

## INDUCED MAPPINGS BETWEEN QUOTIENT SPACES OF N-FOLD HYPERSPACES OF CONTINUA

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ABSTRACT. For a continuum  $X$  the hyperspace of nonempty closed subsets of  $X$  with at most  $n$  components is called the  $n$ -fold hyperspace  $C_n(X)$  and if  $m < n$  then  $C_m(X) \subset C_n(X)$  so it is possible to form a quotient space  $C_n(X)/C_m(X)$  identifying the set  $C_m(X)$  to a point in  $C_n(X)$ . If  $f$  is a mapping from a continuum  $X$  onto a continuum  $Y$  there will be a induced mappings between  $C_n(X)$  and  $C_m(X)$  and between the quotient spaces  $C_n(X)/C_m(X)$  and  $C_n(Y)/C_m(Y)$ . Now if a list of function properties that are of interest to continua theorists is considered, there will be natural questions about when these properties are passed on from the functions between the continua to the induced mappings between the hyperspaces or the induced mappings between the quotients of the hyperspaces. Many of these questions have been considered extensively for the hyperspaces so the main thing that is new here is the questions and answers about the quotient spaces and their induced mappings. Here we consider the following families of mappings: atomic, atriodic, confluent, hereditarily weakly confluent, joining, light, local homeomorphism, locally confluent, locally weakly confluent, monotone, open, OM, semi-confluent and weakly confluent.

### 1. INTRODUCTION

A *continuum* is a nonempty, compact, connected metric space. A *subcontinuum* is a continuum which is a subset of a metric space. A *mapping* is a continuous function. The notation  $\mathbb{N}$  denotes the set of all positive integers. Given a continuum  $X$  and  $n \in \mathbb{N}$ ,  $C_n(X)$  denotes the hyperspace of all nonempty closed subsets of  $X$  having at most  $n$  components. The hyperspace  $C_n(X)$  is considered with the Hausdorff metric,  $\mathcal{H}$ , (see [27, p. 1]). Let

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$n, m \in \mathbb{N}$  be such that  $n > m$ . The notation  $SC_m^n(X)$  denotes the quotient space  $C_n(X)/C_m(X)$  obtained of  $C_n(X)$  identifying the set  $C_m(X)$  to a point. And the function  $\rho_X^{n,m} : C_n(X) \rightarrow SC_m^n(X)$  denotes the quotient mapping.

Given a mapping  $f : X \rightarrow Y$  between continua, the mapping  $C_n(f) : C_n(X) \rightarrow C_n(Y)$  given by  $C_n(f)(A) = f(A)$  is a mapping induced by  $f$ . Let  $SC_m^n(f) : SC_m^n(X) \rightarrow SC_m^n(Y)$  be a function defined such a way that

$$SC_m^n(f)(\rho_X^{n,m}(A)) = \rho_Y^{n,m}(f(A)).$$

By [15, Theorem 4.3],  $SC_m^n(f)$  is a continuous function. Moreover the following diagram is commutative.

$$(1.1) \quad \begin{array}{ccc} C_n(X) & \xrightarrow{C_n(f)} & C_n(Y) \\ \downarrow \rho_X^{n,m} & & \downarrow \rho_Y^{n,m} \\ SC_m^n(X) & \xrightarrow{SC_m^n(f)} & SC_m^n(Y) \end{array}$$

Let  $\mathcal{M}$  be a class of mappings between continua. A general problem in Continuum Theory is to determine all possible relationships among the following statements:

- (1)  $f \in \mathcal{M}$ ;
- (2)  $C_n(f) \in \mathcal{M}$ ;
- (3)  $SC_m^n(f) \in \mathcal{M}$ .

Readers especially interested in this topic are referred to [1]-[14], [16]-[21].

In this paper we study the interrelations among the statements (1)-(3), where  $\mathcal{M}$  is the family of: atomic, atriodic, confluent, hereditarily weakly confluent, joining, light, local homeomorphism, locally confluent, locally weakly confluent, monotone, open, OM, semi-confluent or weakly confluent mappings.

## 2. PRELIMINARIES

Let  $X$  be a continuum with metric  $d$ . Let  $\epsilon > 0$  and  $A \subset X$ , we define  $\mathcal{V}_\epsilon^d(A) = \{x \in X : \text{there exists } y \in A \text{ such that } d(x, y) < \epsilon\}$ , and we use the symbol  $\text{Cl}(A)$  to denote the closure of  $A$  in  $X$ . An *order arc* in  $C_n(X)$  is an arc  $\alpha : [0, 1] \rightarrow C_n(X)$  such that if  $0 \leq s < t \leq 1$ , then  $\alpha(s) \subset \alpha(t)$  and  $\alpha(s) \neq \alpha(t)$ . Given a finite collection  $K_1, \dots, K_r$  of subsets of  $X$ ,  $\langle K_1, \dots, K_r \rangle$ , is used to denote the following subset of  $C_n(X)$ ,

$$\{A \in C_n(X) : A \subset \bigcup_{i=1}^r K_i, A \cap K_i \neq \emptyset \text{ for each } i \in \{1, \dots, r\}\}.$$

It is well known that the family of all subsets of  $C_n(X)$  of the form  $\langle K_1, \dots, K_r \rangle$ , where each  $K_i$  is an open subset of  $X$ , forms a base for a topology for  $C_n(X)$  (see [27, Theorem 0.11]) called the *Vietoris topology*, which

coincide with the topology induced on  $C_n(X)$  by the Hausdorff metric (see [27, Theorem 0.13]).

Given two positive integers  $s > k$ , we define  $C_k^s(X) = C_s(X) \setminus C_k(X)$ .

Notice that  $\rho_X^{s,k}|_{C_k^s(X)} : C_k^s(X) \rightarrow SC_k^s(X) \setminus \rho_X^{s,k}(C_k(X))$  is a homeomorphism.

A surjective mapping  $f : X \rightarrow Y$  between continua is:

- *atriodic* if, for each subcontinuum  $Q$  of  $Y$ , there are two components  $C_1$  and  $C_2$  of  $f^{-1}(Q)$  such that  $f(C_1) \cup f(C_2) = Q$  and for each component  $C$  of  $f^{-1}(Q)$ , we have either  $f(C) = Q$  or  $f(C) \subset f(C_1)$  or  $f(C) \subset f(C_2)$ ;
- *atomic* if, for each subcontinuum  $K$  of  $X$  such that  $f(K)$  is nondegenerate,  $f^{-1}(f(K)) = K$ ;
- *confluent* if, for every subcontinuum  $K$  of  $Y$  and for each component  $M$  of  $f^{-1}(K)$ ,  $f(M) = K$ ;
- *joining* if, for every subcontinuum  $B$  of  $Y$  and every pair of components  $C$  and  $D$  of  $f^{-1}(B)$ ,  $f(C) \cap f(D) \neq \emptyset$ ;
- *light* if  $f^{-1}(y)$  is totally disconnected for each  $y \in Y$ ;
- a *local homeomorphism* if, for each point  $x \in X$ , there exists an open neighbourhood  $U$  of  $x$  in  $X$  such that  $f(U)$  is a open neighbourhood of  $f(x)$  in  $Y$  and the partial mapping  $f|_U : U \rightarrow f(U)$  is a homeomorphism;
- *monotone* if  $f^{-1}(y)$  is connected for each  $y \in Y$ ;
- *open* if  $f(U)$  is open in  $Y$  for each open subset  $U$  of  $X$ ;
- an *OM-mapping* if there exist a continuum  $Z$  and mappings  $g : X \rightarrow Z$  and  $h : Z \rightarrow Y$  such that  $f = h \circ g$ ,  $g$  is monotone and  $h$  is open;
- *semi-confluent* if, for every subcontinuum  $B$  of  $Y$  and every pair of components  $C$  and  $D$  of  $f^{-1}(B)$ , either  $f(C) \subset f(D)$  or  $f(D) \subset f(C)$ ;
- *weakly confluent* if, for each subcontinuum  $K$  of  $Y$ , there exists a subcontinuum  $M$  of  $X$  such that  $f(M) = K$ ;
- *locally confluent (locally weakly confluent)* provided for each point  $y \in Y$ , there exists a closed neighbourhood  $F$  of  $y$  in  $Y$  such that the partial mapping  $f|_{f^{-1}(F)}$  is a confluent mapping (a weakly confluent mapping) of  $f^{-1}(F)$  onto  $F$ ;
- *hereditarily weakly confluent* if, for each nondegenerate subcontinuum  $M$  of  $X$ ,  $f|_M$  is weakly confluent.

As a consequence of the fact that  $C_n(X)$  is arcwise connected [23, Theorem 3.1] and [28, Theorem 3.10] we have the following proposition.

PROPOSITION 2.1. *Let  $X$  be a continuum and  $n \geq 2$ . Then  $SC_m^n(X)$  is an arcwise connected continuum.*

The next theorem is proved in [23, Theorem 3.3].

PROPOSITION 2.2. *Let  $X$  be a continuum, if  $n \geq 2$  then  $C_{n-1}^n(X)$  is dense on  $C_n(X)$ .*

LEMMA 2.3. *Let  $X$  be a continuum and  $n \geq 2$ . If  $\mathcal{U} \subset C_n(X)$  such that  $\mathcal{U} \cap C_m(X) \neq \emptyset$  and  $\rho_X^{n,m}(\mathcal{U})$  is an open subset of  $SC_m^n(X)$ , then  $C_m(X) \subset \text{Cl}(\mathcal{U})$ .*

PROOF. Let  $A \in C_m(X)$  such that  $A \notin \text{Cl}(\mathcal{U})$ . Then there is an  $\epsilon > 0$  such that  $\mathcal{V}_\epsilon^H(A) \cap \mathcal{U} = \emptyset$ . Thus, since  $(\rho_X^{n,m})^{-1}(\rho_X^{n,m}(\mathcal{U})) = \mathcal{U} \cup C_m(X)$  is an open subset of  $C_n(X)$ , there exists  $0 < \delta \leq \epsilon$  such that  $\mathcal{V}_\delta^H(A) \subset C_m(X)$ . This contradicts Proposition 2.2.  $\square$

PROPOSITION 2.4. *Let  $f : X \rightarrow Y$  be a surjective mapping between continua and  $1 \leq r \leq n$ . Let  $K_1, \dots, K_r$  be nonempty disjoint closed subsets of  $Y$ . For each  $i \in \{1, \dots, r\}$ , let  $M_i$  be a component of  $f^{-1}(K_i)$ . Then*

- (1)  $\langle M_1, \dots, M_r \rangle$  is a component of  $C_n(f)^{-1}(\langle K_1, \dots, K_r \rangle)$ .
- (2) If  $M$  is a component of  $f^{-1}(K_i)$  such that  $M \neq M_i$  and  $r < n$ , then  $\langle M_1, \dots, M_r, M \rangle$  is a component of  $C_n(f)^{-1}(\langle K_1, \dots, K_r \rangle)$ .
- (3) If  $r > m$ , then  $\rho_X^{n,m}(\langle M_1, \dots, M_r \rangle)$  is a component of

$$SC_m^n(f)^{-1}(\rho_Y^{n,m}(\langle K_1, \dots, K_r \rangle)).$$

PROOF. We will prove (1). By [2, Proposition 2.3 ],  $\langle M_1, \dots, M_r \rangle$  is connected. Note that  $\langle M_1, \dots, M_r \rangle \subset C_n(f)^{-1}(\langle K_1, \dots, K_r \rangle)$ . Consider  $\mathcal{C}$ , the component of  $C_n(f)^{-1}(\langle K_1, \dots, K_r \rangle)$  containing  $\langle M_1, \dots, M_r \rangle$ . Clearly

$$\bigcup_{i=1}^r M_i \subset \bigcup \mathcal{C} \subset \bigcup_{i=1}^r f^{-1}(K_i).$$

Now, let  $P$  be a component of  $\bigcup \mathcal{C}$ . By [20, Lemma 3.1],  $P \cap M_j \neq \emptyset$  for some  $j \in \{1, \dots, r\}$ . Thus,  $P = M_j$ . Hence, using [20, Lemma 3.1],  $\mathcal{C} = \langle M_1, \dots, M_r \rangle$ .

The proof of (2) is similar to the proof of (1).

To prove (3), consider  $\mathcal{D}$  the component of  $SC_m^n(f)^{-1}(\rho_Y^{n,m}(\langle K_1, \dots, K_r \rangle))$  containing  $\rho_X^{n,m}(\langle M_1, \dots, M_r \rangle)$ . Clearly,  $\langle M_1, \dots, M_r \rangle \subset (\rho_X^{n,m})^{-1}(\mathcal{D})$ . We need to prove that  $(\rho_X^{n,m})^{-1}(\mathcal{D}) \subset C_n(f)^{-1}(\langle K_1, \dots, K_r \rangle)$ . Notice that  $SC_m^n(f)(\rho_X^{n,m}(A)) = \rho_Y^{n,m}(C_n(f)(A))$  for each  $A \in C_m^n(X)$ . Since  $\rho_Y^{n,m}(C_m(Y)) \notin \rho_Y^{n,m}(\langle K_1, \dots, K_r \rangle)$ ,  $\rho_X^{n,m}(C_m(X)) \notin \mathcal{D}$  and  $C_m(X) \cap (\rho_X^{n,m})^{-1}(\mathcal{D}) = \emptyset$ . Then,  $\rho_Y^{n,m}(C_n(f)((\rho_X^{n,m})^{-1}(\mathcal{D}))) \subset \rho_Y^{n,m}(\langle K_1, \dots, K_r \rangle)$ . Thus  $C_n(f)((\rho_X^{n,m})^{-1}(\mathcal{D})) \subset \langle K_1, \dots, K_r \rangle$ . Now, since  $\rho_X^{n,m}$  is monotone,  $(\rho_X^{n,m})^{-1}(\mathcal{D}) \subset \langle M_1, \dots, M_r \rangle$ ,  $\mathcal{D} = \rho_X^{n,m}(\langle M_1, \dots, M_r \rangle)$ .  $\square$

### 3. HOMEOMORPHISMS AND OPEN MAPPINGS

THEOREM 3.1. *Let  $f : X \rightarrow Y$  be a mapping between continua and  $n \geq 2$ . Then the following conditions are equivalent.*

- (1)  $f$  is a homeomorphism.
- (2)  $C_n(f)$  is a homeomorphism.
- (3)  $SC_m^n(f)$  is a homeomorphism.

PROOF. (1)  $\Rightarrow$  (2). It follows from [12, Theorem 46].

It is easy to see (2)  $\Rightarrow$  (3).

In order to prove that (3)  $\Rightarrow$  (1), let  $y \in Y$  and let  $y_1, \dots, y_m \in Y \setminus \{y\}$  such that  $y_i \neq y_j$  if  $i \neq j$ . Let  $B = \{y_1, \dots, y_m, y\} \in C_m^n(Y)$ . Since  $SC_m^n(f)$  is a homeomorphism, there is  $A \in C_m^n(X)$  such that  $SC_m^n(f)(\rho_X^{n,m}(A)) = \rho_Y^{n,m}(B)$ . So,  $C_n(f)(A) = B = f(A)$ . Thus,  $f$  is surjective. Now, let  $x, y \in X$  such that  $x \neq y$  and let  $z_1, \dots, z_m \in X \setminus \{x, y\}$  such that  $A = \{z_1, \dots, z_m, x\}, B = \{z_1, \dots, z_m, y\} \in C_m^n(X)$ . Since  $SC_m^n(f)$  is a homeomorphism,  $\rho_Y^{n,m}(f(A)) \neq \rho_Y^{n,m}(f(B))$ . So  $f(x) \neq f(y)$ . Therefore  $f$  is a homeomorphism.  $\square$

THEOREM 3.2. *Let  $f : X \rightarrow Y$  be a mapping between continua and  $n \geq 2$ . Then the following conditions are equivalent.*

- (1)  $f$  is homeomorphism.
- (2)  $C_n(f)$  is open.
- (3)  $SC_m^n(f)$  is open for every  $1 \leq m < n$ .

PROOF. By [1, Corollary 3.3], (1) and (2) are equivalent.

(1)  $\Rightarrow$  (3) Follows from Theorem 3.1. To prove (3)  $\Rightarrow$  (1), let  $x_1, x_2 \in X$  such that  $x_1 \neq x_2$  and  $f(x_1) = f(x_2)$ . Consider  $A \in C_m^{m+1}(X)$  such that  $\{x_1\}$  and  $\{x_2\}$  are components of  $A$ . Choose  $\delta > 0$  such that  $\mathcal{V}_\delta^{\mathcal{H}}(A) \cap C_m(X) = \emptyset$ . Let  $0 < \epsilon < \delta$ . Then,  $\rho_X^{n,m}(\mathcal{V}_\epsilon^{\mathcal{H}}(A))$  is an open subset of  $SC_m^n(X)$ . Since  $SC_m^n(f)(\rho_X^{n,m}(\mathcal{V}_\epsilon^{\mathcal{H}}(A))) = \rho_Y^{n,m}(C_n(f)(\mathcal{V}_\epsilon^{\mathcal{H}}(A))), \rho_Y^{n,m}(C_n(f)(\mathcal{V}_\epsilon^{\mathcal{H}}(A)))$  is open in  $SC_m^n(Y)$ . By Lemma 2.3,  $C_m(Y) \subset \bigcap_{\epsilon > 0} Cl(C_n(f)(\mathcal{V}_\epsilon^{\mathcal{H}}(A)))$ . Thus,  $C_m(Y) = \{C_n(f)(A)\}$ , a contradiction. Therefore  $f$  is a homeomorphism.  $\square$

Concerning the last theorem, we can see in the following example that there exists an open mapping between continua such that  $SC_m^n(f)$  is not open, in other words, the condition that  $f$  is homeomorphism is essential to prove that  $SC_m^n(f)$  is an open mapping.

EXAMPLE 3.3. There exist continua  $X$  and  $Y$ , and an open mapping  $f : X \rightarrow Y$  such that  $SC_m^n(f)$  is not open for  $n \geq 2$ .

Consider the mapping  $f : [-1, 1] \rightarrow [0, 1]$  defined by  $f(x) = |x|$ . Clearly,  $f$  is open. By Theorem 3.2,  $C_n(f)$  and  $SC_m^n(f)$  are not open for  $n \geq 2$ .

#### 4. MONOTONE MAPPINGS

THEOREM 4.1. *Let  $f : X \rightarrow Y$  be a mapping between continua and  $n \geq 2$ . Then the following conditions are equivalent.*

- (1)  $f$  is monotone.
- (2)  $C_n(f)$  is monotone.
- (3)  $SC_m^n(f)$  is monotone for every  $1 \leq m < n$ .

PROOF. By [12, Theorem 4], (1) and (2) are equivalent.

To prove that (2)  $\Rightarrow$  (3), let  $B \in C_n(Y)$ . Since  $C_n(f)$  and  $\rho_X^{n,m}$  are monotone mappings and

$$\rho_X^{n,m} \left( C_n(f)^{-1} \left( (\rho_Y^{n,m})^{-1} (\rho_Y^{n,m}(B)) \right) \right) = SC_m^n(f)^{-1} (\rho_Y^{n,m}(B)),$$

$SC_m^n(f)^{-1} (\rho_Y^{n,m}(B))$  is connected. Hence,  $SC_m^n(f)$  is monotone.

Now, we show that (3)  $\Rightarrow$  (1). Suppose that there exists  $y \in Y$  such that  $f^{-1}(y)$  is not connected. Let  $K_1$  and  $K_2$  be different components of  $f^{-1}(y)$ . Consider  $\{z_1, \dots, z_{n-1}\} \in Y \setminus \{y\}$  such that  $z_i \neq z_j$  if  $i \neq j$ . Let  $M_i$  be a component of  $f^{-1}(z_i)$ , for each  $i \in \{1, \dots, n-1\}$ . By Proposition 2.4,  $\rho_X^{n,m}(\langle M_1, \dots, M_{n-1}, K_i \rangle)$  is a component of  $SC_m^n(f)^{-1}(\rho_Y^{n,m}(\{z_1, \dots, z_{n-1}, y\}))$ , for each  $i \in \{1, 2\}$ . Since  $SC_m^n(f)$  is a monotone mapping,  $SC_m^n(f)^{-1}(\rho_Y^{n,m}(\{z_1, \dots, z_{n-1}, y\}))$  is connected. Thus,  $\rho_X^{n,m}(\langle M_1, \dots, M_{n-1}, K_1 \rangle) = \rho_X^{n,m}(\langle M_1, \dots, M_{n-1}, K_2 \rangle)$ . Then,  $K_1 = K_2$ . A contradiction.  $\square$

### 5. CONFLUENT MAPPINGS

PROPOSITION 5.1. *Let  $f : X \rightarrow Y$  be a mapping between continua.*

- (1) *If  $SC_m^n(f)$  is confluent, then for each subcontinuum  $\mathcal{B} \subset C_m^n(Y)$  and each component  $\mathcal{A}$  of  $C_n(f)^{-1}(\mathcal{B})$ ,  $C_n(f)(\mathcal{A}) = \mathcal{B}$ .*
- (2) *If  $SC_m^n(f)$  is weakly confluent, then for each subcontinuum  $\mathcal{B} \subset C_m^n(Y)$ , there exists  $\mathcal{A}$  be a subcontinuum of  $C_n(X)$  such that  $C_n(f)(\mathcal{A}) = \mathcal{B}$ .*

PROOF. It follows of the facts that  $\rho_X^{n,m}|_{C_m^n(X)}$  and  $\rho_Y^{n,m}|_{C_m^n(Y)}$  are homeomorphisms and

$$C_n(f)|_{C_n(f)^{-1}(C_m^n(Y))} = (\rho_Y^{n,m})^{-1}|_{C_m^n(Y)} \circ SC_m^n(f) \circ \rho_X^{n,m}|_{C_m^n(X)}.$$

$\square$

THEOREM 5.2. *Let  $f : X \rightarrow Y$  be a mapping between continua and let  $n \geq 2$ . We consider the following conditions:*

- (1)  *$f$  is confluent;*
- (2)  *$C_n(f)$  is confluent;*
- (3)  *$SC_m^n(f)$  is confluent for all  $m < n$ .*

*Then condition (2) implies (3) and each of the conditions (2) and (3) implies (1).*

PROOF. (2)  $\Rightarrow$  (3). Since  $C_n(f)$  is confluent,  $\rho_Y^{n,m}$  is monotone and  $\rho_Y^{n,m} \circ C_n(f) = SC_m^n(f) \circ \rho_X^{n,m}$ , by 5.4 of [24, p. 29],  $SC_m^n(f) \circ \rho_X^{n,m}$  is confluent. Hence by 5.16 of [24, p. 32],  $SC_m^n(f)$  is confluent.

We will prove that (3)  $\Rightarrow$  (1). Let  $B$  be a proper subcontinuum of  $Y$  and  $K$  be a component of  $f^{-1}(B)$ . Let  $\{z_1, \dots, z_m\} \subset Y \setminus B$  such that  $z_i \neq z_j$  if  $i \neq j$ . Let  $M_i$  be a component of  $f^{-1}(z_i)$ , for each  $i \in \{1, \dots, m\}$ . By Proposition 2.4,  $\langle M_1, \dots, M_m, K \rangle$  is a component of  $C_n(f)^{-1}(\langle \{z_1\}, \dots, \{z_m\}, B \rangle)$ .

Notice that  $\langle \{z_1\}, \dots, \{z_m\}, B \rangle \subset C_m^n(Y)$ . Then, by the Proposition 5.1,  $C_n(f)(\langle M_1, \dots, M_m, K \rangle) = \langle \{z_1\}, \dots, \{z_m\}, B \rangle$ . Therefore  $f(K) = B$ .

Clearly (2)  $\Rightarrow$  (1). □

**THEOREM 5.3.** *Let  $f : X \rightarrow Y$  be a mapping between continua and let  $n \geq 4$ . Then the following conditions are equivalent.*

1.  $f$  is a monotone mapping.
2.  $SC_m^n(f)$  is a confluent mapping for  $2 \leq m \leq n - 2$ .

**PROOF.** (1)  $\Rightarrow$  (2). It follows from Theorem 4.1 and (3.2) of [24, p. 13].

To prove that  $f$  is a monotone mapping, suppose to the contrary, that is, there exists  $y \in Y$  such that  $f^{-1}(y)$  is not connected. Let  $M_0$  and  $M_1$  be different components of  $f^{-1}(y)$ . By [28, Corollary 5.5], there exists a subcontinuum  $K$  of  $Y$  such that  $y \notin K$ . Take  $\{z_2, \dots, z_m\} \subset Y \setminus (K \cup \{y\})$  such that  $z_i \neq z_j$  if  $i \neq j$ . Let  $M_i$  be a component of  $f^{-1}(z_i)$ , for each  $i \in \{2, \dots, m\}$  and let  $M_{m+1}$  be a component of  $f^{-1}(K)$ . Notice that  $\langle \{y\}, \{z_2\}, \dots, \{z_m\}, K \rangle \subset C_m^n(Y)$  and  $C_{n-1}^n(Y) \cap \langle \{y\}, \{z_2\}, \dots, \{z_m\}, K \rangle \neq \emptyset$ . By Proposition 2.4,  $\langle M_0, \dots, M_{m+1} \rangle$  is a component of  $C_n(f)^{-1}(\langle \{y\}, \{z_2\}, \dots, \{z_m\}, K \rangle)$ . By Proposition 5.1,  $C_n(f)(\langle M_0, \dots, M_{m+1} \rangle) = \langle \{y\}, \{z_2\}, \dots, \{z_m\}, K \rangle$ . Thus  $\langle \{y\}, \{z_2\}, \dots, \{z_m\}, K \rangle \subset C_{n-1}(Y)$ . A contradiction. □

The next example shows that (1) does not imply (3) in Theorem 5.2.

**EXAMPLE 5.4.** There exist continua  $X$  and  $Y$ , and a confluent mapping  $f : X \rightarrow Y$  such that  $SC_m^n(f)$  is not confluent for  $n \geq 4$ .

Consider the mapping  $f : [-1, 1] \rightarrow [0, 1]$  defined by  $f(x) = |x|$ . Clearly,  $f$  is confluent but not monotone. By Theorem 5.3,  $SC_m^n(f)$  is not confluent for  $n \geq 4$  and  $2 \leq m \leq n - 2$ .

**THEOREM 5.5.** *Let  $f : X \rightarrow Y$  be a mapping between continua and let  $n \geq 2$ . Consider the following conditions:*

- (1)  $f$  is weakly confluent;
- (2)  $C_n(f)$  is weakly confluent;
- (3)  $SC_m^n(f)$  is weakly confluent.

*Then condition (2) implies (3) and each of the conditions (2) and (3) implies (1).*

**PROOF.** We will prove that (2)  $\Rightarrow$  (3). Since  $C_n(f)$  is weakly confluent,  $\rho_Y^{n,m}$  is monotone and  $\rho_Y^{n,m} \circ C_n(f) = SC_m^n(f) \circ \rho_X^{n,m}$ , by (5.4) of [24, p. 29],  $SC_m^n(f) \circ \rho_X^{n,m}$  is weakly confluent. By 5.16 of [24, p. 32],  $SC_m^n(f)$  is weakly confluent.

Now, we will show that (3)  $\Rightarrow$  (1). Let  $B$  be a proper subcontinuum of  $Y$ . Consider  $\{z_1, \dots, z_{n-1}\} \subset Y \setminus B$  such that  $z_i \neq z_j$  if  $i \neq j$ . Note that  $\langle \{z_1\}, \dots, \{z_{n-1}\}, B \rangle \subset C_{n-1}^n(Y)$ . Then, by Proposition 5.1, there exists a component  $\mathcal{K}$  of  $C_n(f)^{-1}(\langle \{z_1\}, \dots, \{z_{n-1}\}, B \rangle)$  such that  $C_n(f)(\mathcal{K}) =$

$\langle \{z_1\}, \dots, \{z_{n-1}\}, B \rangle$ . Since  $\mathcal{K} \subset C_{n-1}^n(X)$ , we can find subset  $M_1, \dots, M_n$  of  $X$ , such that  $M_i$  is a component of  $f^{-1}(z_i)$ , for each  $i \in \{1, \dots, n-1\}$ ,  $M_n$  is a component of  $f^{-1}(B)$  and  $\langle M_1, \dots, M_n \rangle \cap \mathcal{K} \neq \emptyset$ . Now, by Proposition 2.4,  $\langle M_1, \dots, M_n \rangle$  is a component of  $C_n(f)^{-1}(\langle \{z_1\}, \dots, \{z_{n-1}\}, B \rangle)$ . Thus  $\mathcal{K} = \langle M_1, \dots, M_n \rangle$ . Hence  $f(M_n) = B$ .

Clearly (2)  $\Rightarrow$  (1). □

**THEOREM 5.6.** *Let  $f : X \rightarrow Y$  be a mapping between continua and let  $n \geq 2$ . Consider the following conditions:*

- (1)  $f$  is hereditarily weakly confluent;
- (2)  $C_n(f)$  is hereditarily weakly confluent;
- (3)  $SC_m^n(f)$  is hereditarily weakly confluent.

*Then condition (2) implies (3) and each of the conditions (2) and (3) implies (1).*

**PROOF.** (2)  $\Rightarrow$  (3). By [12, Theorem 44],  $f$  is a homeomorphism. Then, by Theorem 3.1,  $SC_m^n(f)$  is a homeomorphism. Thus  $SC_m^n(f)$  is hereditarily weakly confluent.

Now we will prove (3)  $\Rightarrow$  (1). Let  $D$  be a subcontinuum of  $X$ , consider two cases,  $f(D) = Y$  or  $f(Y) \neq X$ . Suppose  $f(D) = Y$ . Since  $SC_m^n(f)|_{SC_m^n(D)}$  is weakly confluent, by Theorem 5.5,  $f|_D$  is weakly confluent. Now, suppose that  $f(D)$  is a subcontinuum proper of  $Y$ . Let  $B$  be a subcontinuum of  $f(D)$ . Consider  $\{z_1, \dots, z_{n-1}\} \subset Y \setminus B$  such that  $z_i \neq z_j$  if  $i \neq j$ . Let  $M_i$  be a component of  $f^{-1}(z_i)$ , for each  $i \in \{1, \dots, n-1\}$ . Notice that  $\mathcal{D} = \rho_X^{n,m}(\langle M_1, \dots, M_{n-1}, D \rangle)$  is a subcontinuum of  $SC_m^n(X)$ . Since  $SC_m^n(f)|_{\mathcal{D}}$  is weakly confluent and  $\rho_Y^{n,m}(\langle \{z_1\}, \dots, \{z_{n-1}\}, B \rangle)$  is a subcontinuum of  $SC_m^n(f)(\mathcal{D})$ , there exists a subcontinuum  $\mathcal{W}$  of  $\mathcal{D}$  such that  $SC_m^n(f)(\mathcal{W}) = \rho_Y^{n,m}(\langle \{z_1\}, \dots, \{z_{n-1}\}, B \rangle)$ . Moreover,  $(\rho_X^{n,m})^{-1}(\mathcal{W})$  is a subcontinuum of  $\langle M_1, \dots, M_{n-1}, D \rangle$  such that  $C_n(f)((\rho_X^{n,m})^{-1}(\mathcal{W})) = \langle \{z_1\}, \dots, \{z_{n-1}\}, B \rangle$ . By [20, Lemma 3.1 ],  $K = \bigcup (\rho_X^{n,m})^{-1}(\mathcal{W}) \cap D$  is a nonempty connected set. Thus  $f(K) = B$ . □

**THEOREM 5.7.** *Let  $f : X \rightarrow Y$  be a mapping between continua and  $n \geq 2$ . Consider the following conditions:*

- (1)  $f$  is semi-confluent;
- (2)  $C_n(f)$  is semi-confluent;
- (3)  $SC_m^n(f)$  is semi-confluent.

*Then (2) implies (3) and each of the conditions (2) and (3) implies (1).*

**PROOF.** (2)  $\Rightarrow$  (3). Since  $\rho_Y^{n,m}$  is monotone, by Statement 1.1 of [10, p. 41],  $\rho_Y^{n,m} \circ C_n(f)$  is semi-confluent. Thus  $SC_m^n(f) \circ \rho_X^{n,m}$  is semi-confluent. By 5.16 of [24, p. 32],  $SC_m^n(f)$  is semi-confluent.

We will prove that (3)  $\Rightarrow$  (1). Let  $B$  be a proper subcontinuum of  $Y$ . Let  $C_1$  and  $C_2$  are components of  $f^{-1}(B)$ . Consider  $\{z_1, \dots, z_m\} \subset Y \setminus B$



such that  $z_i \neq z_j$  if  $i \neq j$ . Let  $M_i$  be a component of  $f^{-1}(z_i)$ , for each  $i \in \{1, \dots, m\}$ . By Proposition 2.4,  $\rho_X^{n,m}(\langle M_1, \dots, M_m, C_i \rangle)$  is a component of  $SC_m^n(f)^{-1}(\rho_Y^{n,m}(\langle \{z_1\}, \dots, \{z_m\}, B \rangle))$ , for each  $i \in \{1, 2\}$ . Without loss of generality we can suppose that  $SC_m^n(f)(\rho_X^{n,m}(\langle M_1, \dots, M_m, C_1 \rangle)) \subset SC_m^n(f)(\rho_X^{n,m}(\langle M_1, \dots, M_m, C_2 \rangle))$ . Thus

$$C_n(f)(\langle M_1, \dots, M_m, C_1 \rangle) \subset C_n(f)(\langle M_1, \dots, M_m, C_2 \rangle).$$

Hence  $f(C_1) \subset f(C_2)$ .

(2)  $\Rightarrow$  (1) is obvious. □

### 6. OM MAPPINGS

In this part we use the following definition to limit superior of a sequence of subsets

DEFINITION 6.1. *Let  $X$  be a continuum. Given a sequence  $\{A_m\}_{m \in \mathbb{N}}$  of subsets of  $X$  we define  $\limsup_{m \rightarrow \infty} A_m$  as the set of points  $x \in X$  such that there exists a sequence of positive numbers  $m_1 < m_2 < \dots$  and points  $x_{m_k} \in A_{m_k}$  such that  $\lim x_{m_k} = x$ .*

The following characterization of OM mappings was showed in [12, p. 788].

LEMMA 6.2. *A mapping  $f : X \rightarrow Y$  between continua is an OM-mapping if and only if, for each point  $y \in Y$  and each sequence of points  $y_m \in Y$  tending to  $y$  the set  $\limsup_{m \rightarrow \infty} f^{-1}(y_m)$  meets each component of  $f^{-1}(y)$ .*

THEOREM 6.3. *Let  $f : X \rightarrow Y$  be a mapping between continua and let  $n \geq 2$ . Consider the following conditions:*

- (1)  *$f$  is an OM-mapping;*
- (2)  *$C_n(f)$  is an OM-mapping;*
- (3)  *$SC_m^n(f)$  is an OM-mapping.*

*Then condition (2) implies (3) and each of the conditions (2) and (3) implies (1).*

PROOF. (2)  $\Rightarrow$  (3). Since  $C_n(f)$  is an OM-mapping,  $\rho_Y^{n,m}$  is monotone and  $\rho_Y^{n,m} \circ C_n(f) = SC_m^n(f) \circ \rho_X^{n,m}$ , by 5.4 of [24, p. 28],  $SC_m^n(f) \circ \rho_X^{n,m}$  is an OM-mapping. Hence by 5.6 of [24, p. 33],  $SC_m^n(f)$  is an OM-mapping.

To prove (3)  $\Rightarrow$  (1), let  $\{y_r\}_{r \in \mathbb{N}}$  be a sequence of points in  $Y$  converging to a point  $y \in Y$ . Consider  $\{z_1, \dots, z_{n-1}\} \subset Y \setminus (\{y\} \cup \{y_r\}_{r \in \mathbb{N}})$  such that  $z_i \neq z_j$  if  $i \neq j$ . Let  $M_i$  be a component of  $f^{-1}(z_i)$ , for each  $i \in \{1, \dots, n-1\}$  and let  $M_n$  be a component of  $f^{-1}(y)$ . By Proposition 2.4,  $\rho_X^{n,m}(\langle M_1, \dots, M_n \rangle)$  is a component of  $SC_m^n(f)^{-1}(\rho_Y^{n,m}(\{z_1, \dots, z_{n-1}, y\}))$ . Clearly the sequence  $\{\rho_Y^{n,m}(\{z_1, \dots, z_{n-1}, y_r\})\}_{r \in \mathbb{N}}$  converges to  $\rho_Y^{n,m}(\{z_1, \dots, z_{n-1}, y\})$ .

Since  $SC_m^n(f)$  is an OM-mapping, by Lemma 6.2,

$$\rho_X^{n,m}(\langle M_1, \dots, M_n \rangle) \cap \limsup_{s \rightarrow \infty} SC_m^n(f)^{-1}(\rho_Y^{n,m}(\{z_1, \dots, z_{n-1}, y_s\})) \neq \emptyset.$$

Let  $A \in \langle M_1, \dots, M_n \rangle$  such that

$$\rho_X^{n,m}(A) \in \limsup_{s \rightarrow \infty} SC_m^n(f)^{-1}(\rho_Y^{n,m}(\{z_1, \dots, z_{n-1}, y_s\})).$$

Then, there exists a subsequence  $\{\rho_X^{n,m}(A_{s_k})\}_{k \in \mathbb{N}}$  such that for each positive integer  $k$ ,  $\rho_X^{n,m}(A_{s_k}) \in SC_m^n(f)^{-1}(\rho_Y^{n,m}(\{z_1, \dots, z_{n-1}, y_{s_k}\}))$  and

$$\lim_{k \rightarrow \infty} \rho_X^{n,m}(A_{s_k}) = \rho_X^{n,m}(A).$$

Let  $a \in A \cap M_n$ . Since  $\lim_{k \rightarrow \infty} A_{s_k} = A_n$ , there exists a sequence  $\{a_{s_k}\}_{k \in \mathbb{N}}$ , with  $a_{s_k} \in A_{s_k}$ , such that converges to  $a \in A$ . Thus exists a positive integer  $l_0$  such that  $f(a_{s_k}) = y_{s_k}$  for each  $k \geq l_0$ . Hence

$$a \in A \cap M_n \cap \limsup_{s \rightarrow \infty} f^{-1}(y_s).$$

Therefore, by Lemma 6.2,  $f$  is  $OM$  mapping.

Clearly (2)  $\Rightarrow$  (1). □

**COROLLARY 6.4.** *Let  $f : X \rightarrow Y$  be a mapping between continua. Then the following conditions are equivalent.*

- (1)  $f$  is an  $OM$ -mapping.
- (2)  $C_2(f)$  is an  $OM$ -mapping.
- (3)  $SC_1^2(f)$  is an  $OM$ -mapping.

**PROOF.** It follows from [12, Theorem 14] and Theorem 6.3. □

**THEOREM 6.5.** *Let  $f : X \rightarrow Y$  be a mapping between continua and let  $n \geq 3$ . Then the following conditions are equivalent.*

- (1)  $f$  is monotone.
- (2)  $C_n(f)$  is monotone.
- (3)  $C_n(f)$  is  $OM$ -mapping.
- (4)  $SC_m^n(f)$  is an  $OM$ -mapping,  $2 \leq m \leq n - 2$ .

**PROOF.** By Theorem 4.1, (1) and (2) are equivalent. (1)  $\Rightarrow$  (3) follows from Theorem 4.1. By Theorem 6.3, (3)  $\Rightarrow$  (4). We will prove that (4)  $\Rightarrow$  (1). Since  $SC_m^n(f)$  is an  $OM$ -mapping, by (4.9) of [24, p. 16],  $SC_m^n(f)$  is confluent. Thus, by Theorem 5.3,  $f$  is monotone. □

The next example shows that (1) does not imply (3) in Theorem 6.3.

**EXAMPLE 6.6.** There exist continua  $X$  and  $Y$  and an  $OM$ -mapping  $f : X \rightarrow Y$  such that  $SC_m^n(f)$  is not an  $OM$ -mapping.

Consider the mapping  $f : [-1, 1] \rightarrow [0, 1]$  defined by  $f(x) = |x|$ . Clearly,  $f$  is an  $OM$ -mapping but not monotone. By Theorem 6.5,  $SC_m^n(f)$  is not an  $OM$ -mapping for  $n \geq 4$  and  $2 \leq m \leq n - 2$ .

7. JOINING MAPPINGS

THEOREM 7.1. *Let  $f : X \rightarrow Y$  be a mapping between continua and let  $n \geq 2$ . Consider the following conditions:*

- (1)  $f$  is joining;
- (2)  $C_n(f)$  is joining;
- (3)  $SC_m^n(f)$  is joining.

*Then condition (2) implies (3) and each of the conditions (2) and (3) implies (1).*

PROOF. (2)  $\Rightarrow$  (3). Since  $\rho_Y^{n,m}$  is monotone and  $C_n(f)$  is a joining mapping,  $\rho_Y^{n,m} \circ C_n(f)$  is a joining mapping. Thus  $SC_m^n(f) \circ \rho_X^{n,m}$  is a joining mapping. By (5.21) of [24, p. 33],  $SC_m^n(f)$  is a joining mapping.

We will prove that (3)  $\Rightarrow$  (1). Let  $B$  be a proper subcontinuum of  $Y$ . Let  $C_1$  and  $C_2$  components of  $f^{-1}(B)$ . Consider  $\{z_1, \dots, z_m\} \subset Y \setminus B$  such that  $z_i \neq z_j$  if  $i \neq j$ . For each  $i \in \{1, \dots, m\}$ , let  $M_i$  be a component of  $f^{-1}(z_i)$ . By Proposition 2.4  $\rho_X^{n,m}(\langle M_1, \dots, M_m, C_i \rangle)$  is be a component of  $SC_m^n(f)^{-1}(\rho_Y^{n,m}(\{\{z_1\}, \dots, \{z_m\}, B\}))$  for  $i \in \{1, 2\}$ . Since  $SC_m^n(f)$  is a joining mapping,

$$SC_m^n(f)(\rho_X^{n,m}(\langle M_1, \dots, M_m, C_1 \rangle)) \cap SC_m^n(f)(\rho_X^{n,m}(\langle M_1, \dots, M_m, C_2 \rangle)) \neq \emptyset.$$

So  $C_n(f)(\langle M_1, \dots, M_m, C_1 \rangle) \cap C_n(f)(\langle M_1, \dots, M_m, C_2 \rangle) \neq \emptyset$ . Thus  $f(C_1) \cap f(C_2) \neq \emptyset$ .

Clearly (2)  $\Rightarrow$  (1). □

8. LIGHT MAPPINGS

THEOREM 8.1. *Let  $f : X \rightarrow Y$  be a mapping between continua and let  $n \geq 2$ . Consider the following conditions:*

- (1)  $f$  is light;
- (2)  $C_n(f)$  is light;
- (3)  $SC_m^n(f)$  is light.

*Then each of the conditions (2) and (3) implies (1).*

PROOF. (2)  $\Rightarrow$  (1). It follows from  $C(f) = C_n(f)|_{C(X)}$  and [6, Theorem 3.10].

(3)  $\Rightarrow$  (1). To prove that  $f$  is a light mapping, suppose that exists  $y \in Y$  such that  $f^{-1}(y)$  is not totally disconnected. Let  $M$  be a nondegenerate component of  $f^{-1}(y)$ . Consider  $\{y_1, \dots, y_{n-1}\} \subset Y \setminus \{y\}$  such that  $y_i \neq y_j$  if  $i \neq j$ . Let  $M_i$  be a component of  $f^{-1}(y_i)$ , for each  $i \in \{1, \dots, n - 1\}$ . By Proposition 2.4  $\rho_X^{n,m}(\langle M_1, \dots, M_{n-1}, M \rangle)$  is a subcontinuum nondegenerate of  $SC_m^n(f)^{-1}(\rho_Y^{n,m}(\{\{y_1, \dots, y_{n-1}, y\}\}))$ , a contradiction. □

The next example shows that (1) does not imply (3) in Theorem 8.1.

EXAMPLE 8.2. There exist continua  $X$  and  $Y$  and a light mapping  $f : X \rightarrow Y$  such that  $SC_m^n(f) : SC_m^n(X) \rightarrow SC_m^n(Y)$  is not light.

Let  $\mathbb{C}$  be the complex plane and  $S^1 \subset \mathbb{C}$  the unit circle centered at the origin. Fix  $n$  and  $m$ . Define  $f : S^1 \rightarrow S^1$  by  $f(z) = z^n$ . Clearly  $f$  is a light mapping. Consider

$$\mathcal{A} = \left\{ \rho_X^{n,m} \left( \left\{ e^{i\theta}, e^{i(\frac{2\pi}{n}+\theta)}, \dots, e^{i(\frac{2(n-1)\pi}{n}+\theta)} \right\} \right) \in SC_m^n(X) : 0 \leq \theta \leq \frac{2\pi}{n} \right\}.$$

Notice that  $\mathcal{A} \subset SC_m^n(f)^{-1}(\rho_Y^{n,m}(C_m(S^1)))$ . It is easy to show that  $\mathcal{A}$  is an arc in  $SC_m^n(X)$ . Therefore  $SC_m^n(f)$  is not a light mapping.

9. ATOMIC MAPPINGS

THEOREM 9.1. *Let  $f : X \rightarrow Y$  be a mapping between continua and  $n \geq 2$ . The conditions are equivalent.*

- (1)  $f$  is homeomorphism.
- (2)  $C_n(f)$  is atomic.
- (3)  $SC_m^n(f)$  is atomic.

PROOF. By Theorem 3.1, (1) implies (2) and (3).

(2)  $\Rightarrow$  (1) is proved in [12, Theorem 43].

(3)  $\Rightarrow$  (1) follows from Proposition 2.1 and (6.3) of [24, p. 51]. □

10. ATRIODIC MAPPINGS

THEOREM 10.1. *Let  $f : X \rightarrow Y$  be a mapping between continua and  $n \geq 2$ . If  $SC_m^n(f)$  is atriodic, then  $f$  is atriodic.*

PROOF. Let  $D$  be a proper subcontinuum of  $Y$ . Consider  $\{z_1, \dots, z_{n-1}\} \subset Y \setminus D$  such that  $z_i \neq z_j$  if  $i \neq j$ . Then there exist two components  $\mathcal{D}_1$  and  $\mathcal{D}_2$  of  $SC_m^n(f)^{-1}(\rho_Y^{n,m}(\langle \{z_1\}, \dots, \{z_{n-1}\}, D \rangle))$  such that

- 1.  $SC_m^n(f)(\mathcal{D}_1) \cup SC_m^n(f)(\mathcal{D}_2) = \rho_Y^{n,m}(\langle \{z_1\}, \dots, \{z_{n-1}\}, D \rangle)$ ,
- 2. for each component  $\mathcal{K}$  of  $SC_m^n(f)^{-1}(\rho_Y^{n,m}(\langle \{z_1\}, \dots, \{z_{n-1}\}, D \rangle))$ , we have either  $SC_m^n(f)(\mathcal{K}) = \rho_Y^{n,m}(\langle \{z_1\}, \dots, \{z_{n-1}\}, D \rangle)$  or  $SC_m^n(f)(\mathcal{K}) \subset SC_m^n(f)(\mathcal{D}_1)$  or  $SC_m^n(f)(\mathcal{K}) \subset SC_m^n(f)(\mathcal{D}_2)$ .

Notice that for each  $j \in \{1, 2\}$ ,  $(\rho_X^{n,m})^{-1}(\mathcal{D}_j) \subset C_{n-1}^n(X)$ . So, we can find subsets  $M_1^j, \dots, M_n^j$  of  $X$  such that for each  $i \in \{1, \dots, n-1\}$ ,  $M_i^j$  is be a component of  $f^{-1}(z_i)$ ,  $M_n^j$  is a component of  $f^{-1}(D)$  and since  $\mathcal{D}_j$  is a component of  $SC_m^n(f)^{-1}(\rho_Y^{n,m}(\langle \{z_1\}, \dots, \{z_{n-1}\}, D \rangle))$  then  $\mathcal{D}_j \cap \rho_X^{n,m}(\langle M_1^j, \dots, M_n^j \rangle) \neq \emptyset$ .

It follows from Proposition 2.4 that  $\mathcal{D}_j = \rho_X^{n,m}(\langle M_1^j, \dots, M_n^j \rangle)$ . Thus, by 1.,  $f(M_n^1) \cup f(M_n^2) = D$ .

Now, let  $K$  be a component of  $f^{-1}(D)$ . Since  $\rho_X^{n,m}(\langle M_1^1, \dots, M_{n-1}^1, K \rangle)$  is a component of  $SC_m^n(f)^{-1}(\rho_Y^{n,m}(\langle \{y_1\}, \dots, \{y_{n-1}\}, D \rangle))$  by 2.,  $f(K) = D$  or  $f(K) \subset f(M_n^1)$  or  $f(K) \subset f(M_n^2)$ . □

11. LOCALLY MAPPINGS

**THEOREM 11.1.** *Let  $f : X \rightarrow Y$  be a mapping between continua and let  $n \geq 3$ . If  $SC_m^n(f)$  is a local homeomorphism then,  $f$  is a homeomorphism.*

**PROOF.** It follows from (4.26) of [24, p. 20] and Theorem 3.2. □

S. Nadler Jr. in [26, Lemma 2.3], defines a metric for a quotient space obtained identifying a subcontinuum to a point. In a similar way we are going to define a metric on the  $SC_m^n(X)$ . Let

$$\Gamma_{n,m} = \{C_m(X) \cup \{A\} : A \in C_n(X)\}.$$

Define  $G_{n,m} : SC_m^n(X) \rightarrow \Gamma_{n,m}$  by

$$G_{n,m}(\rho_X^{n,m}(A)) = C_m(X) \cup (\rho_X^{n,m})^{-1}(\rho_X^{n,m}(A)).$$

Then  $G_{n,m}$  is a homeomorphism. Next, define

$$\sigma_{n,m} : SC_m^n(X) \times SC_m^n(X) \rightarrow [0, \infty)$$

by

$$\sigma_{n,m}(\rho_X^{n,m}(A), \rho_X^{n,m}(B)) = \mathcal{H}^2(G_{n,m}(\rho_X^{n,m}(A)), G_{n,m}(\rho_X^{n,m}(B))),$$

where  $\mathcal{H}^2$  is the Hausdorff metric induced by the Hausdorff metric  $\mathcal{H}$  on  $C_n(X)$ . Thus  $\sigma_{n,m}$  is a metric.

**LEMMA 11.2.** *Let  $X$  be a continuum,  $\varepsilon > 0$  and  $n \geq 2$ . Let  $A, B \in C_n(X)$  such that  $\mathcal{H}(A, B) < \varepsilon$ , then  $\sigma_{n,m}(\rho_X^{n,m}(A), \rho_X^{n,m}(B)) < \varepsilon$ .*

**PROOF.** Let  $A, B \in C_n(X)$  such that  $\mathcal{H}(A, B) < \varepsilon$ , then  $\{B\} \in \mathcal{V}_\varepsilon^{\mathcal{H}}(\{A\})$  and  $\{A\} \in \mathcal{V}_\varepsilon^{\mathcal{H}}(\{B\})$ . Notice that  $G_{n,m}(\rho_X^{n,m}(A)) \subset \mathcal{V}_\varepsilon^{\mathcal{H}}(G_{n,m}(\rho_X^{n,m}(B)))$  and  $G_{n,m}(\rho_X^{n,m}(B)) \subset \mathcal{V}_\varepsilon^{\mathcal{H}}(G_{n,m}(\rho_X^{n,m}(A)))$ . Thus

$$\mathcal{H}^2(G_{n,m}(\rho_X^{n,m}(A)), G_{n,m}(\rho_X^{n,m}(B))) < \varepsilon.$$

□

**LEMMA 11.3.** *Let  $X$  be a continuum,  $\varepsilon > 0$  and  $n \geq 2$ . Take  $A \in C(X)$  such that  $\text{diam}(A) < \varepsilon$  and let  $\{z_1, \dots, z_m\} \subset X \setminus A$  such that  $z_i \neq z_j$  if  $i \neq j$ . Then  $\text{diam}(\rho_X^{n,m}(\{\{z_1\}, \dots, \{z_m\}, A\})) < \varepsilon$ .*

**PROOF.** Let  $B, D \in \langle \{z_1\}, \dots, \{z_m\}, A \rangle$ . Consider  $D_1 = D \cap A$  and  $B_1 = B \cap A$ . Notice that  $d(x, c) < \text{diam}(A)$  for each  $c \in D_1$  and for each  $x \in B_1$ . Then  $B \subset \mathcal{V}_\varepsilon^d(D)$  and  $D \subset \mathcal{V}_\varepsilon^d(B)$ . Thus  $\mathcal{H}(B, D) < \varepsilon$ . By Lemma 11.2,  $\sigma_{n,m}(\rho_X^{n,m}(B), \rho_X^{n,m}(D)) < \varepsilon$ . Therefore  $\text{diam}(\rho_X^{n,m}(\{\{z_1\}, \dots, \{z_m\}, A\})) < \varepsilon$ . □

**THEOREM 11.4.** *Let  $f : X \rightarrow Y$  be a mapping between continua and  $n \geq 2$ . If  $SC_m^n(f)$  is locally confluent then,  $f$  is locally confluent.*

PROOF. By [24, Theorem 4.38], there exists  $\varepsilon > 0$  such that for each subcontinuum  $\mathcal{A}$  of  $SC_m^n(Y)$  of diameter less than  $\varepsilon$  and each component  $\mathcal{D}$  of  $SC_m^n(f)^{-1}(\mathcal{A})$  is mapped by  $SC_m^n(f)$  onto  $\mathcal{A}$ . Let  $B$  be a subcontinuum of  $Y$  of diameter less than  $\varepsilon$  and  $D$  be a component of  $f^{-1}(B)$ . Consider  $\{z_1, \dots, z_m\} \subset Y \setminus B$  such that  $z_i \neq z_j$  if  $i \neq j$ . Let  $M_i$  be a component of  $f^{-1}(z_i)$ , for each  $i \in \{1, \dots, m\}$ . By Lemma 11.3,  $\rho_Y^{n,m}(\langle \{z_1\}, \dots, \{z_m\}, B \rangle)$  has diameter less than  $\varepsilon$ . By Proposition 2.4,  $\rho_X^{n,m}(\langle M_1, \dots, M_m, D \rangle)$  is a component of  $SC_m^n(f)^{-1}(\rho_Y^{n,m}(\langle \{z_1\}, \dots, \{z_m\}, B \rangle))$ . Thus

$$SC_m^n(f)(\rho_X^{n,m}(\langle M_1, \dots, M_m, D \rangle)) = (\rho_Y^{n,m}(\langle \{z_1\}, \dots, \{z_m\}, B \rangle)).$$

We conclude that  $f(D) = B$ . By [24, Theorem 4.38],  $f$  is locally confluent. □

THEOREM 11.5. *Let  $f : X \rightarrow Y$  be a mapping between continua and  $n \geq 2$ . If  $SC_m^n(f)$  is locally weakly confluent then,  $f$  is locally weakly confluent.*

PROOF. By 4.37 of [24, p. 23], there exists  $\varepsilon > 0$  such that for each subcontinuum  $\mathcal{A}$  of  $SC_m^n(Y)$  of diameter less than  $\varepsilon$ , there exists a component  $\mathcal{D}$  of  $SC_m^n(f)^{-1}(\mathcal{A})$  such that  $SC_m^n(f)(\mathcal{D}) = \mathcal{A}$ . Let  $B$  be a subcontinuum of  $Y$  of diameter less than  $\varepsilon$ . Consider  $\{z_1, \dots, z_{n-1}\} \subset Y \setminus B$  such that  $z_i \neq z_j$  if  $i \neq j$ . By Lemma 11.3,  $\rho_Y^{n,m}(\langle \{z_1\}, \dots, \{z_{n-1}\}, B \rangle)$  has diameter less than  $\varepsilon$ . Since  $SC_m^n(f)$  is a locally weakly confluent mapping, there exists a component  $\mathcal{B}$  of  $SC_m^n(f)^{-1}(\rho_Y^{n,m}(\langle \{z_1\}, \dots, \{z_{n-1}\}, B \rangle))$  such that  $SC_m^n(f)(\mathcal{B}) = \rho_Y^{n,m}(\langle \{z_1\}, \dots, \{z_{n-1}\}, B \rangle)$ . Notice that  $(\rho_Y^{n,m})^{-1}(\mathcal{B}) \subset C_{n-1}^n(X)$ . Thus, we can find subset  $M_1, \dots, M_n$  of  $X$ , such that for each  $i \in \{1, \dots, n-1\}$ ,  $M_i$  is a component of  $f^{-1}(z_i)$ ,  $M_n$  is a component of  $f^{-1}(B)$  and  $\rho_X^{n,m}(\langle M_1, \dots, M_n \rangle) \cap \mathcal{B} \neq \emptyset$ . It follows from Proposition 2.4 that  $\rho_X^{n,m}(\langle M_1, \dots, M_n \rangle)$  is a component of

$$SC_m^n(f)^{-1}(\rho_Y^{n,m}(\langle \{z_1\}, \dots, \{z_{n-1}\}, B \rangle)).$$

Hence  $\mathcal{B} = \rho_X^{n,m}(\langle M_1, \dots, M_n \rangle)$ . Therefore  $f(M_n) = B$ . By [24, Theorem 4.37],  $f$  is locally weakly confluent. □

## 12. PROBLEMS

- (1) Is there weakly confluent mapping  $f : X \rightarrow Y$  such that either  $C_n(f)$  is not weakly confluent or  $SC_m^n(f)$  is not weakly confluent?
- (2) Is there hereditarily weakly confluent mapping  $f : X \rightarrow Y$  such that either  $C_n(f)$  is not hereditarily weakly confluent or  $SC_m^n(f)$  is not hereditarily weakly confluent?
- (3) Is there semi-confluent mapping  $f : X \rightarrow Y$  such that either  $C_n(f)$  is not semi-confluent or  $SC_m^n(f)$  is not semi-confluent?
- (4) Is there  $OM$ -mapping  $f : X \rightarrow Y$  such that either  $C_n(f)$  is not  $OM$  or  $SC_m^n(f)$  is not  $OM$  for  $n > 2$ ?

- (5) Is there joining mapping  $f : X \rightarrow Y$  such that either  $C_n(f)$  is not joining or  $SC_m^n(f)$  is not joining?

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## REFERENCES

- [1] J. Camargo, *Openness of the induced map  $C_n(f)$* , Bol. Mat. (N.S.) **16** (2009), 115–123.
- [2] J. Camargo and S. Macías, *On strongly freely decomposable and induced maps*, Glas. Mat. Ser. III **48** (2013), 429–442.
- [3] F. Barragán, *Induced maps on  $n$ -fold symmetric product suspensions*, Topology Appl. **158** (2011), 1192–1205.
- [4] J. J. Charatonik, *Recent results on induced mappings between hyperspaces of continua*, Topology. Proc. **22** (1997), 103–122.
- [5] J. J. Charatonik and W. J. Charatonik, *Hereditarily weakly confluent induced mappings are homeomorphisms*, Colloq. Math. **75** (1998), 195–203.
- [6] J. J. Charatonik and W. J. Charatonik, *Lightness of induced mappings*, Tsukuba J. Math. **22** (1998), 179–192.
- [7] J. J. Charatonik and W. J. Charatonik, *Induced MO-mappings*, Tsukuba J. Math. **23** (1999), 245–252.
- [8] J. J. Charatonik and W. J. Charatonik, *Limit properties of induced mappings*, Topology Appl. **100** (2000), 103–118.
- [9] J. J. Charatonik and W. J. Charatonik, *Inverse limits and openness of the induced mappings*, Topology Appl. **114** (2001), 235–260.
- [10] J. J. Charatonik and W. J. Charatonik *Semi-confluent mappings*, Math. Pannon. **12** (2001), 39–54.
- [11] J. J. Charatonik, W. J. Charatonik and A. Illanes, *Openness of induced mappings*, Topology Appl. **98** (1999), 67–80.
- [12] J. J. Charatonik, A. Illanes and S. Macías, *Induced mappings on the hyperspaces  $C_n(X)$  of a continuum  $X$* , Houston J. Math. **28** (2002), 781–805.
- [13] W. J. Charatonik, *Arc approximation property and confluence of induced mappings*, Rocky Mountain J. Math. **28** (1998), 107–154.
- [14] W. J. Charatonik, *Openness and monotoneity of induced mappings*, Proc. Amer. Math. Soc. **127** (1999), 3729–3731.
- [15] J. Dugundji, Topology, Allyn and Bacon, Inc., Boston, 1966.
- [16] G. Higuera and A. Illanes, *Induced mappings on symmetric products*, Topology Proc. **37** (2011), 367–401.
- [17] H. Hosokawa, *Induced mappings between hyperspaces*, Bull. Tokyo Gakugei Univ. (4) **41** (1989), 1–6.
- [18] H. Hosokawa, *Mappings of hyperspaces induced by refinable mappings*, Bull. Tokyo Gakugei Univ. (4) **42** (1990), 1–8.
- [19] H. Hosokawa, *Induced mappings between hyperspaces. II*, Bull. Tokyo Gakugei Univ. (4) **44** (1992), 1–7.
- [20] H. Hosokawa, *Induced mappings on hyperspaces*, Tsukuba J. Math. **21** (1997), 239–259.

- [21] H. Hosokawa, *Induced mappings on hyperspaces. II*, Tsukuba J. Math. **21** (1997), 773–783.
- [22] A. Illanes and S. B. Nadler, Jr., *Hyperspaces. Fundamentals and recent advances*, Marcel Dekker, New York, 1999.
- [23] S. Macías, *On the hyperspaces  $C_n(X)$  of a continuum  $X$* , Topology Appl. **109** (2001), 237–256.
- [24] T. Maćkowiak, *Continuous mappings on continua*, Dissertationes Math. (Rozprawy Mat.) **158** (1979), 95 pp.
- [25] T. Maćkowiak, *Singular arc-like continua*, Dissertationes Math. (Rozprawy Mat.) **257** (1986), 40 pp.
- [26] S. B. Nadler, Jr., *A fixed point theorem for hyperspace suspensions*, Houston J. Math. **5** (1979), 125–132.
- [27] S. B. Nadler, Jr., *Hyperspaces of Sets*, Marcel Dekker, New York, 1978.
- [28] S. B. Nadler Jr., *Continuum theory. An introduction*, Marcel Dekker, New York, 1992.

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