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Decompositions of the Cauchy and Ferrers-Jackson polynomials

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Abstract. Recently, Witula and Slota have given decompositions of the Cauchy and Ferrers-Jackson polynomials [Cauchy, Ferrers-Jackson and Chebyshev polynomials and identities for the powers of elements of some conjugate recurrence sequences, Central Europan J. Math., 2006]. Our main purpose is to derive a different decomposition of the Cauchy and Ferrers-Jackson polynomials. Our approach is to use the Waring formula and the Saalschütz identity to prove the claimed results. Also, we obtain generalizations of the results of Carlitz, Hunter and Koshy as corollaries of our results about sums and differences of powers of the Fibonacci and Lucas numbers.

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1. Introduction

For $n \in \mathbb{N}$, the Cauchy and Ferrers-Jackson polynomials are defined by

$$p_n(x,y) := (x+y)^{2n+1} - x^{2n+1} - y^{2n+1}$$

and

$$q_n(x,y) := (x+y)^{2n} + x^{2n} + y^{2n},$$

respectively. Some authors have studied their decompositions. Recently, Witula and Slota [8] have obtained the following decompositions

$$p_n(x,y) = \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \frac{2n+1}{n-k} \binom{n-k}{2k+1} (xy(x+y))^{2k+1} \left(x^2+xy+y^2\right)^{n-3k-1} \tag{1}$$

and

$$q_n(x,y) = \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \frac{2n}{n-k} \binom{n-k}{2k} (xy(x+y))^{2k} \left(x^2+xy+y^2\right)^{n-3k}. \tag{2}$$

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The proof of these decompositions was given by induction based on simple recurrence dependence between polynomials $p_n(x,y)$ and $q_n(x,y)$.

In this paper, our main purpose is to derive alternative approach to obtain different decompositions of the Cauchy and Ferrers-Jackson polynomials. To prove the claimed result, our approach is to use the Waring formula and the Saalschütz identity given by

$$a^m + b^m = \sum_{k=0}^{\lfloor m/2 \rfloor} (-1)^k \frac{m}{m-k} \binom{m-k}{k} (ab)^k (a+b)^{m-2k}, \quad m > 0,$$

and for $n \geq 0$

$$F\left(\begin{matrix} a, b, -n \\ c, a+b-c-n+1 \end{matrix} \middle| 1 \right) = \frac{(c-a)^{\overline{n}} (c-b)^{\overline{n}}}{c^{\overline{n}} (c-a-b)^{\overline{n}}},$$

respectively, where $\overline{n}$ is the rising factorial power: $\overline{n} = x(x+1)\dots(x+n-1)$ (for more details about the Saalschütz identity and the rising factorial power, we refer to [4]).

It would be much valuable to note that the decompositions of the Cauchy and Ferrers-Jackson polynomials are very closely related to the identities given by Carlitz, Hunter and Koshy on differences and sums of powers of the Fibonacci and Lucas numbers up to seventh and eighth powers, respectively.

We also present generalizations of the identities of Carlitz [1], Hunter and Koshy [5] including the Fibonacci and Lucas numbers for any odd and even powers as applications of our main results

Recall that the well-known Fibonacci sequence $\{F_n\}$ is defined by the recurrence

$$F_n = F_{n-1} + F_{n-2}, \quad n > 1$$

with initials $F_0 = 0$ and $F_1 = 1$. The Lucas sequence $\{L_n\}$ satisfies the same recurrence relation but the initials are $L_0 = 2$ and $L_1 = 1$. A formula for the generating functions of powers of general cases of these sequences and squaring terms of an ℓ-sequence were also given in [6] and [7], respectively.

Now we recall the results of Carlitz [1]. He proposed the following interesting identities as advanced problems for $n > 1$

$$\begin{aligned}
F_{n+1}^3 - F_n^3 - F_{n-1}^3 &= 3F_{n+1}F_nF_{n-1}, \\
L_{n+1}^3 - L_n^3 - L_{n-1}^3 &= 3L_{n+1}L_nL_{n-1}, \\
F_{n+1}^5 - F_n^5 - F_{n-1}^5 &= 5F_{n+1}F_nF_{n-1}\left(2F_n^2 + (-1)^n\right), \\
L_{n+1}^5 - L_n^5 - L_{n-1}^5 &= 5L_{n+1}L_nL_{n-1}\left(2L_n^2 - 5(-1)^n\right), \\
F_{n+1}^7 - F_n^7 - F_{n-1}^7 &= 7F_{n+1}F_nF_{n-1}\left(2F_n^2 + (-1)^n\right)^2, \\
L_{n+1}^7 - L_n^7 - L_{n-1}^7 &= 7L_{n+1}L_nL_{n-1}\left(2L_n^2 - 5(-1)^n\right)^2. \qquad (3)
\end{aligned}$$

Then Carlitz proved his propositions two years later. Also, Charles Wall independently solved the problem at the same time (see [2]). For the sums of powers of Fibonacci and Lucas numbers, Carlitz, Hunter and Koshy gave the following identities,

$$\begin{aligned}
F_{n+1}^4 + F_n^4 + F_{n-1}^4 &= 2\left(2F_n^2 + (-1)^n\right)^2,\\
L_{n+1}^4 + L_n^4 + L_{n-1}^4 &= 2\left(2L_n^2 - 5(-1)^n\right)^2,\\
F_{n+1}^6 + F_n^6 + F_{n-1}^6 &= 2\left(2F_n^2 + (-1)^n\right)^3 + 3F_{n-1}^2F_n^2F_{n+1}^2,\\
F_{n+1}^8 + F_n^8 + F_{n-1}^8 &= 2\left(2F_n^2 + (-1)^n\right)^4 + 8F_{n-1}^2F_n^2\\
&\quad\times\left(F_{n-1}^4 + F_n^4 + 4F_{n-1}^2F_n^2 + 3F_{n-1}F_nF_{2n-1}\right). \qquad (4)
\end{aligned}$$

2. Decomposition of the Cauchy and Ferrers-Jackson polynomials

Before our main results, we give a lemma playing a crucial point.

Lemma 1. *For $m > 0$ and $r \in \{0,1\}$, then*

$$\sum_{k=0}^{\lfloor\frac{t-r+1}{3}\rfloor} \binom{t-k+1}{t-3k-r+1}_{3,2}\binom{t-3k-r+1}{m-2k} = \frac{2t+r+2}{2t-m+2}\binom{2t-m+2}{m+r},$$

where

$$\binom{n}{k}_{3,2} = \frac{3n-k}{n}\binom{n}{k}.$$

Proof. Assume that $r=1$. In a much clear form we have to prove that

$$\sum_{k=0}^{\lfloor t/3\rfloor} \binom{t+1-k}{t-3k}\binom{t-3k}{m-2k}\frac{1}{t+1-k} = \frac{1}{m+1}\binom{2t-m+1}{m}.$$

Consider the LHS of the claimed identity

$$\sum_{k=0}^{\lfloor t/3\rfloor} \binom{t+1-k}{t-3k}\binom{t-3k}{m-2k}\frac{1}{t+1-k} = \sum_{k=0}^{\lfloor t/3\rfloor}\frac{(t-k)!}{(2k+1)!\,(m-2k)!\,(t-m-k)!}.$$

Define

$$T_k := \frac{(t-k)!}{(2k+1)!\,(m-2k)!\,(t-m-k)!}.$$

Consider

$$\begin{aligned}
\frac{T_{k+1}}{T_k} &= \frac{(t-k-1)!\,(2k+1)!\,(m-2k)!\,(t-m-k)!}{(2k+3)!\,(m-2k-2)!\,(t-m-k-1)!\,(t-k)!}\\
&= \frac{(m-2k)\,(m-2k-1)\,(t-m-k)}{(2k+3)\,(2k+2)\,(t-k)}\\
&= \frac{\left(k-\frac{m}{2}\right)\left(k-\frac{m-1}{2}\right)(k+m-t)}{(k-t)\left(k+\frac{3}{2}\right)(k+1)}.
\end{aligned}$$

Since

$$\sum_{k=0}^{\lfloor t/3\rfloor}\binom{t+1-k}{t-3k}\binom{t-3k}{m-2k}\frac{1}{t+1-k}=T_0F\left(\begin{matrix}-\frac{m}{2}, & -\frac{m-1}{2}, & m-t\\ \frac{3}{2}, & -t & \end{matrix}\middle|1\right),$$

the LHS of the above sum yields the Saalschütz identity. Thus

$$\begin{aligned}\sum_{k=0}^{\lfloor t/3\rfloor}\binom{t+1-k}{t-3k}\binom{t-3k}{m-2k}\frac{1}{t+1-k}&=\binom{t}{m}F\left(\begin{matrix}-\frac{m}{2}, & -\frac{m-1}{2}, & m-t\\ \frac{3}{2}, & -t & \end{matrix}\middle|1\right)\\&=\binom{t}{m}\frac{\left(\frac{m+3}{2}\right)^{\overline{t-m}}\left(\frac{m+2}{2}\right)^{\overline{t-m}}}{\left(\frac{3}{2}\right)^{\overline{t-m}}(m+1)^{\overline{t-m}}}\\&=\frac{1}{m+1}\binom{2t-m+1}{m},\end{aligned}$$

as claimed.

Now suppose that $r=0$. Thus the claim takes the form

$$\sum_{k=0}^{\lfloor\frac{t+1}{3}\rfloor}\binom{t-k+1}{t-3k+1}\binom{t-3k+1}{m-2k}\frac{1}{t-k+1}=\frac{1}{2t-m+2}\binom{2t-m+2}{m}.$$

Define

$$T_k:=\binom{t-k+1}{t-3k+1}\binom{t-3k+1}{m-2k}\frac{1}{t-k+1}.$$

Thus

$$\begin{aligned}T_k&=\frac{(t-k+1)!}{(t-3k+1)!\,(2k)!}\frac{(t-3k+1)!}{(m-2k)!\,(t-m-k+1)!}\frac{1}{t-k+1}\\&=\frac{(t-k)!}{(2k)!\,(m-2k)!\,(t-m-k+1)!}\end{aligned}$$

and so

$$\begin{aligned}\frac{T_{k+1}}{T_k}&=\frac{(t-k-1)!}{(2k+2)!\,(m-2k-2)!\,(t-m-k)!}\frac{(2k)!\,(m-2k)!\,(t-m-k+1)!}{(t-k)!}\\&=\frac{(m-2k)\,(m-2k-1)\,(t-m-k+1)}{(t-k)\,(2k+2)\,(2k+1)}\\&=\frac{\left(k-\frac{m}{2}\right)\left(k-\frac{m-1}{2}\right)(k+m-t-1)}{(k-t)\,(k+1)\left(k+\frac{1}{2}\right)}.\end{aligned}$$

Thus we write

$$\sum_{k=0}^{\lfloor\frac{t+1}{3}\rfloor}\binom{t-k+1}{t-3k+1}\binom{t-3k+1}{m-2k}\frac{1}{t-k+1}=T_0F\left(\begin{matrix}-\frac{m}{2}, & -\frac{m-1}{2}, & m-t-1\\ \frac{1}{2}, & -t & \end{matrix}\middle|1\right),$$

which, by the Saalschütz identity and $T_0 = \binom{t+1}{m}\frac{1}{t+1}$, equals

$$\binom{t+1}{m}\frac{1}{t+1}\frac{\left(\frac{m+1}{2}\right)^{\overline{t-m+1}}\left(\frac{m}{2}\right)^{\overline{t-m+1}}}{\left(\frac{1}{2}\right)^{\overline{t-m+1}}m^{\overline{t-m+1}}} = \frac{1}{2t-m+2}\binom{2t-m+2}{m},$$

as claimed. □

Now we are ready to give our main first result:

Theorem 1. *For $m \geq 0$ and $r \in \{0,1\}$; then*

$$\begin{aligned}&(a+b)^{2m+r} + (-a)^{2m+r} + (-b)^{2m+r}\\ &\quad = \sum_{i=0}^{\lfloor (m-r)/3 \rfloor} \binom{m-i}{m-3i-r}_{3,2} (ab\,(a+b))^{2i+r}\left(a^2+ab+b^2\right)^{m-3i-r}.\end{aligned} \tag{5}$$

Proof. Firstly, assume that $r = 0$. By the Waring formula, we have

$$\begin{aligned}&(a+b)^{2m} + a^{2m} + b^{2m}\\ &\quad = (a+b)^{2m} + \sum_{k=0}^{m} \frac{2m}{2m-k}\binom{2m-k}{k}(-1)^k (ab)^k (a+b)^{2m-2k}\end{aligned}$$

which, after an arrangement, equals

$$\begin{aligned}&2\,(a+b)^{2m} + \sum_{k=1}^{m} \frac{2m}{2m-k}\binom{2m-k}{k}(-1)^k (ab)^k (a+b)^{2m-2k}\\ &\quad = \binom{m}{m}_{3,2}(a+b)^{2m} + \sum_{k=1}^{m} \frac{2m}{2m-k}\binom{2m-k}{k}(-1)^k (ab)^k (a+b)^{2m-2k},\end{aligned}$$

which, by Lemma 1, yields

$$\begin{aligned}&\sum_{k=0}^{m}\sum_{i=0}^{\lfloor \frac{m}{3} \rfloor}\binom{m-i}{m-3i}_{3,2}\binom{m-3i}{k-2i}(-1)^k (ab)^k (a+b)^{2m-2k}\\ &\quad = \sum_{i=0}^{\lfloor \frac{m}{3} \rfloor}\binom{m-i}{m-3i}_{3,2}\sum_{k=2i}^{m-i}\binom{m-3i}{k-2i}(-1)^k (ab)^k (a+b)^{2m-2k}\\ &\quad = \sum_{i=0}^{\lfloor \frac{m}{3} \rfloor}\binom{m-i}{m-3i}_{3,2}\sum_{k=0}^{m-3i}\binom{m-3i}{k}(-1)^k (ab)^{k+2i} (a+b)^{2m-2k-4i}\\ &\quad = \sum_{i=0}^{\lfloor \frac{m}{3} \rfloor}\binom{m-i}{m-3i}_{3,2}(ab+(a+b))^{2i}\left((a+b)^2-ab\right)^{m-3i},\end{aligned}$$

which completes the proof for the case $r = 0$.

Consider the case $r=1$:

$$\begin{aligned}
&(a+b)^{2m+1}-a^{2m+1}-b^{2m+1}\\
&\quad=(a+b)^{2m+1}-\sum_{k=0}^{m}(-1)^k\,\frac{2m+1}{2m+1-k}\binom{2m+1-k}{k}(ab)^k\,(a+b)^{2m+1-2k}\\
&\quad=\sum_{k=1}^{m}(-1)^{k+1}\,\frac{2m+1}{2m+1-k}\binom{2m+1-k}{k}(ab)^k\,(a+b)^{2m+1-2k}\\
&\quad=\sum_{k=0}^{m-1}(-1)^k\,\frac{2m+1}{2m-k}\binom{2m-k}{k+1}(ab)^{k+1}\,(a+b)^{2m-2k-1},
\end{aligned}$$

which, by the Lemma 1, equals

$$\begin{aligned}
&\sum_{k=0}^{m-1}(-1)^k\sum_{i=0}^{\lfloor(m-1)/3\rfloor}\binom{m-i}{m-3i-1}_{3,2}\binom{m-3i-1}{k-2i}(ab)^{k+1}\,(a+b)^{2m-2k-1}\\
&\quad=\sum_{i=0}^{\lfloor(m-1)/3\rfloor}\binom{m-i}{m-3i-1}_{3,2}\sum_{k=0}^{m-1}\binom{m-3i-1}{k-2i}(-1)^k\,(ab)^{k+1}\,(a+b)^{2m-2k-1}\\
&\quad=\sum_{i=0}^{\lfloor(m-1)/3\rfloor}\binom{m-i}{m-3i-1}_{3,2}(ab\,(a+b))^{2i+1}\left((a+b)^2-ab\right)^{m-3i-1}\\
&\quad=\sum_{i=0}^{\lfloor(m-1)/3\rfloor}\binom{m-i}{m-3i-1}_{3,2}(ab\,(a+b))^{2i+1}\left((a+b)^2-ab\right)^{m-3i-1}.
\end{aligned}$$

□

2.1. Generalizations of the results of Carlitz

In this section, we give a general form of identities (3) and (4) as corollaries of Theorem 1. If we take $a=F_n$ and $b=F_{n-1}$ in (5) with the case $r=1$, then we get

$$\begin{aligned}
&F_{n+1}^{2m+1}-F_n^{2m+1}-F_{n-1}^{2m+1}\\
&\quad=\sum_{i=0}^{\lfloor(m-1)/3\rfloor}\binom{m-i}{m-3i-1}_{3,2}(F_{n+1}F_nF_{n-1})^{2i+1}\left(F_{n+1}^2-F_nF_{n-1}\right)^{m-1-3i}\\
&\quad=\sum_{i=0}^{\lfloor(m-1)/3\rfloor}\binom{m-i}{m-3i-1}_{3,2}(F_{n+1}F_nF_{n-1})^{2i+1}\left(2F_n^2+(-1)^n\right)^{m-1-3i}. \quad (6)
\end{aligned}$$

Similarly, when $a=L_n$ and $b=L_{n-1}$, we obtain

$$
\begin{aligned}
&L_{n+1}^{2m+1} - L_n^{2m+1} - L_{n-1}^{2m+1} \\
&\quad = \sum_{i=0}^{\lfloor (m-1)/3 \rfloor} \binom{m-i}{m-3i-1}_{3,2} \left(L_{n+1}L_nL_{n-1}\right)^{2i+1} \left(L_{n+1}^2 - L_nL_{n-1}\right)^{m-1-3i} \\
&\quad = \sum_{i=0}^{\lfloor (m-1)/3 \rfloor} \binom{m-i}{m-3i-1}_{3,2} \left(L_{n+1}L_nL_{n-1}\right)^{2i+1} \left(2L_n^2 - 5\,(-1)^n\right)^{m-1-3i}. \qquad (7)
\end{aligned}
$$

For $r = 0$, we have the following identities

$$
\begin{aligned}
&F_{n+1}^{2m} + F_n^{2m} + F_{n-1}^{2m} \\
&\quad = \sum_{i=0}^{\lfloor m/3 \rfloor} \binom{m-i}{m-3i}_{3,2} \left(F_{n+1}F_nF_{n-1}\right)^{2i} \left(F_{n+1}^2 - F_nF_{n-1}\right)^{m-3i} \\
&\quad = \sum_{i=0}^{\lfloor m/3 \rfloor} \binom{m-i}{m-3i}_{3,2} \left(F_{n+1}F_nF_{n-1}\right)^{2i} \left(2F_n^2 + (-1)^n\right)^{m-3i} \qquad (8)
\end{aligned}
$$

and

$$
\begin{aligned}
&L_{n+1}^{2m} + L_n^{2m} + L_{n-1}^{2m} \\
&\quad = \sum_{i=0}^{\lfloor m/3 \rfloor} \binom{m-i}{m-3i}_{3,2} \left(L_{n+1}L_nL_{n-1}\right)^{2i} \left(L_{n+1}^2 - L_nL_{n-1}\right)^{m-3i} \\
&\quad = \sum_{i=0}^{\lfloor m/3 \rfloor} \binom{m-i}{m-3i}_{3,2} \left(L_{n+1}L_nL_{n-1}\right)^{2i} \left(2L_n^2 - 5\,(-1)^n\right)^{m-3i}. \qquad (9)
\end{aligned}
$$

The case $m = 1, 2$ and 3 coincides with Carlitz's, Hunter's and Koshy's propositions.

Now, we give modular identities belonging to the sequence of Fibonacci and Lucas without proofs.

Corollary 1. *For $m, n \geq 0$, the following identities hold*

$$E_{n+1}^{2m+1} - E_n^{2m+1} - E_{n-1}^{2m+1} \equiv 0 \pmod{E_{n+1}E_nE_{n-1}}$$

and

$$E_{n+1}^{2m} + E_n^{2m} + E_{n-1}^{2m} \equiv 2\left(E_{n+1}^2 - E_nE_{n-1}\right)^m \pmod{E_{n+1}E_nE_{n-1}},$$

where $\{E_n\}_{n\geq 0}$ is the Fibonacci or Lucas sequence.

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