/K. Since q_X is continuous, it follows that $q_X^{-1}(\mathcal{U})$ and $q_X^{-1}(\mathcal{V})$ are nonempty open subsets of X. Since f is mixing, there exists $N \in \mathbb{N}$ such that

$$f^k(q_X^{-1}(\mathcal{U})) \cap q_X^{-1}(\mathcal{V}) \neq \emptyset, \text{ for each } k \geq N.$$

Let $k \geq N$. We have that there exists $a \in q_X^{-1}(\mathcal{U})$ such that $f^k(a) \in q_X^{-1}(\mathcal{V})$. This implies that $q_X(a) \in \mathcal{U}$ and $q_X(f^k(a)) \in \mathcal{V}$. Hence, by Proposition 3.2 (a), we obtain that $q_X(a) \in \mathcal{U}$ and $(f_\star)^k(q_X(a)) \in \mathcal{V}$. In consequence, $(f_\star)^k(q_X(a)) \in (f_\star)^k(\mathcal{U})$ and $(f_\star)^k(q_X(a)) \in \mathcal{V}$. Thus, $(f_\star)^k(\mathcal{U}) \cap \mathcal{V} \neq \emptyset$. Therefore, $f_\star$ is mixing.

To verify that if f is weakly mixing, then $f_\star$ is weakly mixing, we use a similar argument to the proof given in the previous paragraph.

Suppose that f is transitive, we prove that $f_\star$ is transitive. Let $\mathcal{U}$ and $\mathcal{V}$ be nonempty open subsets of X/K. Since q_X is continuous, $q_X^{-1}(\mathcal{U})$ and $q_X^{-1}(\mathcal{V})$ are nonempty open subsets in X. Since f is transitive, there exists $k \in \mathbb{N}$ such that $f^k(q_X^{-1}(\mathcal{U})) \cap q_X^{-1}(\mathcal{V}) \neq \emptyset$. Hence, there exists $a \in q_X^{-1}(\mathcal{U})$ such that $f^k(a) \in q_X^{-1}(\mathcal{V})$. It follows that, $q_X(a) \in \mathcal{U}$ and $q_X(f^k(a)) \in \mathcal{V}$. By Proposition 3.2 (a), we have that $q_X(a) \in \mathcal{U}$ and $(f_\star)^k(q_X(a)) \in \mathcal{V}$. In consequence, $(f_\star)^k(q_X(a)) \in (f_\star)^k(\mathcal{U})$ and $(f_\star)^k(q_X(a)) \in \mathcal{V}$. Thus, $(f_\star)^k(\mathcal{U}) \cap \mathcal{V} \neq \emptyset$. Therefore, $f_\star$ is transitive.

Assume that f is totally transitive, we prove that $f_\star$ is totally transitive. For this, let $s \in \mathbb{N}$ and let $\mathcal{U}$ and $\mathcal{V}$ be nonempty open subsets in X/K. Since q_X is continuous, we have that $q_X^{-1}(\mathcal{U})$ and $q_X^{-1}(\mathcal{V})$ are nonempty open subsets in X. Since f is totally transitive, f^s is transitive. In consequence, there exists $k \in \mathbb{N}$ such that $(f^s)^k(q_X^{-1}(\mathcal{U})) \cap q_X^{-1}(\mathcal{V}) \neq \emptyset$. Thus, there exists $a \in q_X^{-1}(\mathcal{U})$ such that $(f^s)^k(a) \in q_X^{-1}(\mathcal{V})$. It follows that $q_X(a) \in \mathcal{U}$ and $q_X((f^s)^k(a)) \in \mathcal{V}$. Hence, by Proposition 3.2 (e), $q_X(a) \in \mathcal{U}$ and $((f_\star)^s)^k(q_X(a)) \in \mathcal{V}$. This implies that:

$$((f_\star)^s)^k(q_X(a)) \in ((f_\star)^s)^k(\mathcal{U}) \text{ and } ((f_\star)^s)^k(q_X(a))) \in \mathcal{V}.$$

Thus, $((f_\star)^s)^k(\mathcal{U}) \cap \mathcal{V} \neq \emptyset$. This proves that $(f_\star)^s$ is transitive. Therefore, $f_\star$ is totally transitive.

Next, suppose that f is strongly transitive, we see that $f_\star$ is strongly transitive. Let $\mathcal{U}$ be a nonempty open subset of X/K. By continuity of q_X, we have that $q_X^{-1}(\mathcal{U})$ is a nonempty open subset of X. Since f is strongly transitive, there exists $s \in \mathbb{N}$ such that $X = \bigcup_{k=0}^{s} f^k(q_X^{-1}(\mathcal{U}))$. Hence, since q_X is surjective, we have that:

$$X/K = \bigcup_{k=0}^{s} q_X(f^k(q_X^{-1}(\mathcal{U}))).$$

So, by Proposition 3.2 (a) and since q_X is surjective, we have that:

$$X/K = \bigcup_{k=0}^{s} (f_\star)^k(\mathcal{U}).$$

Therefore, $f_\star$ is strongly transitive.

Now, assume that f is chaotic. From the previous paragraph, we have that $f_\star$ is transitive. Furthermore, by Lemma 3.3, we have that $\mathrm{per}(f_\star)$ is dense in X/K. This proves that $f_\star$ is chaotic.

Finally, suppose that f is minimal, we prove that $f_\star$ is minimal. Let $\chi \in X/K$ and let $\mathcal{U}$ be a nonempty open subset of X/K. Let $x \in X$ such that $q_X(x) = \chi$. Since f is minimal, $q_X^{-1}(\mathcal{U}) \cap \mathrm{orb}(x, f) \neq \emptyset$. Hence, there exist $w \in q_X^{-1}(\mathcal{U})$ and $k \in \mathbb{N} \cup \{0\}$ such that $w = f^k(x)$. Thus, $q_X(w) \in \mathcal{U}$ and $q_X(w) = q_X(f^k(x))$. By Proposition 3.2 (a), $q_X(w) \in \mathcal{U}$ and $q_X(w) = (f_\star)^k(q_X(x))$. Hence, $q_X(w) \in \mathcal{U}$ and $q_X(w) \in \mathrm{orb}(q_X(x), f_\star)$. It follows that, $\mathcal{U} \cap \mathrm{orb}(\chi, f_\star) \neq \emptyset$. Therefore, $f_\star$ is minimal. ∎

4. Dynamical properties related to transitivity

Let X be a continuum, let n be an integer greater than or equal to two, and let $f : X \to X$ be a map. Observe that $\mathcal{F}_1(X)$ is a subcontinuum of $\mathcal{F}_n(X)$ such that $\mathcal{F}_1(X)$ is strongly invariant under $\mathcal{F}_n(f)$. By Remark 3.1, we can consider the dynamical system $(\mathcal{SF}_n(X), \mathcal{SF}_n(f))$.

PROPOSITION 4.1. *Let X be a continuum, let n be an integer greater than or equal to two, and let $f : X \to X$ be a map. Then, for each $k, s \in \mathbb{N}$, the following holds:*

(a) $(\mathcal{F}_n(f))^k(A) = f^k(A)$, *for every* $A \in \mathcal{F}_n(X)$,
(b) $q \circ (\mathcal{F}_n(f))^k = (\mathcal{SF}_n(f))^k \circ q$,
(c) $((\mathcal{F}_n(f))^s)^k = (\mathcal{F}_n(f))^{sk}$,
(d) $q \circ ((\mathcal{F}_n(f))^s)^k = ((\mathcal{SF}_n(f))^s)^k \circ q$.

PROOF. Part (a) follows directly from the definition of $\mathcal{F}_n(f)$, and parts (b), (c) and (d) follow from Proposition 3.2. ∎

REMARK 4.2. Let (X,d) be a continuum, let $f : X \to X$ be a map, and let $k \in \mathbb{N}$. If f is an isometry, then for any $x, y \in X$, $d(x,y) = d(f^k(x), f^k(y))$.

THEOREM 4.3. *Let (X,d) be a nondegenerate continuum, let n be an integer greater than or equal to two, and let $f : X \to X$ be a map. If f is an isometry, then $\mathcal{SF}_n(f)$ is not transitive.*

PROOF. Suppose that f is an isometry and that $\mathcal{SF}_n(f)$ is transitive. Let $x, y \in X$ such that $x \neq y$. Let $r = d(x,y) > 0$. We define $U_1 = V_{\frac{r}{4}}(x)$ and $U_2 = V_{\frac{r}{4}}(y)$. We consider V_1 and V_2 nonempty open subsets of X such that $V_1 \cup V_2 \subseteq U_1$ and $V_1 \cap V_2 = \emptyset$. It follows that $\langle U_1, U_2\rangle_n$ and $\langle V_1, V_2\rangle_n$ are nonempty open subsets of $\mathcal{F}_n(X)$ such that $\langle U_1, U_2\rangle_n \cap \mathcal{F}_1(X) = \emptyset$ and $\langle V_1, V_2\rangle_n \cap \mathcal{F}_1(X) = \emptyset$. By Remark 2.2, we have that $q(\langle U_1, U_2\rangle_n)$ and $q(\langle V_1, V_2\rangle_n)$ are nonempty open subsets of $\mathcal{SF}_n(X)$. Since $\mathcal{SF}_n(f)$ is transitive, there exists $k \in \mathbb{N}$ such that:

$$(\mathcal{SF}_n(f))^k(q(\langle U_1, U_2\rangle_n)) \cap q(\langle V_1, V_2\rangle_n) \neq \emptyset.$$

By Proposition 4.1 (b), we obtain that:

$$q((\mathcal{F}_n(f))^k(\langle U_1, U_2\rangle_n)) \cap q(\langle V_1, V_2\rangle_n) \neq \emptyset.$$

Let $B \in (\mathcal{F}_n(f))^k(\langle U_1, U_2\rangle_n)$ such that $q(B) \in q(\langle V_1, V_2\rangle_n)$. We consider $A \in \langle V_1, V_2\rangle_n$ such that $q(A) = q(B)$. By Remark 2.2, we have that $A = B$. Let $C \in \langle U_1, U_2\rangle_n$ such that $(\mathcal{F}_n(f))^k(C) = B$. Thus, $(\mathcal{F}_n(f))^k(C) = A$. By Proposition 4.1 (a), $f^k(C) = A$. Let $c_1 \in C \cap U_1$ and $c_2 \in C \cap U_2$. Hence, $d(x,y) \leq d(x,c_1) + d(c_1,c_2) + d(c_2,y) < \frac{r}{2} + d(c_1,c_2)$. This implies that $\frac{r}{2} < d(c_1,c_2)$. On the other hand, $f^k(c_1), f^k(c_2) \in f^k(C) \subseteq V_1 \cup V_2 \subseteq U_1$. Thus, $d(f^k(c_1), f^k(c_2)) \leq \frac{r}{2}$. In consequence, $d(f^k(c_1), f^k(c_2)) < d(c_1,c_2)$, which is a contradiction (see Remark 4.2). Therefore, we conclude that $\mathcal{SF}_n(f)$ is not transitive. □

By Theorem 4.3 and Diagram 1, we obtain:

THEOREM 4.4. *Let X be a nondegenerate continuum, let n be an integer greater than or equal to two, and let $f : X \to X$ be a map. Let $\mathcal{M}$ be one of the following classes of maps: exact, mixing, weakly mixing, transitive, totally transitive, strongly transitive, chaotic and minimal. If f is an isometry, then $\mathcal{SF}_n(f) \notin \mathcal{M}$.*

Our next result follows from Theorem 3.4 and Theorem 4.4.

THEOREM 4.5. *Let X be a nondegenerate continuum, let n be an integer greater than or equal to two, and let $f : X \to X$ be a map. Let $\mathcal{M}$ be one of the following classes of maps: exact, mixing, weakly mixing, transitive, totally transitive, strongly transitive, chaotic and minimal. If f is an isometry, then $\mathcal{F}_n(f) \notin \mathcal{M}$.*

Let $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ denote the set of rational numbers, real numbers and complex numbers, respectively. We denote by S^1 the set $\{e^{2\pi i\theta} \in \mathbb{C} \mid \theta \in [0,1]\}$.

EXAMPLE 4.6. Let $f : S^1 \to S^1$ be the map defined by $f(e^{2\pi i\theta}) = e^{2\pi i(\theta+\alpha)}$, where $\alpha \in \mathbb{R}\setminus\mathbb{Q}$. Let $\mathcal{M}$ be one of the following classes of maps: exact, mixing, weakly mixing, transitive, totally transitive, strongly transitive, chaotic and minimal. It is know that f is an isometry. Thus, by Theorem 4.4, we have that $\mathcal{SF}_n(f) \notin \mathcal{M}$. Moreover, by Theorem 4.5, we obtain that $\mathcal{F}_n(f) \notin \mathcal{M}$. However, we have that f is transitive, totally transitive, strongly transitive and minimal (see [22, p. 261]).

The following result includes [10, Theorem 13] with an alternative proof.

THEOREM 4.7. *Let X be a nondegenerate continuum, let n be an integer greater than or equal to two, and let $f : X \to X$ be a map. Then the following are equivalent:*

(1) *f is exact;*
(2) *$\mathcal{F}_n(f)$ is exact;*
(3) *$\mathcal{SF}_n(f)$ is exact.*

PROOF. Suppose that f is exact, we prove that $\mathcal{F}_n(f)$ is exact. Let $\mathcal{U}$ be a nonempty open subset of $\mathcal{F}_n(X)$. We see that $(\mathcal{F}_n(f))^k(\mathcal{U}) = \mathcal{F}_n(X)$, for some $k \in \mathbb{N}$. By [11, Lemma 4.2], there exist nonempty open subsets $U_1, U_2, \dots, U_n$ of X such that $\langle U_1, U_2, \dots, U_n\rangle_n \subseteq \mathcal{U}$. Thus, it is enough verify that there exists $k \in \mathbb{N}$ such that

$$(\mathcal{F}_n(f))^k(\langle U_1, U_2, \dots, U_n\rangle_n) = \mathcal{F}_n(X).$$

Since f is exact, for each $i \in \{1, 2, \dots, n\}$ there exists $k_i \in \mathbb{N}$ such that $f^{k_i}(U_i) = X$. We define $k = \max\{k_1, k_2, \dots, k_n\}$. Note that, for each $i \in \{1, 2, \dots, n\}$, $f^k(U_i) = X$.

Let $B \in \mathcal{F}_n(X)$. We put $B = \{b_1, b_2, \dots, b_r\}$ with $r \leq n$. Define $C = \{b_1, b_2, \dots, b_r, b_{r+1}, \dots, b_n\}$, where $b_r = b_{r+1} = \cdots = b_n$. In consequence, for each $i \in \{1, 2, \dots, n\}$, $b_i \in f^k(U_i)$. Hence, for each $i \in \{1, 2, \dots, n\}$, let $a_i \in U_i$ such that $f^k(a_i) = b_i$. Define $A = \{a_1, a_2, \dots, a_n\}$. It follows that $A \in \langle U_1, U_2, \dots, U_n\rangle_n$ and $(\mathcal{F}_n(f))^k(A) = C = B$. Thus, we obtain that $B \in (\mathcal{F}_n(f))^k(\langle U_1, U_2, \dots, U_n\rangle_n)$. This implies that $\mathcal{F}_n(X) \subseteq (\mathcal{F}_n(f))^k(\langle U_1, U_2, \dots, U_n\rangle_n)$. Thus, $(\mathcal{F}_n(f))^k(\langle U_1, U_2, \dots, U_n\rangle_n) = \mathcal{F}_n(X)$. Therefore, $\mathcal{F}_n(f)$ is exact.

It follows from Theorem 3.4 that if $\mathcal{F}_n(f)$ is exact, then $\mathcal{SF}_n(f)$ is exact.

Finally, assume that $\mathcal{SF}_n(f)$ is exact, we see that f is exact. For this end, let U be a nonempty open subset of X. We consider two open subsets U_1 and U_2 of X such that $U_1 \cup U_2 \subseteq U$ and $U_1 \cap U_2 = \emptyset$. It follows that $\langle U_1, U_2\rangle_n$ is a nonempty open subset of $\mathcal{F}_n(X)$ such that $\langle U_1, U_2\rangle_n \cap \mathcal{F}_1(X) = \emptyset$. Thus, by Remark 2.2, we have that $q(\langle U_1, U_2\rangle_n)$ is a nonempty open subset of $\mathcal{SF}_n(X)$. Since $\mathcal{SF}_n(f)$ is exact, there exists $k \in \mathbb{N}$ such that

$(\mathcal{SF}_n(f))^k(q(\langle U_1, U_2\rangle_n)) = \mathcal{SF}_n(X)$. By Proposition 4.1 (b), we obtain that $q((\mathcal{F}_n(f))^k(\langle U_1, U_2\rangle_n)) = \mathcal{SF}_n(X)$.

Now, let $x \in X$. We consider a point $y \in X$ such that $x \neq y$ and define $A = \{x, y\}$. Note that $A \in \mathcal{F}_n(X) \setminus \mathcal{F}_1(X)$. Thus, $q(A) \neq F_X$. Since $q(A) \in \mathcal{SF}_n(X)$, there exists $B \in (\mathcal{F}_n(f))^k(\langle U_1, U_2\rangle_n)$ such that $q(B) = q(A)$. Hence, by Remark 2.2, we obtain that $B = A$. Let $C \in \langle U_1, U_2\rangle_n$ such that $(\mathcal{F}_n(f))^k(C) = B$. By Proposition 4.1 (a), we deduce that $f^k(C) = B$. Since $A = B$ and $C \subseteq U$, it follows that $x \in f^k(U)$. Thus, $X \subseteq f^k(U)$. In consequence, $f^k(U) = X$. Therefore, f is exact. ∎

Our next result follows from Theorem 4.7 and Diagram 1 (compare with [7, Proposition 3.3]).

Corollary 4.8. *Let X be a nondegenerate continuum, let n be an integer greater than or equal to two, and let $f : X \to X$ be a map. Then the following statements hold.*

(1) *If f is exact, then $\mathcal{F}_n(f)$ is transitive.*
(2) *If f is exact, then $\mathcal{SF}_n(f)$ is transitive.*

Theorem 4.9. *Let X be a continuum, let n be an integer greater than or equal to two, and let $f : X \to X$ be a map. Then the following are equivalent:*

(1) *f is mixing;*
(2) *$\mathcal{F}_n(f)$ is mixing;*
(3) *$\mathcal{SF}_n(f)$ is mixing.*

Proof. As a consequence from [11, Theorem 4.3], we have that (1) implies (2); and by Theorem 3.4, it follows that (2) implies (3).

Finally, suppose that $\mathcal{SF}_n(f)$ is mixing, we see that f is mixing. Let U and V be nonempty open subsets of X. We consider nonempty open subsets U_1, U_2, V_1 and V_2 of X such that $U_1 \cup U_2 \subseteq U$, $V_1 \cup V_2 \subseteq V$, $U_1 \cap U_2 = \emptyset$ and $V_1 \cap V_2 = \emptyset$. Hence, $\langle U_1, U_2\rangle_n$ and $\langle V_1, V_2\rangle_n$ are nonempty open subsets of $\mathcal{F}_n(X)$ such that $\langle U_1, U_2\rangle_n \cap \mathcal{F}_1(X) = \emptyset$ and $\langle V_1, V_2\rangle_n \cap \mathcal{F}_1(X) = \emptyset$. By Remark 2.2, $q(\langle U_1, U_2\rangle_n)$ and $q(\langle V_1, V_2\rangle_n)$ are nonempty open subsets of $\mathcal{SF}_n(X)$ such that $F_X \notin q(\langle U_1, U_2\rangle_n)$ and $F_X \notin q(\langle V_1, V_2\rangle_n)$. Since $\mathcal{SF}_n(f)$ is mixing, there exists $N \in \mathbb{N}$ such that for each $k \geq N$, $(\mathcal{SF}(f))^k(q(\langle U_1, U_2\rangle_n)) \cap q(\langle V_1, V_2\rangle_n) \neq \emptyset$. Fix $k \geq N$ and let $\chi \in q(\langle U_1, U_2\rangle_n)$ such that $(\mathcal{SF}_n(f))^k(\chi) \in q(\langle V_1, V_2\rangle_n)$. Let $A \in \langle U_1, U_2\rangle_n$ such that $q(A) = \chi$ and let $B \in \langle V_1, V_2\rangle_n$ such that $(\mathcal{SF}_n(f))^k(\chi) = q(B)$. It follows that $(\mathcal{SF}_n(f))^k(q(A)) = q(B)$. In consequence, by Proposition 4.1 (b), we have that $q((\mathcal{F}_n(f))^k(A)) = q(B)$. Hence, by Remark 2.2, $(\mathcal{F}_n(f))^k(A) = B$. Now, by Proposition 4.1 (a), we have that $f^k(A) = B$. Let $a \in A \cap U_1$. Thus, $f^k(a) \in f^k(A)$. Since $A \subseteq U$ and $B \subseteq V$, we have that $a \in U$ and $f^k(a) \in V$. Hence, $f^k(U) \cap V \neq \emptyset$. Therefore, f is mixing. ∎

THEOREM 4.10. *Let X be a nondegenerate continuum, let n be an integer greater than or equal to two, and let $f : X \to X$ be a map. Consider the following statements:*

(1) *f is transitive;*
(2) *$\mathcal{F}_n(f)$ is transitive;*
(3) *$\mathcal{SF}_n(f)$ is transitive.*

Then (2) and (3) are equivalent, (2) implies (1), (3) implies (1), (1) does not imply (2) and (1) does not imply (3).

PROOF. As a consequence from Theorem 3.4, we have that (2) implies (3).

Suppose that $\mathcal{SF}_n(f)$ is transitive, we prove that $\mathcal{F}_n(f)$ is transitive. To this end, let $\mathcal{U}$ and $\mathcal{V}$ be nonempty open subsets of $\mathcal{F}_n(X)$. By [11, Lemma 4.2], there exist nonempty open subsets $U_1, U_2, \ldots, U_n$ and $V_1, V_2, \ldots, V_n$ of X such that

$$\langle U_1, U_2, \ldots, U_n \rangle_n \subseteq \mathcal{U} \text{ and } \langle V_1, V_2, \ldots, V_n \rangle_n \subseteq \mathcal{V}.$$

For each $i \in \{1, 2, \ldots, n\}$, let W_i be a nonempty open subset of X such that $W_i \subseteq U_i$ and for each $i, j \in \{1, 2, \ldots, n\}$, $W_i \cap W_j = \emptyset$, if $i \neq j$. Similarly, for each $i \in \{1, 2, \ldots, n\}$, let O_i be a nonempty open subset of X such that $O_i \subseteq V_i$ and for each $i, j \in \{1, 2, \ldots, n\}$, $O_i \cap O_j = \emptyset$, if $i \neq j$. Note that $\langle U_1, U_2, \ldots, U_n \rangle_n$ and $\langle V_1, V_2, \ldots, V_n \rangle_n$ are nonempty open subsets of $\mathcal{F}_n(X)$ such that $\langle W_1, W_2, \ldots, W_n \rangle_n \subseteq \langle U_1, U_2, \ldots, U_n \rangle_n \subseteq \mathcal{U}$, $\langle O_1, O_2, \ldots, O_n \rangle_n \subseteq \langle V_1, V_2, \ldots, V_n \rangle_n \subseteq \mathcal{V}$, $\langle W_1, W_2, \ldots, W_n \rangle_n \cap \mathcal{F}_1(X) = \emptyset$ and $\langle O_1, O_2, \ldots, O_n \rangle_n \cap \mathcal{F}_1(X) = \emptyset$. By Remark 2.2, $q(\langle W_1, W_2, \ldots, W_n \rangle_n)$ and $q(\langle O_1, O_2, \ldots, O_n \rangle_n)$ are nonempty open subsets of $\mathcal{SF}_n(X)$ such that $F_X \notin q(\langle W_1, W_2, \ldots, W_n \rangle_n)$ and $F_X \notin q(\langle O_1, O_2, \ldots, O_n \rangle_n)$. Since $\mathcal{SF}_n(f)$ is transitive, there exists $k \in \mathbb{N}$ such that:

$$((\mathcal{SF}_n(f))^k(q(\langle W_1, W_2, \ldots, W_n \rangle_n)) \cap q(\langle O_1, O_2, \ldots, O_n \rangle_n) \neq \emptyset.$$

By Proposition 4.1 (b), it follows that:

$$q((\mathcal{F}_n(f))^k(\langle W_1, W_2, \ldots, W_n \rangle_n)) \cap q(\langle O_1, O_2, \ldots, O_n \rangle_n) \neq \emptyset.$$

From Remark 2.2, we obtain that:

$$(\mathcal{F}_n(f))^k(\langle W_1, W_2, \ldots, W_n \rangle_n) \cap \langle O_1, O_2, \ldots, O_n \rangle_n \neq \emptyset.$$

This implies that $(\mathcal{F}_n(f))^k(\mathcal{U}) \cap \mathcal{V} \neq \emptyset$. Therefore, $\mathcal{F}_n(f)$ is transitive.

On the other hand, by [11, Theorem 4.5] and Diagram 1, it follows that (2) implies (1). Moreover, since (2) and (3) are equivalent, we obtain that (3) implies (1).

Finally, by Example 4.6, we deduce that (1) does not imply (2) and that (1) does not imply (3). □

THEOREM 4.11. *Let X be a nondegenerate continuum, let n be an integer greater than or equal to two, and let $f : X \to X$ be a map. Then the following are equivalent:*

(1) *f is weakly mixing;*
(2) *$\mathcal{F}_n(f)$ is weakly mixing;*
(3) *$\mathcal{F}_n(f)$ is transitive;*
(4) *$\mathcal{SF}_n(f)$ is weakly mixing;*
(5) *$\mathcal{SF}_n(f)$ is transitive.*

PROOF. By [11, Theorem 4.5], we have that (1), (2) and (3) are equivalent and by Theorem 4.10, it follows that (3) and (5) are equivalent. By Diagram 1, we obtain that (4) implies (5). Moreover, by Theorem 3.4, we conclude that (2) implies (4). Therefore, for complete the proof of the theorem it suffices to prove that (5) implies (1).

Suppose that $\mathcal{SF}_n(f)$ is transitive, we prove that f is weakly mixing. For this, let U, V_1 and V_2 be nonempty open subsets of X. By [11, Theorem 4.4], it suffices to show that there exists $k \in \mathbb{N}$ such that $f^k(U) \cap V_1 \neq \emptyset$ and $f^k(U) \cap V_2 \neq \emptyset$. For this end, let U_1 and U_2 be nonempty open subsets of X such that $U_1 \cap U_2 = \emptyset$ and $U_1 \cup U_2 \subseteq U$. On the other hand, let W_1 and W_2 be nonempty open subsets of X such that $W_1 \subseteq V_1$, $W_2 \subseteq V_2$ and $W_1 \cap W_2 = \emptyset$. Note that $\langle U_1, U_2 \rangle_n$ and $\langle W_1, W_2 \rangle_n$ are nonempty open subsets of $\mathcal{F}_n(X)$ such that $\langle U_1, U_2 \rangle_n \cap \mathcal{F}_1(X) = \emptyset$ and $\langle W_1, W_2 \rangle_n \cap \mathcal{F}_1(X) = \emptyset$. By Remark 2.2, it follows that $q(\langle U_1, U_2 \rangle_n)$ and $q(\langle W_1, W_2 \rangle_n)$ are nonempty open subsets of $\mathcal{SF}_n(X)$ such that $F_X \notin q(\langle U_1, U_2 \rangle_n)$ and $F_X \notin q(\langle W_1, W_2 \rangle_n)$. Since $\mathcal{SF}_n(f)$ is transitive, there exists $k \in \mathbb{N}$ such that $(\mathcal{SF}(f))^k(q(\langle U_1, U_2 \rangle_n)) \cap q(\langle W_1, W_2 \rangle_n) \neq \emptyset$. Let $\chi \in q(\langle U_1, U_2 \rangle_n)$ such that $(\mathcal{SF}_n(f))^k(\chi) \in q(\langle W_1, W_2 \rangle_n)$. Let $A \in \langle U_1, U_2 \rangle_n$ such that $q(A) = \chi$ and let $B \in \langle W_1, W_2 \rangle_n$ such that $q(B) = (\mathcal{SF}_n(f))^k(\chi)$. This implies that $q(B) = (\mathcal{SF}_n(f))^k(q(A))$. By Proposition 4.1 (b), $q(B) = q((\mathcal{F}_n(f))^k(A))$. Since $q(B) \neq F_X$, we have that $(\mathcal{F}_n(f))^k(A) \in \mathcal{F}_n(X) \setminus \mathcal{F}_1(X)$. Moreover, $A \in \mathcal{F}_n(X) \setminus \mathcal{F}_1(X)$. In consequence, by Remark 2.2, it follows that $B = (\mathcal{F}_n(f))^k(A)$. Hence, by Proposition 4.1 (a), we obtain that $f^k(A) = B$.

Now, since $B \in \langle W_1, W_2 \rangle_n$, we have that $B \cap W_1 \neq \emptyset$. Let $b \in B \cap W_1$. Thus, $b \in V_1$ and $b \in f^k(A)$. In consequence, there exists $a \in A$ such that $f^k(a) = b$. Furthermore, since $a \in U$, we have that $b \in f^k(U) \cap V_1$. Thus, $f^k(U) \cap V_1 \neq \emptyset$. With a similar argument, we obtain that $f^k(U) \cap V_2 \neq \emptyset$. Hence, by [11, Theorem 4.4], we conclude that f is weakly mixing. ∎

THEOREM 4.12. *Let X be a continuum, let n be an integer greater than or equal to two, and let $f : X \to X$ be a map. Consider the following statements:*

(1) *f is totally transitive;*
(2) *$\mathcal{F}_n(f)$ is totally transitive;*
(3) *$\mathcal{SF}_n(f)$ is totally transitive.*

Then (2) and (3) are equivalents, (3) implies (1), (2) implies (1), (1) does not imply (2) and (1) does not imply (3).

PROOF. By Theorem 3.4, we have that (2) implies (3).

Suppose that $\mathcal{SF}_n(f)$ is totally transitive, we prove that $\mathcal{F}_n(f)$ is totally transitive. For this end, let $s \in \mathbb{N}$. We see that $(\mathcal{F}_n(f))^s$ is transitive. Let $\mathcal{U}$ and $\mathcal{V}$ be nonempty open subsets of $\mathcal{F}_n(X)$. By [11, Lemma 4.2], there exist nonempty open subsets $U_1, U_2, \ldots, U_n$ and $V_1, V_2, \ldots, V_n$ of X such that $\langle U_1, U_2, \ldots, U_n\rangle_n \subseteq \mathcal{U}$ and $\langle V_1, V_2, \ldots, V_n\rangle_n \subseteq \mathcal{V}$. For each $i \in \{1, 2, \ldots, n\}$, let W_i be a nonempty open subset of X such that $W_i \subseteq U_i$ and for each $i, j \in \{1, 2, \ldots, n\}$, $W_i \cap W_j = \emptyset$, if $i \neq j$. Also, for each $i \in \{1, 2, \ldots, n\}$, let O_i be a nonempty open subset of X such that $O_i \subseteq V_i$ and for each $i, j \in \{1, 2, \ldots, n\}$, $O_i \cap O_j = \emptyset$, if $i \neq j$. Note that $\langle U_1, U_2, \ldots, U_n\rangle_n$ and $\langle V_1, V_2, \ldots, V_n\rangle_n$ are nonempty open subsets of $\mathcal{F}_n(X)$ such that:

$$\langle W_1, W_2, \ldots, W_n\rangle_n \subseteq \langle U_1, U_2, \ldots, U_n\rangle_n \subseteq \mathcal{U}$$

and

$$\langle O_1, O_2, \ldots, O_n\rangle_n \subseteq \langle V_1, V_2, \ldots, V_n\rangle_n \subseteq \mathcal{V}.$$

Note that $\langle W_1, W_2, \ldots, W_n\rangle_n \cap \mathcal{F}_1(X) = \emptyset$ and $\langle O_1, O_2, \ldots, O_n\rangle_n \cap \mathcal{F}_1(X) = \emptyset$. Hence, by Remark 2.2, we have that:

$$q(\langle W_1, W_2, \ldots, W_n\rangle_n) \text{ and } q(\langle O_1, O_2, \ldots, O_n\rangle_n)$$

are open subsets of $\mathcal{SF}_n(X)$ such that $F_X \notin q(\langle W_1, W_2, \ldots, W_n\rangle_n)$ and $F_X \notin q(\langle O_1, O_2, \ldots, O_n\rangle_n)$. Since $\mathcal{SF}_n(f)$ is totally transitive, $(\mathcal{SF}_n(f))^s$ is transitive. Thus, there exists $k \in \mathbb{N}$ such that:

$$((\mathcal{SF}_n(f))^s)^k(q(\langle W_1, W_2, \ldots, W_n\rangle_n)) \cap q(\langle O_1, O_2, \ldots, O_n\rangle_n) \neq \emptyset.$$

By Proposition 4.1 (d), it follows that:

$$q(((\mathcal{F}_n(f))^s)^k(\langle W_1, W_2, \ldots, W_n\rangle_n)) \cap q(\langle O_1, O_2, \ldots, O_n\rangle_n) \neq \emptyset.$$

We take $B \in ((\mathcal{F}_n(f))^s)^k(\langle W_1, W_2, \ldots, W_n\rangle_n)$ with the following property: $q(B) \in q(\langle O_1, O_2, \ldots, O_n\rangle_n)$. Hence, we consider $A \in \langle O_1, O_2, \ldots, O_n\rangle_n$ such that $q(A) = q(B)$. By Remark 2.2, we obtain that $A = B$. Thus, it follows that:

$$(\mathcal{F}_n(f))^s)^k(\langle W_1, W_2, \ldots, W_n\rangle_n) \cap \langle O_1, O_2, \ldots, O_n\rangle_n \neq \emptyset.$$

This implies that $((\mathcal{F}_n(f))^s)^k(\mathcal{U}) \cap \mathcal{V} \neq \emptyset$. In consequence, $(\mathcal{F}_n(f))^s$ is transitive. Therefore, $\mathcal{F}_n(f)$ is totally transitive.

Now, suppose that $\mathcal{SF}_n(f)$ is totally transitive, we see that f is totally transitive. Fix $s \in \mathbb{N}$. We prove that f^s is transitive. For this, let U and V be nonempty open subsets of X. We consider U_1, U_2, V_1 and V_2 nonempty open subsets of X such that $U_1 \cup U_2 \subseteq U$, $U_1 \cap U_2 = \emptyset$, $V_1 \cup V_2 \subseteq V$ and $V_1 \cap V_2 = \emptyset$. Hence, $\langle U_1, U_2\rangle_n$ and $\langle V_1, V_2\rangle_n$ are nonempty open subsets of $\mathcal{F}_n(X)$ such that $\langle U_1, U_2\rangle_n \cap \mathcal{F}_1(X) = \emptyset$ and $\langle V_1, V_2\rangle_n \cap \mathcal{F}_1(X) = \emptyset$. By Remark 2.2, we obtain that $q(\langle U_1, U_2\rangle_n)$ and $q(\langle V_1, V_2\rangle_n)$ are nonempty open subsets of $\mathcal{SF}_n(X)$

such that $F_X \notin q(\langle U_1, U_2 \rangle_n)$ and $F_X \notin q(\langle V_1, V_2 \rangle_n)$. Since $\mathcal{SF}_n(f)$ is totally transitive, $(\mathcal{SF}_n(f))^s$ is transitive, in consequence, there exists $k \in \mathbb{N}$ such that:

$$((\mathcal{SF}_n(f))^s)^k(q(\langle U_1, U_2 \rangle_n)) \cap q(\langle V_1, V_2 \rangle_n) \neq \emptyset.$$

By Proposition 4.1 (d), it follows that:

$$q(((\mathcal{F}_n(f))^s)^k(\langle U_1, U_2 \rangle_n)) \cap q(\langle V_1, V_2 \rangle_n) \neq \emptyset.$$

Let $B \in ((\mathcal{F}_n(f))^s)^k(\langle U_1, U_2 \rangle_n)$ such that $q(B) \in q(\langle V_1, V_2 \rangle_n)$. We consider an element $A \in \langle V_1, V_2 \rangle_n$ such that $q(B) = q(A)$. By Remark 2.2, we obtain that $A = B$. This implies that:

$$((\mathcal{F}_n(f))^s)^k(\langle U_1, U_2 \rangle_n) \cap \langle V_1, V_2 \rangle_n \neq \emptyset.$$

Let $C \in \langle U_1, U_2 \rangle_n$ such that $((\mathcal{F}_n(f))^s)^k(C) \in \langle V_1, V_2 \rangle_n$. We consider $D \in \langle V_1, V_2 \rangle_n$ such that $((\mathcal{F}_n(f))^s)^k(C) = D$. By Proposition 4.1 (c), we have that $(\mathcal{F}_n(f))^{sk}(C) = D$. Thus, by Proposition 4.1 (a), $f^{sk}(C) = D$. Now, we consider an element $a \in C \cap U_1$. It follows that $f^{sk}(a) \in D$. Since $C \subseteq U$ and $D \subseteq V$, we have that $f^{sk}(a) \in f^{sk}(U) \cap V$. By Proposition 3.2 (d), $(f^s)^k(a) \in (f^s)^k(U) \cap V$. Hence, $(f^s)^k(U) \cap V \neq \emptyset$. In consequence, f^s is transitive. Therefore, f is totally transitive.

On the other hand, since (2) implies (3) and (3) implies (1), we obtain that (2) implies (1).

Finally, by Example 4.6, we see that (1) does not imply (2) and (1) does not imply (3). ∎

THEOREM 4.13. *Let X be a nondegenerate continuum, let n be an integer greater than or equal to two, and let $f : X \to X$ be a map. Consider the following statements:*

(1) *f is strongly transitive;*
(2) *$\mathcal{F}_n(f)$ is strongly transitive;*
(3) *$\mathcal{SF}_n(f)$ is strongly transitive.*

Then (2) implies (3), (3) implies (1), (2) implies (1), (1) does not imply (2) and (1) does not imply (3).

PROOF. By Theorem 3.4, we have that (2) implies (3).

Now, suppose that $\mathcal{SF}_n(f)$ is strongly transitive, we see that f is strongly transitive. Let U be a nonempty open subset of X. We consider U_1 and U_2 nonempty open subsets of X such that $U_1 \cup U_2 \subseteq U$ and $U_1 \cap U_2 = \emptyset$. It follows that $\langle U_1, U_2 \rangle_n$ is a nonempty open subset of $\mathcal{F}_n(X)$ such that $\langle U_1, U_2 \rangle_n \cap \mathcal{F}_1(X) = \emptyset$. By Remark 2.2, we have that $q(\langle U_1, U_2 \rangle_n)$ is a nonempty open subset of $\mathcal{SF}_n(X)$ such that $F_X \notin q(\langle U_1, U_2 \rangle_n)$. Since $\mathcal{SF}_n(f)$ is strongly transitive, there exists $s \in \mathbb{N}$ such that:

$$\mathcal{SF}_n(X) = \bigcup_{k=0}^{s} (\mathcal{SF}_n(f))^k(q(\langle U_1, U_2 \rangle_n)).$$

By Proposition 4.1 (b), we obtain that:

$$\mathcal{SF}_n(X) = \bigcup_{k=0}^{s} q((\mathcal{F}_n(f))^k(\langle U_1, U_2\rangle_n)).$$

We prove that $X = \bigcup_{k=0}^{s} f^k(U)$. For this end, let $x \in X$. We fix $y \in X \setminus \{x\}$ and we consider $A = \{x, y\}$. Thus, $A \in \mathcal{F}_n(X) \setminus \mathcal{F}_1(X)$. In consequence, $q(A) \in \mathcal{SF}_n(X) \setminus \{F_X\}$. This implies that there exists $j \in \{0, 1, \ldots, s\}$ such that $q(A) \in q((\mathcal{F}_n(f))^j(\langle U_1, U_2\rangle_n))$. Hence, there exists $B \in (\mathcal{F}_n(f))^j(\langle U_1, U_2\rangle_n)$ such that $q(B) = q(A)$. Note that, by Remark 2.2, $A = B$. On the other hand, there exists $C \in \langle U_1, U_2\rangle_n$ such that $(\mathcal{F}_n(f))^j(C) = B$. By Proposition 4.1 (a), $f^j(C) = B$. Moreover, since $C \subseteq U$, it follows that $A \subseteq f^j(U)$. In consequence, $x \in f^j(U)$. Thus, $X \subseteq \bigcup_{k=0}^{s} f^k(U)$. Hence, $X = \bigcup_{k=0}^{s} f^k(U)$. Therefore, f is strongly transitive.

Since (2) implies (3) and (3) implies (1), we obtain that (2) implies (1).

Finally, in Example 4.6, we have that (1) does not implies (2) and that (1) does not implies (3). ∎

As a consequence of Diagram 1 and Theorem 4.10, we have the follows:

Corollary 4.14. *Let X be a nondegenerate continuum, let n be an integer greater than or equal to two, and let $f : X \to X$ be a map. If $\mathcal{SF}_n(f)$ is strongly transitive, then $\mathcal{F}_n(f)$ is transitive.*

Moreover, by Corollary 4.14 and Theorem 4.11, we obtain:

Corollary 4.15. *Let X be a nondegenerate continuum, let n be an integer greater than or equal to two, and let $f : X \to X$ be a map. If $\mathcal{SF}_n(f)$ is strongly transitive, then f, $\mathcal{F}_n(f)$ and $\mathcal{SF}_n(f)$ are weakly mixing.*

Question 4.1. *Let X be a continuum, let n be an integer greater than or equal to two, and let $f : X \to X$ be a map. If $\mathcal{SF}_n(f)$ is strongly transitive, then is $\mathcal{F}_n(f)$ strongly transitive?*

The following lemma is used in the proof of Theorem 4.17.

Lemma 4.16. *Let X be a continuum, let n be an integer greater than or equal to two, and let $f : X \to X$ be a surjective map. Then the following are equivalent:*

(1) *per(f) is dense in X;*
(2) *per$(\mathcal{F}_n(f))$ is dense in $\mathcal{F}_n(X)$;*
(3) *per$(\mathcal{SF}_n(f))$ is dense in $\mathcal{SF}_n(X)$.*

Proof. By [11, Theorem 4.7], we obtain that (1) and (2) are equivalent and, by Lemma 3.3, we have that (2) implies (3).

Suppose that $\mathrm{per}(\mathcal{SF}_n(f))$ is dense in $\mathcal{SF}_n(X)$, we see that $\mathrm{per}(\mathcal{F}_n(f))$ is dense in $\mathcal{F}_n(X)$. Let $\mathcal{U}$ be a nonempty open subset of $\mathcal{F}_n(X)$. We prove that $\mathcal{U} \cap \mathrm{per}(\mathcal{F}_n(f)) \neq \emptyset$. In other words, we see that there exist $A \in \mathcal{U}$

and $k \in \mathbb{N}$ such that $(\mathcal{F}_n(f))^k(A) = A$. By [11, Lemma 4.2], there exist nonempty open subsets $U_1, U_2, \ldots, U_n$ of X such that $\langle U_1, U_2, \ldots, U_n\rangle_n \subseteq \mathcal{U}$. For each $i \in \{1, 2, \ldots, n\}$, let W_i be a nonempty open subset of X such that $W_i \subseteq U_i$ and for each $i, j \in \{1, 2, \ldots, n\}$, $W_i \cap W_j \neq \emptyset$, if $i \neq j$. It follows that $\langle W_1, W_2, \ldots, W_n\rangle_n$ is a nonempty open subset of $\mathcal{F}_n(X)$ such that:

$$\langle W_1, W_2, \ldots, W_n\rangle_n \subseteq \langle U_1, U_2, \ldots, U_n\rangle_n \subseteq \mathcal{U}$$

and

$$\langle W_1, W_2, \ldots, W_n\rangle_n \cap \mathcal{F}_1(X) = \emptyset.$$

By Remark 2.2, we have that $q(\langle W_1, W_2, \ldots, W_n\rangle_n)$ is a nonempty open subset of $\mathcal{SF}_n(X)$ such that $F_X \notin q(\langle W_1, W_2, \ldots, W_n\rangle_n)$. Since $\mathrm{per}(\mathcal{SF}_n(f))$ is dense in $\mathcal{SF}_n(X)$, there exist $A \in \langle W_1, W_2, \ldots, W_n\rangle_n$ and $k \in \mathbb{N}$ such that $(\mathcal{SF}_n(f))^k(q(A)) = q(A)$. By Proposition 4.1 (b), $q((\mathcal{F}_n(f))^k(A)) = q(A)$. Furthermore, by Remark 2.2, $(\mathcal{F}_n(f))^k(A) = A$. Hence, $\mathcal{U} \cap \mathrm{per}(\mathcal{F}_n(f)) \neq \emptyset$. Therefore, $\mathrm{per}(\mathcal{F}_n(f))$ is dense in $\mathcal{F}_n(X)$. ∎

THEOREM 4.17. *Let X be a continuum, let n be an integer greater than or equal to two, and let $f : X \to X$ be a map. Then the following are equivalent:*

(1) *f is chaotic and weakly mixing;*
(2) *$\mathcal{F}_n(f)$ is chaotic;*
(3) *$\mathcal{SF}_n(f)$ is chaotic.*

PROOF. By [11, Theorem 4.9], we have that (1) and (2) are equivalent. On the other hand, by Theorem 4.10 and by Lemma 4.16, we conclude that (2) and (3) are equivalent. ∎

THEOREM 4.18. *Let X be a nondegenerate continuum, let n be an integer greater than or equal to two, and let $f : X \to X$ be a map. Consider the following statements:*

(1) *f is minimal;*
(2) *$\mathcal{F}_n(f)$ is minimal;*
(3) *$\mathcal{SF}_n(f)$ is minimal.*

Then (2) implies (3), (3) implies (1), (2) implies (1), (1) does not imply (2) and (1) does not imply (3).

PROOF. By Theorem 3.4, we have that (2) implies (3).

Suppose that $\mathcal{SF}_n(f)$ is minimal, we see that f is minimal. For this end, let $x \in X$ and let U be a nonempty open subset of X. We consider nonempty open subsets V_1 and V_2 of X such that $V_1 \cup V_2 \subseteq U$ and $V_1 \cap V_2 = \emptyset$. Hence, $\langle V_1, V_2\rangle_n$ is a nonempty open subset of $\mathcal{F}_n(X)$ such that $\langle V_1, V_2\rangle_n \cap \mathcal{F}_1(X) = \emptyset$. By Remark 2.2, $q(\langle V_1, V_2\rangle_n)$ is a nonempty open subset of $\mathcal{SF}_n(X)$. Note that $F_X \notin q(\langle V_1, V_2\rangle_n)$. Let $y \in X \setminus \{x\}$ and we consider $A = \{x, y\}$. Clearly $A \in \mathcal{F}_n(X) \setminus \mathcal{F}_1(X)$. Since $\mathcal{SF}_n(f)$ is minimal, $\mathrm{orb}(q(A), \mathcal{SF}_n(f))$ is dense in $\mathcal{SF}_n(X)$. Thus, $q(\langle V_1, V_2\rangle_n) \cap \mathrm{orb}(q(A), \mathcal{SF}_n(f)) \neq \emptyset$. Let $C \in \langle V_1, V_2\rangle_n$

and let $k \in \mathbb{N} \cup \{0\}$ such that $(\mathcal{SF}_n(f))^k(q(A)) = q(C)$. By Proposition 4.1 (b), $q((\mathcal{F}_n(f))^k(A)) = q(C)$. Hence, by Remark 2.2, $(\mathcal{F}_n(f))^k(A) = C$. Thus, by Proposition 4.1 (a), $f^k(A) = C$. Since $x \in A$ and $C \subseteq U$, it follows that $f^k(x) \in U$. In consequence, $U \cap \mathrm{orb}(x, f) \neq \emptyset$. This implies that f is minimal.

Since (2) implies (3) and (3) implies (1), we have that (2) implies (1).

Finally, in Example 4.6, we obtain that (1) does not implies (2) and that (1) does not implies (3). ◻

Note that as a consequence of Diagram 1 and Theorem 4.10, we obtain that.

Corollary 4.19. *Let X be a nondegenerate continuum, let n be an integer greater than or equal to two, and let $f : X \to X$ be a map. If $\mathcal{SF}_n(f)$ is minimal, then $\mathcal{F}_n(f)$ is transitive.*

Moreover, by Corollary 4.19 and Theorem 4.11, we obtain:

Corollary 4.20. *Let X be a nondegenerate continuum, let n be an integer greater than or equal to two, and let $f : X \to X$ be a map. If $\mathcal{SF}_n(f)$ is minimal, then f, $\mathcal{F}_n(f)$ and $\mathcal{SF}_n(f)$ are weakly mixing.*

Question 4.2. *Let X be a continuum, let n be an integer greater than or equal to two, and let $f : X \to X$ be a map. If $\mathcal{SF}_n(f)$ is minimal, then is $\mathcal{F}_n(f)$ minimal?*

Recall that $f : X \to X$ is an *open* map if for each open subset A in X, $f(A)$ is an open subset in X.

Corollary 4.21. *Let X be a nondegenerate continuum, let n be an integer greater than or equal to two, and let $f : X \to X$ be a map. If $\mathcal{F}_n(f)$ is minimal and open, then f is a homeomorphism.*

Proof. The result follows from Theorem 4.18, [4, Theorem 5.3] and [14, Theorem 2.4 (2)]. ◻

5. Other dynamical properties

We begin this section with the following result.

Theorem 5.1. *Let X be a nondegenerate continuum, let n be an integer greater than or equal to two, and let $f : X \to X$ be a map. Consider the following statements:*

(1) *f is irreducible;*
(2) *$\mathcal{F}_n(f)$ is irreducible;*
(3) *$\mathcal{SF}_n(f)$ is irreducible.*

Then (2) implies (1) and (3) implies (1).

PROOF. Suppose that $\mathcal{F}_n(f)$ is irreducible, we prove that f is irreducible. Let A be a nonempty closed subset of X such that $f(A) = X$. It follows that $\langle A\rangle_n$ is a nonempty closed subset of $\mathcal{F}_n(X)$ such that $\mathcal{F}_n(f)(\langle A\rangle_n) = \mathcal{F}_n(X)$. Since $\mathcal{F}_n(f)$ is irreducible, we have that $\langle A\rangle_n = \mathcal{F}_n(X)$. Thus, $\mathcal{F}_1(X) \subseteq \langle A\rangle_n$. This implies that $X = A$. Hence, f is irreducible.

Now, suppose that $\mathcal{SF}_n(f)$ is irreducible, we prove that f is irreducible. Let A be a nonempty closed subset of X such that $f(A) = X$. It follows that $\langle A\rangle_n$ is a nonempty closed subset of $\mathcal{F}_n(X)$ such that $\mathcal{F}_n(f)(\langle A\rangle_n) = \mathcal{F}_n(X)$. Hence, $q(\mathcal{F}_n(f)(\langle A\rangle_n)) = \mathcal{SF}_n(X)$. In consequence, by Proposition 4.1 (b), $\mathcal{SF}_n(f)(q(\langle A\rangle_n)) = \mathcal{SF}_n(X)$. Since $q(\langle A\rangle_n)$ is a nonempty closed subset of $\mathcal{SF}_n(X)$ and $\mathcal{SF}_n(f)$ is irreducible, we have that $q(\langle A\rangle_n) = \mathcal{SF}_n(X)$. Now, let $x \in X$ and let $y \in X \setminus \{x\}$. Let $B = \{x, y\}$. Since $q(B) \in \mathcal{SF}_n(X)$, there exists $C \in \langle A\rangle_n$ such that $q(C) = q(B)$. By Remark 2.2, $C = B$. Hence, $x \in A$. In consequence, $X = A$. Therefore, f is irreducible. ∎

COROLLARY 5.2. *Let X be a nondegenerate continuum, let n be an integer greater than or equal to two, and let $f : X \to X$ be a map. Then $\mathcal{F}_n(f)$ is irreducible and open if and only if f is a homeomorphism.*

PROOF. The corollary follows easily from Theorem 5.1, [4, Theorem 5.3] and [14, Lemma 2.2]. ∎

Note that feebly open maps are also know semi-open maps in the literature. The following result is [6, Theorem 10.1].

THEOREM 5.3. *Let X be a continuum, let n be an integer greater than or equal to two, and let $f : X \to X$ be a surjective map. Then the following are equivalent:*

(1) *f is feebly open;*
(2) *$\mathcal{F}_n(f)$ is feebly open;*
(3) *$\mathcal{SF}_n(f)$ is feebly open.*

Using the Diagram 1 and Theorem 5.3, we obtain the following result.

COROLLARY 5.4. *Let X be a continuum, let n be an integer greater than or equal to two, and let $f : X \to X$ be a map. Then the following statements hold.*

(1) *If f is irreducible, then $\mathcal{F}_n(f)$ is feebly open.*
(2) *If f is irreducible, then $\mathcal{SF}_n(f)$ is feebly open.*
(3) *If $\mathcal{F}_n(f)$ is irreducible, then $\mathcal{SF}_n(f)$ is feebly open.*
(4) *If $\mathcal{SF}_n(f)$ is irreducible, then $\mathcal{F}_n(f)$ is feebly open.*

QUESTIONS 5.5. *Let X be a continuum, let n be an integer greater than or equal to two, and let $f : X \to X$ be a map.*

(1) *If f is irreducible, then is $\mathcal{F}_n(f)$ irreducible?*
(2) *If f is irreducible, then is $\mathcal{SF}_n(f)$ irreducible?*

(3) *If $\mathcal{F}_n(f)$ is irreducible, then is $\mathcal{SF}_n(f)$ irreducible?*
(4) *If $\mathcal{SF}_n(f)$ is irreducible, then is $\mathcal{F}_n(f)$ irreducible?*

THEOREM 5.6. *Let X be a nondegenerate continuum, let n be an integer greater than or equal to two, and let $f : X \to X$ be a surjective map. Consider the following statements:*

(1) *f is turbulent;*
(2) *$\mathcal{F}_n(f)$ is turbulent;*
(3) *$\mathcal{SF}_n(f)$ is turbulent.*

Then (1) implies (2) and (3).

PROOF. Suppose that f is turbulent, we prove that $\mathcal{SF}_n(f)$ is turbulent. Let K and C be nondegenerate compact subsets of X such that $K \cap C$ has at most one point and $K \cup C \subseteq f(K) \cap f(C)$. It is easy to see that $\langle K \rangle_n$ and $\langle C \rangle_n$ are nondegenerate compact subsets of $\mathcal{F}_n(X)$. Let $\Lambda = q(\langle K \rangle_n)$ and $\Gamma = q(\langle C \rangle_n)$. It follows that Λ and Γ are nondegenerate compact subsets of $\mathcal{SF}_n(X)$.

We show that $\Lambda \cap \Gamma$ has at most one point. For this, note that if $K \cap C = \emptyset$, then $\langle K \rangle_n \cap \langle C \rangle_n = \emptyset$. Since $\mathcal{F}_1(K) \subseteq \langle K \rangle_n$ and $\mathcal{F}_1(C) \subseteq \langle C \rangle_n$, it follows that $F_X \in \Lambda \cap \Gamma$.

On the other hand, if $K \cap C = \{a\}$, we obtain that $\langle K \rangle_n \cap \langle C \rangle_n = \{\{a\}\}$. Thus, $F_X \in \Lambda \cap \Gamma$. Now, if $\chi \in (\Lambda \cap \Gamma) \setminus \{F_X\}$, then there exist $A \in \langle K \rangle_n \setminus \mathcal{F}_1(X)$ and $B \in \langle C \rangle_n \setminus \mathcal{F}_1(X)$ such that $q(A) = \chi = q(B)$. By Remark 2.2, $A = B$. This proves that $A \subseteq K \cap C$. Thus, $K \cap C$ has at least two elements, which is a contradiction. Therefore, $\Lambda \cap \Gamma$ has at most one point.

Now, we see that $\Lambda \cup \Gamma \subseteq \mathcal{SF}_n(f)(\Lambda) \cap \mathcal{SF}_n(f)(\Gamma)$. Let $\chi \in \Lambda \cup \Gamma$. Hence, there exits $A \in \langle K \rangle_n \cup \langle C \rangle_n$ such that $q(A) = \chi$. Since $A \subseteq K \cup C$ and $K \cup C \subseteq f(K) \cap f(C)$, we have that $A \subseteq f(K) \cap f(C)$. This implies that $A \in \langle f(K) \cap f(C) \rangle_n$. Thus, $A \in \langle f(K) \rangle_n \cap \langle f(C) \rangle_n$. In consequence, $q(A) \in q(\langle f(K) \rangle_n) \cap q(\langle f(C) \rangle_n)$. Since $q(A) = \chi$, we have that $\chi \in q(\mathcal{F}_n(f)(\langle K \rangle_n)) \cap q(\mathcal{F}_n(f)(\langle C \rangle_n))$. By Proposition 4.1 (b), $\chi \in \mathcal{SF}_n(f)(q(\langle K \rangle_n)) \cap \mathcal{SF}_n(f)(q(\langle C \rangle_n))$. It follows that $\chi \in \mathcal{SF}_n(f)(\Lambda) \cap \mathcal{SF}_n(f)(\Gamma)$. In consequence, $\Lambda \cup \Gamma \subseteq \mathcal{SF}_n(f)(\Lambda) \cap \mathcal{SF}_n(f)(\Gamma)$. Therefore, $\mathcal{SF}_n(f)$ is turbulent.

The proof (1) implies (2) is similar to the proof (1) implies (3). □

We end this paper with following questions.

QUESTIONS 5.7. *Let X be a continuum, let n be an integer greater than or equal to two, and let $f : X \to X$ be a map.*

(i) *If $\mathcal{F}_n(f)$ is turbulent, then is f turbulent?*
(ii) *If $\mathcal{F}_n(f)$ is turbulent, then is $\mathcal{SF}_n(f)$ turbulent?*
(iii) *If $\mathcal{SF}_n(f)$ is turbulent, then is f turbulent?*
(iv) *If $\mathcal{SF}_n(f)$ is turbulent, then is $\mathcal{F}_n(f)$ turbulent?*

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