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Univalence criteria for linear fractional differential operators associated with a generalized Bessel function

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Abstract. In this paper, our aim is to establish some generalizations upon the sufficient conditions for linear fractional differential operators involving the normalized forms of the generalized Bessel functions of the first kind to be univalent in the open unit disk as investigated recently by [E. Deniz, H. Orhan, H. M. Srivastava, *Some sufficient conditions for univalence of certain families of integral operators involving generalized Bessel functions*, Taiwanese J. Math. **15**(2011), No. 2, 883–917] and [Á. Baricz, B. Frasin, *Univalence of integral operators involving Bessel functions*, Appl. Math. Letters **23**(2010), No. 4, 371–376]. Our method uses certain Luke's bounding inequalities for hypergeometric functions ${}_{p+1}F_p$ and ${}_pF_p$.
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1. Introduction and preparation

Several applications of Bessel functions arise naturally in a wide variety of problems in applied mathematics, statistics, operational research, theoretical physics and engineering sciences. Bessel functions are series solutions to a second order differential equation that ascend in many and diverse situations. Bessel's differential equation of order ν is defined as [26, p. 97, Eq. (3)]:

$$z^2w^2 + bzw + [cz^2 - \nu^2 + (1-b)\nu]w = 0, \qquad b, c, \nu \in \mathbb{C}. \tag{1}$$

A particular solution of differential equation (1), denoted by $w_{\nu,b,c}(z)$, is called *the generalized Bessel function of the first kind of order ν*. In fact, we have the following familiar series representation for the function $w_{\nu,b,c}(z)$:

$$w_{\nu,b,c}(z) = \sum_{m\geq 0} \frac{(-c)^m}{m!\,\Gamma(\nu + \frac{b+1}{2} + m)} \left(\frac{z}{2}\right)^{2m+\nu}, \qquad z \in \mathbb{C}\,, \tag{2}$$

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where $\Gamma(z)$ stands for the Euler gamma function. The series in (2) permits us to study the Bessel, the modified Bessel and the spherical Bessel functions in a unified manner. Each of these particular cases of the function $w_{\nu,b,d}(z)$ is worthing mention here. So, for $b = c = 1$ in (2) we obtain the *Bessel function of the first kind* $J_\nu(z)$ of order ν, defined by [26] (also see [6])

$$J_\nu(z) = \sum_{m\geq 0} \frac{(-1)^m}{m!\,\Gamma(\nu+m+1)} \left(\frac{z}{2}\right)^{2m+\nu}, \qquad z\in\mathbb{C}, \tag{3}$$

while if $b = -c = 1$ in (2), we obtain the *modified Bessel function of the first kind* $I_\nu(z)$ defined by (see [26] and [6])

$$I_\nu(z) = \sum_{m\geq 0} \frac{1}{m!\,\Gamma(\nu+m+1)} \left(\frac{z}{2}\right)^{2m+\nu}, \qquad z\in\mathbb{C}. \tag{4}$$

Now, consider the function $u_{\nu,b,c}\colon \mathbb{C}\mapsto\mathbb{C}$, defined by the transformation

$$u_{\nu,b,c}(z) = 2^\nu\,\Gamma\left(\nu+\tfrac{b+1}{2}\right)\, z^{-\frac{\nu}{2}}\, w_{\nu,b,c}(\sqrt{z}).$$

By using the Pochhammer (Appell, or shifted factorial) symbol, defined in terms of the Euler gamma function,

$$(a)_m = \frac{\Gamma(a+m)}{\Gamma(a)} = a(a+1)\cdots(a+m-1)$$

and $(a)_0 = 1$, for the function $u_{\nu,b,c}$ we obtain the following representation

$$u_{\nu,b,c}(z) = \sum_{m\geq 0} \frac{(-\frac{c}{4})^m}{\left(\nu+\frac{b+1}{2}\right)_m} \frac{z^m}{m!},$$

where $\nu+\frac{b+1}{2}\neq 0,-1,-2,\cdots$. This function is analytic in $\mathbb{C}$ and satisfies the second order linear differential equation

$$4z^2u''(z) + 2(2\nu+b+1)zu'(z) + c\,z\,u(z) = 0.$$

We now introduce the function $\varphi_{\nu,b,c}(z) = z\,u_{\nu,b,c}(z)$ defined in terms of the generalized Bessel function $w_{\nu,b,c}(z)$ (and in a hypergeometric form, too) by

$$\begin{aligned}\varphi_{\nu,b,c}(z) &= 2^\nu\Gamma\left(\nu+\tfrac{b+1}{2}\right)\, z^{1-\frac{\nu}{2}}\, w_{\nu,b,c}(\sqrt{z})\\ &= z + z\sum_{m\geq 1}\frac{\left(-\frac{cz}{4}\right)^m}{(\kappa)_m\, m!} = z\,{}_0F_1\Big(-;\kappa;-\frac{cz}{4}\Big), \quad \kappa = \nu+\tfrac{1}{2}(b+1).\end{aligned} \tag{5}$$

Let $\mathscr{A}$ denote the class of analytic functions f defined in the open unit disk $\mathbb{U} = \{z\colon |z|<1\}$ and let it have the form $f(z) = z+\sum_{k\geq 2} a_k z^k$. For functions $f(z) = z+\sum_{m\geq 2} a_m z^m$ and $g(z) = z+\sum_{m\geq 2} b_m z^m$, the Hadamard product (or convolution) $f*g$ is defined as usual, by $(f*g)(z) = z+\sum_{m\geq 2} a_m b_m z^m$.

This paper deals with the linear fractional differential operator $D_\lambda^{n,\gamma}$, where for for $0 \le \gamma < 1, \lambda \ge 0$ [3, p. 658, Eq. (1.6)]

$$\begin{aligned} D_\lambda^{n,\gamma} f(z) &= (\underbrace{D_\lambda^{1,\gamma} * D_\lambda^{1,\gamma} * \cdots * D_\lambda^{1,\gamma}}_{n} * f)(z) \\ D_\lambda^{1,\gamma} f(z) &= \Gamma(2-\gamma)\, z^\gamma D_z^\gamma\, f(z) * g_\lambda(z), \quad \gamma \notin \mathbb{N}_2 = \{2,3,\cdots\} \\ g_\lambda(z) &= \frac{z-(1-\lambda)z^2}{(1-z)^2} = \sum_{m\ge 0} \big(1+\lambda m\big) z^{m+1} \\ D_z^\alpha f(z) &= \frac{1}{\Gamma(1-\alpha)} \frac{\mathrm{d}}{\mathrm{d}z} \int_0^z \frac{f(t)}{(z-t)^\alpha} \mathrm{d}t, \quad 0 \le \alpha < 1. \end{aligned}$$

Using the fractional derivative D_z^α of order α, Owa [15], and later Owa and Srivastava [16] introduced the operator $\Omega^\alpha \colon \mathscr{A} \to \mathscr{A}$, which is known as an extension of fractional derivative and fractional integral, as follows

$$\Omega^\alpha f(z) = \Gamma(2-\alpha) z^\alpha D_z^\alpha f(z) = z + \sum_{m\ge 2} \mathrm{B}(m+1, 2-\alpha) a_m z^m, \qquad \alpha \notin \mathbb{N}_2,$$

where $\mathrm{B}(s,t)$ stands for the Euler beta function, recalling

$$\mathrm{B}(s,t) = \int_0^1 x^{s-1}(1-x)^{t-1}\mathrm{d}x = \frac{\Gamma(s)\Gamma(t)}{\Gamma(s+t)}, \quad \min\{\Re(s), \Re(t)\} > 0.$$

In [3], the authors introduced the operator $D_\lambda^{n,\alpha} \colon \mathscr{A} \to \mathscr{A}$ as follows:

$$D_\lambda^{n,\gamma} f(z) = z + \sum_{m \ge 2} \mathrm{B}(m+1, 2-\gamma)\big(1+\lambda(m-1)\big)^n a_m z^m. \tag{6}$$

When $\alpha = 0$, we get Al-Oboudi's differential operator [2], when $\alpha = 0$ and $\lambda = 1$, (6) covers Sălăgean's differential operator [21]; while the special cases $D_0^{n,\alpha}, D_\lambda^{0,\alpha}$ either $n \in \mathbb{N}_0$ or $\lambda \in \mathbb{C}$ mutually coincide with the Owa-Srivastava fractional differential operator Ω^α studied in [16].

Now, let the linear fractional differential operator $D_\lambda^{n,\gamma}\varphi_{\nu,b,c} \colon \mathscr{A} \to \mathscr{A}$ *viz*

$$D_\lambda^{n,\gamma}\varphi_{\nu,b,c}(z) = 2^\nu \Gamma\left(\nu + \tfrac{b+1}{2}\right) D_\lambda^{n,\gamma}\left[z^{1-\frac{\nu}{2}}\, w_{\nu,b,c}(\sqrt{z}\,)\right].$$

In the recently growing and developing area of Geometric function theory, general families of integral operators were introduced and studied [7, 11, 13, 19, 22, 23, 24], among others

$$H_{\alpha_1,\alpha_2,\cdots,\alpha_m;\beta}(z) = \left\{\beta \int_0^z t^{\beta-1} \prod_{j=1}^m \left(\frac{h_j(t)}{t}\right)^{\frac{1}{\alpha_i}} \mathrm{d}t\right\}^{\frac{1}{\beta}} \tag{7}$$

$$F_{m,\gamma}(z) = \left\{(m\gamma+1) \int_0^z \prod_{j=1}^m f_j^\gamma(t)\mathrm{d}t\right\}^{\frac{1}{(m\gamma+1)}} \tag{8}$$

$$G_\lambda(z) = \left\{\lambda \int_0^z t^{\lambda-1} \mathrm{e}^{\lambda g(t)}\mathrm{d}t\right\}^{\frac{1}{\lambda}}, \tag{9}$$

where the functions $h_1, \cdots, h_m; f_1, \cdots, f_m; g \in \mathscr{A}$ and the parameters $\alpha_1, \cdots, \alpha_m$ come from the punctured complex plane $\mathbb{C} \setminus \{0\}$, while β, γ, λ are complex numbers for which the integrals in (7), (8) and (9) converge. Here, and throughout this article, all multiple valued functions are taken *conventionally* with the principle branch.

Two of the most important and widely known univalence criteria for analytic functions defined in the open unit disk $\mathbb{U}$ were obtained by Ahlfors [1] and Becker [9, 10]. Some extensions of these two univalence criteria were given by Pescar [18] and Pascu [17]. Bulut [12] obtained sufficient conditions for the univalence of the integral operator

$$I_\beta^{n,\gamma}(f_1, \cdots, f_k) = \left\{ \beta \int_0^z t^{\beta-1} \prod_{j=1}^k \left(\frac{D_\lambda^{n,\gamma} f_j(t)}{t} \right)^{\alpha_j} \mathrm{d}t \right\}^{\frac{1}{\beta}}, \tag{10}$$

where $z \in \mathbb{U}; n \in \mathbb{N}_0, k \in \mathbb{N}; \beta \in \mathbb{C}$ with $\Re(\beta) > 0$ and $\alpha_j \in \mathbb{C}, j = \overline{1, k}$.

Recently, Szász and Kupán [25] investigated the univalence of the normalized Bessel function of the first kind $\mathfrak{g}_\nu \colon \mathbb{D} \to \mathbb{C}$ defined by

$$\mathfrak{g}_\nu(z) = 2^\nu \Gamma(\nu+1) z^{1-\frac{\nu}{2}} J_\nu\left(\sqrt{z}\,\right) = z + z \sum_{m \geq 1} \frac{(-1)^m}{4^m (\nu+1)_m} \frac{z^m}{m!}.$$

Families of integral operators of types (7) and (9) which involve the normalized forms of the generalized Bessel functions of the first kind were investigated in [7, 4, 5] and [8] to obtain sufficient conditions for integral operators to be univalent in the open unit disk. Also, Prajapat's results [20, Theorem 1] were mentioned.

The main objective of this paper is to extend and refine the parametric space and to give an alternative hypergeometric approach to deriving more sensitive forms in questions treated in the aforementioned results of [7] and [13] and also in the related references therein. For this purpose, in Section 2, we prove *bi-lateral bounding inequalities* for compositions of the operator $D_\lambda^{n,\alpha}$ with the normalized and transformed Bessel function of the first kind $\varphi_{\nu,b,c}$ in terms of generalized hypergeometric functions. In Section 3, we present univalence criteria for three linear fractional differential operators (37), (38) and (39).

2. Bounding inequalities

By using the familiar Pochhammer symbol we obtain the following series representation for $D_\lambda^{n,\gamma}$ in the form

$$D_\lambda^{n,\gamma} f(z) = \begin{cases} z + \sum\limits_{m \geq 2} \left[\frac{\left(1+\frac{1}{\lambda}\right)_{m-1} (2)_{m-1}}{\left(\frac{1}{\lambda}\right)_{m-1} (2-\gamma)_{m-1}} \right]^n a_m z^m, & \lambda > 0 \\ z + \sum\limits_{m \geq 2} \left[\frac{(2)_{m-1}}{(2-\gamma)_{m-1}} \right] a_k z^k, & \lambda = 0 \end{cases}. \tag{11}$$

For the formulation of the bi-lateral bounding inequality results we recall the definition of the *generalized hypergeometric function* with p numerator and q denominator parameters, as the series

$$ {}_pF_q\left[\begin{matrix} a_1, \cdots, a_p \\ b_1, \cdots, b_q \end{matrix}\Big| z\right] = {}_pF_q\left[\begin{matrix} a_p \\ b_q \end{matrix}\Big| z\right] := \sum_{m\geq 0} \frac{(a_1)_m \cdots (a_p)_m}{(b_1)_m \cdots (b_q)_m} \frac{z^m}{m!}, \tag{12} $$

where $b_j \in \mathbb{C}\backslash\mathbb{Z}_0^-, j = \overline{1,q}$. The series converges for all $z \in \mathbb{C}$ if $p \leq q$. It is divergent for all $z \neq 0$ when $p > q+1$, unless at least one numerator parameter is a negative integer in which case (14) is a polynomial. If $p = q+1$, the series converges on $|z| = 1$ when $\Re\big(\sum b_j - \sum a_j\big) > 0$.

For the sake of simplicity, by $((\mu))^n$ we tacitly denote a product of n Pochhammer symbols with the same base parameter μ occurring inside ${}_pF_q$.

Theorem 1. *When $\nu, b \in \mathbb{R}$ and $c \in \mathbb{C}$ are so constrained that $\kappa = \nu + \frac{b+1}{2} > 0$, then the function $z^{-1}D_\lambda^{n,\gamma}\varphi_{\nu,b,c}(z)\colon \mathbb{U} \to \mathbb{C}$ satisfies the inequality*

$$ L_1 := 2 - R_1 \leq \left|z^{-1}\, D_\lambda^{n,\gamma}\varphi_{\nu,b,c}(z)\right| \leq R_1, \tag{13} $$

where

$$ R_1 = {}_{2n}F_{2n+1}\left[\begin{matrix} ((1+\frac{1}{\lambda}))^n, ((2))^n \\ \kappa, ((\frac{1}{\lambda}))^n, ((2-\gamma))^n \end{matrix}\Big| \frac{|c|}{4}\right], $$

provided $R_1 \leq 1$. Moreover, (13) *is reduced to the equality for $c = 0$.*

Proof. Firstly, by virtue of representation (11) we obtain:

$$ \begin{aligned} D_\lambda^{n,\gamma}\varphi_{\nu,b,c}(z) &= z\sum_{m\geq 0} \frac{(-1)^m}{(\kappa)_m} \left[\frac{\left(1+\frac{1}{\lambda}\right)_m (2)_m}{\left(\frac{1}{\lambda}\right)_m (2-\gamma)_m}\right]^n \frac{(cz)^m}{4^m m!} \\ &= z \cdot {}_{2n}F_{2n+1}\left[\begin{matrix} ((1+\frac{1}{\lambda}))^n, ((2))^n \\ \kappa, ((\frac{1}{\lambda}))^n, ((2-\gamma))^n \end{matrix}\Big| -\frac{cz}{4}\right]. \end{aligned} $$

By the triangle inequality, inside $\mathbb{U}$ we conclude

$$ |D_\lambda^{n,\gamma}\varphi_{\nu,b,c}(z)| \leq |z|_{2n}F_{2n+1}\left[\begin{matrix} ((1+\frac{1}{\lambda}))^n, ((2))^n \\ \kappa, ((\frac{1}{\lambda}))^n, ((2-\gamma))^n \end{matrix}\Big| \frac{|c|}{4}\right], \tag{14} $$

with $\kappa > 0, \lambda > 0$ and $\gamma < 2$. On the other hand, we arrive at

$$ |D_\lambda^{n,\gamma}\varphi_{\nu,b,c}(z)| \geq |z| \left(2 - {}_{2n}F_{2n+1}\left[\begin{matrix} ((1+\frac{1}{\lambda}))^n, ((2))^n \\ \kappa, ((\frac{1}{\lambda}))^n, ((2-\gamma))^n \end{matrix}\Big| \frac{|c|}{4}\right]\right). \tag{15} $$

Indeed, rewriting

$$ \frac{D_\lambda^{n,\gamma}\varphi_{\nu,b,c}(z)}{z} = 1 - \sum_{m\geq 1} \frac{(-1)^{m-1}}{(k)_m} \left[\frac{\left(1+\frac{1}{\lambda}\right)_m (2)_m}{\left(\frac{1}{\lambda}\right)_m (2-\gamma)_m}\right]^n \frac{\left(\frac{cz}{4}\right)^m}{m!}, $$

using the classical inequality $|z-w| \geq \big||z|-|w|\big|;\ z, w \in \mathbb{C}$ (15) readily follows. The rest is obvious with ${}_{2n}F_{2n+1}[0] = 1$. $\square$

Despite the elegance of two-sided bounds in (13), we are looking for certain more practical bounds avoiding the higher transcendental generalized hypergeometric building blocks. To do this, we have the classical Luke's article [14] at disposal. Firstly, let us recall the appropriate rational bound results. Here and in what follows we use the shorthand notation

$$\theta = \frac{\max\limits_{1\le j\le p} a_j}{\min\limits_{1\le j\le p} b_j}, \qquad \psi = \frac{1+\max\limits_{1\le j\le p} a_j}{1+\min\limits_{1\le j\le p} b_j}. \tag{16}$$

Assume that $b_j \ge a_j > 0, j = \overline{1,p}$ and $\sigma > 0$. Then for all $x \in (0,1)$ we have [14, p. 55, Theorem 13, Eqs. (4.21), (4.23)]

$$(1-\theta x)^{-\sigma} < {}_{p+1}F_p\left[\begin{matrix}\sigma,\, a_p\\ b_p\end{matrix}\,\middle|\, x\right] < 1-\theta+\theta(1-x)^{-\sigma} \tag{17}$$

$$\begin{aligned} 1+\sigma\theta x\left(1-\frac{\psi x}{2}\right)^{-\sigma-1} &< {}_{p+1}F_p\left[\begin{matrix}\sigma,\, a_p\\ b_p\end{matrix}\,\middle|\, x\right] \\ &< 1+\sigma\theta\left(1-\frac{\psi}{2}\right)x+\frac{\sigma\theta\psi x}{2(1-x)^{\sigma+1}}. \end{aligned} \tag{18}$$

In conjunction with the estimate (13) in Theorem 1 we clearly deduce

Theorem 2. *Let κ, λ be positive, $\gamma \in (0,2)$, $\max\{\frac{1}{\lambda},1\}+1 \le \min\{\kappa, \frac{1}{\lambda}, 2-\gamma\}$ and*

$$\frac{|c|\lambda}{4\kappa} \le 1-\left(\frac{\theta}{1+\theta}\right)^{\frac{1}{1+\max\{1,\lambda^{-1}\}}}. \tag{19}$$

Then for all $z \in \mathbb{U}$ we have

$$\left|\left|\frac{D_\lambda^{n,\gamma}\varphi_{\nu,b,c}(z)}{z}\right|-1\right| \le \theta\left\{\left(1-\frac{|c|\lambda}{4\kappa}\right)^{-1-\max\{1,\lambda^{-1}\}}-1\right\}, \tag{20}$$

where

$$\theta = \frac{1+\max\{1,\lambda^{-1}\}}{\min\{\lambda^{-1},2-\gamma\}}, \qquad \psi = \frac{2+\max\{1,\lambda^{-1}\}}{1+\min\{\lambda^{-1},2-\gamma\}}.$$

Moreover, when we replace (19) *with the constraint*

$$\frac{|c|\lambda}{4\kappa} \le 1-\left[\frac{\theta\psi\lambda|c|\,\min\{2,1+\lambda^{-1}\}}{8\kappa-\theta(2-\psi)\lambda|c|\,\min\{2,1+\lambda^{-1}\}}\right]^{\frac{1}{2+\max\{1,\lambda^{-1}\}}}, \tag{21}$$

it holds

$$2-R_2'' \le \left|\frac{D_\lambda^{n,\gamma}\varphi_{\nu,b,c}(z)}{z}\right| \le R_2'', \tag{22}$$

where

$$R_2'' = 1+\min\{2,1+\lambda^{-1}\}\,\frac{|c|\lambda\theta}{8\kappa}\left(2-\psi+\psi\left(1-\frac{|c|\lambda}{4\kappa}\right)^{-2-\max\{1,\lambda^{-1}\}}\right).$$

Proof. Firstly, to apply Luke's estimate (17) and more sophisticated one (18) we have to have an ${}_{p+1}F_p$ generalized hypergeometric function, that is, two Pochhammer expressions in the numerator of ${}_{2n}F_{2n+1}$ should be transformed into a power expression *via* the obvious $(s)_m \geq s^m$ for a suitable s and $m \in \mathbb{N}_0$. Choosing $(\frac{1}{\lambda})_m \geq \lambda^{-m}, (2-\gamma)_m \geq (2-\gamma)^m$ for that purpose, we get

$$\begin{aligned}
&{}_{2n}F_{2n+1}\Big[\begin{smallmatrix} ((1+\frac{1}{\lambda}))^n,\, ((2))^n \\ \kappa,\, ((\frac{1}{\lambda}))^n,\, ((2-\gamma))^n \end{smallmatrix}\Big| \frac{|c|}{4}\Big] \\
&\qquad \leq {}_{2n}F_{2n-1}\Big[\begin{smallmatrix} ((1+\frac{1}{\lambda}))^n,\, ((2))^n \\ \kappa,\, ((\frac{1}{\lambda}))^{n-1},\, ((2-\gamma))^{n-1} \end{smallmatrix}\Big| \frac{|c|\lambda}{4(2-\gamma)}\Big] =: C_2.
\end{aligned}$$

Next, by means of Luke's upper bounds (17) and (18), we infer

$$C_2 \leq \begin{cases} 1-\theta+\theta\left(1-\dfrac{|c|\lambda}{4\kappa}\right)^{-2-\max\{1,\lambda^{-1}\}} \\ 1+\min\{2,1+\lambda^{-1}\}\,\dfrac{|c|\lambda\theta}{8\kappa}\left(2-\psi+\psi\left(1-\dfrac{|c|\lambda}{4\kappa}\right)^{-2-\max\{1,\lambda^{-1}\}}\right) \end{cases},$$

where R_2 stands for the right-hand side bound. The rest is obvious.

Now, considering the lower bound L_1 in Theorem 1, we see that in this case the bounds upon C_2 again have to be used in estimating $\left|z^{-1}D_\lambda^{n,\gamma}\varphi_{\nu,b,c}(z)\right|$ from below. Therefore

$$\left|\frac{D_\lambda^{n,\gamma}\varphi_{\nu,b,c}(z)}{z}\right| \geq 2-R_2.$$

Finally, we have to check the non-negativity of the lower bounds in both two-sided inequalities. In turn, their non-negativity is controlled by the assumed constraints (19) and (21) respectively, which completes the proof. □

Remark 1. *The proof of* Theorem 2 *is performed by adapting the* ${}_{2n}F_{2n+1}$ *hypergeometric function terms into* ${}_{2n}F_{2n-1}$ *by transforming two of its denominator Pochhammer expressions into powers.*

However, there are another two suitable choices applying the same procedure mutually either to the couples $(\kappa)_m, (\frac{1}{\lambda})_m$ *and* $(\kappa)_m, (2-\gamma)_m$*; or to* $(\frac{1}{\lambda})_m$*, that is,* $(2-\gamma)_m$ *twice in both cases. By this approach we can derive another* **eight** *two-sided inequalities similar to the ones presented above by* (20) *and* (22)*. Since the derivation technique does not change from the exposed one, we leave their development to the interested reader.*

In [14, p. 56 *et seq.*], Luke studied among others the problem of two-sided inequalities for a ${}_pF_p$ type generalized hypergeometric function where the bounds consist of polynomials and/or exponential expressions. Now, we recall his appropriate results. Keeping the meaning of θ and ψ from (16) alive, if $b_j \geq a_j > 0, j = \overline{1,p}$, for all $x > 0$, we have [14, p. 57, Theorem 16, Eq. (5.6)]

$$\mathrm{e}^{\theta x} < {}_pF_p\Big[\begin{smallmatrix} a_p \\ b_p \end{smallmatrix}\Big| x\Big] < 1-\theta\big(1-\mathrm{e}^x\big), \tag{23}$$

together with the companion estimate [14, p. 57, Theorem 16, Eq. (5.8)]

$$1+\theta x \mathrm{e}^{\frac{\psi}{2}x} < {}_pF_p\left[\begin{matrix} a_p \\ b_p\end{matrix}\middle| x\right] < 1+\theta x\left(1-\frac{\psi}{2}+\frac{\psi}{2}\mathrm{e}^x\right). \tag{24}$$

Theorem 3. *Let κ, λ be positive, $\gamma \in (0,2)$. If* $\max\{\lambda^{-1}, 1\}+1 \le \min\{\lambda^{-1}, 2-\gamma\}$ *and $\theta \ge (\mathrm{e}-1)^{-1}$, then for all $z \in \mathbb{U}$ we have*

$$\left|\left|\frac{D_\lambda^{n,\gamma}\varphi_{\nu,b,c}(z)}{z}\right|-1\right| \le \theta\Big(\mathrm{e}^{\frac{|c|}{4\kappa}}-1\Big), \tag{25}$$

Moreover, if we replace the constraint $\theta \ge (\mathrm{e}-1)^{-1}$ with

$$\frac{4\kappa}{\theta\,|c|} \ge 1-\frac{\psi}{2}+\frac{\psi}{2}\,\mathrm{e}^{\frac{|c|}{4\kappa}},$$

then there holds

$$\left|\left|\frac{D_\lambda^{n,\gamma}\varphi_{\nu,b,c}(z)}{z}\right|-1\right| \le \frac{\theta\,|c|}{4\kappa}\left(1-\frac{\psi}{2}+\frac{\psi}{2}\mathrm{e}^{\frac{|c|}{4\kappa}}\right), \tag{26}$$

where in both bilateral inequalities θ and ψ remain the same as in (16).

Proof. The natural choice for transforming one denominator Pochhammer symbol in the displays (14) is $(\kappa)_m \ge \kappa^m$. By this we achieve

$$|D_\lambda^{n,\gamma}\varphi_{\nu,b,c}(z)| \le |z|{}_{2n}F_{2n}\left[\begin{matrix} ((1+\frac{1}{\lambda}))^n, ((2))^n \\ ((\frac{1}{\lambda}))^n, ((2-\gamma))^n\end{matrix}\middle|\frac{|c|}{4\kappa}\right] =: C_3.$$

An obvious use of the upper bound (23) implies

$$C_3|z|^{-1} \le 1-\theta\Big(1-\mathrm{e}^{\frac{|c|}{4\kappa}}\Big) =: R_3,$$

so the upper bound in (25). Following the same steps of the proof of the previous theorems we deduce the lower bound $\left|z^{-1}\,D_\lambda^{n,\gamma}\varphi_{\nu,b,c}(z)\right| \ge 2-R_3$, whose non-negativity is ensured by assuming $\theta \ge (\mathrm{e}-1)^{-1}$.

The proving procedure of (26) is synthesized from the presented steps and Luke's estimate (24). □

Remark 2. *Replacing the estimate $(\kappa)_m \ge \kappa^m$ either with $(\frac{1}{\lambda})_m \ge \lambda^{-m}$ or $(2-\gamma)_m \ge (2-\gamma)^m$ in the proof of* Theorem 3, *we can develop additional* **four** *similar two-sided inequalities close to the ones exposed in* (25) *and* (26).

Theorem 4. *If the parameters $\nu, b \in \mathbb{R}$ are so constrained that $\kappa = \nu+\frac{b+1}{2} > 0$ and $\lambda > 0$, $\gamma \in (0,2)$, $c \in \mathbb{C}$, then $D_\lambda^{n,\gamma}\varphi_{\nu,b,c}(z)\colon \mathbb{U}\to\mathbb{C}$ satisfies the following inequality*

$$\begin{aligned}\left|(D_\lambda^{n,\gamma}\varphi_{\nu,b,c}(z))'-\frac{D_\lambda^{n,\gamma}\varphi_{\nu,b,c}(z)}{z}\right| &\le \frac{[2(1+\lambda)]^n|c|}{4\kappa(2-\gamma)^n} \\ &\times {}_{2n}F_{2n+1}\left[\begin{matrix} ((2+\frac{1}{\lambda}))^n, ((3))^n \\ \kappa+1, ((1+\frac{1}{\lambda}))^n, ((3-\gamma))^n\end{matrix}\middle|\frac{|c|}{4}\right].\end{aligned} \tag{27}$$

Moreover, we have

$$\max\{0, 2-R_5\} \le \left|z\left(D_\lambda^{n,\gamma}\varphi_{\nu,b,c}(z)\right)'\right| \le R_5, \tag{28}$$

where

$$R_5 = 1+\frac{|c|}{4\kappa}\left[\frac{2(1+\lambda)}{2-\gamma}\right]^n \left\{ {}_{2n}F_{2n+1}\left[\begin{matrix}((2+\frac{1}{\lambda}))^n, ((3))^n \\ \kappa+1, ((1+\frac{1}{\lambda}))^n, ((3-\gamma))^n\end{matrix}\,\Big|\,\frac{|c|}{4}\right]\right.$$
$$\left. + {}_{2n+1}F_{2n+2}\left[\begin{matrix}1,\ ((2+\frac{1}{\lambda}))^n, ((3))^n \\ 2,\ \kappa+1, ((1+\frac{1}{\lambda}))^n, ((3-\gamma))^n\end{matrix}\,\Big|\,\frac{|c|}{4}\right]\right\}. \tag{29}$$

Finally, there holds

$$\left|z^2\left(D_\lambda^{n,\gamma}\varphi_{\nu,b,c}(z)\right)''\right| \le \Lambda_n' \left\{ {}_{2n}F_{2n+1}\left[\begin{matrix}((2+\frac{1}{\lambda}))^n, ((3))^n \\ \kappa+1, ((1+\frac{1}{\lambda}))^n, ((3-\gamma))^n\end{matrix}\,\Big|\,\frac{|c|}{4}\right]\right.$$
$$\left. +\Lambda_n'' {}_{2n}F_{2n+1}\left[\begin{matrix}((3+\frac{1}{\lambda}))^n, ((4))^n \\ \kappa+2, ((2+\frac{1}{\lambda}))^n, ((4-\gamma))^n\end{matrix}\,\Big|\,\frac{|c|}{4}\right]\right\}, \tag{30}$$

where

$$\Lambda_n' = \left[\frac{2(1+\lambda)}{2-\gamma}\right]^n \frac{|c|}{2\kappa}, \qquad \Lambda_n'' = \left[\frac{3(1+2\lambda)}{(1+\lambda)(3-\gamma)}\right]^n \frac{|c|}{8(\kappa+1)}.$$

Proof. Firstly, we have

$$\begin{aligned}
\Delta_\varphi &:= \left|\left(D_\lambda^{n,\gamma}\varphi_{\nu,b,c}(z)\right)' - \frac{D_\lambda^{n,\gamma}\varphi_{\nu,b,d}(z)}{z}\right| \\
&= \left|\sum_{m\ge1}\frac{m}{(k)_m}\left[\frac{\left(1+\frac{1}{\lambda}\right)_m (2)_m}{\left(\frac{1}{\lambda}\right)_m (2-\gamma)_m}\right]^n \frac{\left(-\frac{cz}{4}\right)^m}{m!}\right| \\
&\le \frac{\Lambda_n'}{2}\sum_{m\ge0}\frac{1}{(\kappa+1)_m}\left[\frac{\left(2+\frac{1}{\lambda}\right)_m (3)_m}{\left(1+\frac{1}{\lambda}\right)_m (3-\gamma)_m}\right]^n \frac{\left(\frac{|c|}{4}\right)^m}{m!} \\
&= \frac{\Lambda_n'}{2}\,{}_{2n}F_{2n+1}\left[\begin{matrix}((2+\frac{1}{\lambda}))^n, ((3))^n \\ \kappa+1, ((1+\frac{1}{\lambda}))^n, ((3-\gamma))^n\end{matrix}\,\Big|\,\frac{|c|}{4}\right] =: C_4.
\end{aligned} \tag{31}$$

With $z \in \mathbb{U}$, we obtained the asserted upper bound (27).

In order to prove assertion (28) we make use of the series representation

$$z\left(D_\lambda^{n,\gamma}\varphi_{\nu,b,c}(z)\right)' = z\left(1+\sum_{m\ge1}\frac{m+1}{(\kappa)_m}\left[\frac{\left(1+\frac{1}{\lambda}\right)_m (2)_m}{\left(\frac{1}{\lambda}\right)_m (2-\gamma)_m}\right]^n \frac{\left(-\frac{cz}{4}\right)^m}{m!}\right).$$

Splitting the series into a sum writing $\frac{m+1}{(\kappa)_m} = \frac{m}{(\kappa)_m} + \frac{1}{(\kappa)_m}$, we conclude that

$$\left|z\left(D_\lambda^{n,\gamma}\varphi_{\nu,b,c}(z)\right)'\right| \le 1+\frac{\Lambda_n'}{2}\,{}_{2n}F_{2n+1}\left[\begin{matrix}((2+\frac{1}{\lambda}))^n, ((3))^n \\ \kappa+1, ((1+\frac{1}{\lambda}))^n, ((3-\gamma))^n\end{matrix}\,\Big|\,\frac{|c|}{4}\right]$$
$$+\frac{\Lambda_n'}{2}\,{}_{2n+1}F_{2n+2}\left[\begin{matrix}1,\ ((2+\frac{1}{\lambda}))^n, ((3))^n \\ 2,\ \kappa+1, ((1+\frac{1}{\lambda}))^n, ((3-\gamma))^n\end{matrix}\,\Big|\,\frac{|c|}{4}\right] =: R_5,$$

which confirms the upper bound in (28). Similarly, using $|1-z| \geq \big|1-|z|\big|$ once more, we have that $\big|z\,(D_\lambda^{n,\gamma}\varphi_{\nu,b,c}(z))'\,\big| \geq 2 - R_5 =: L_5$.

It remains to prove statement (30). By direct calculations for all $z \in \mathbb{U}$ it follows

$$\begin{aligned}
\big|z^2\,(D_\lambda^{n,\gamma}\varphi_{\nu,b,c}(z))''\,\big| &= |z|\left|\sum_{m\geq 1}\frac{(m+1)m}{(\kappa)_m}\left[\frac{\left(1+\frac{1}{\lambda}\right)_m(2)_m}{\left(\frac{1}{\lambda}\right)_m(2-\gamma)_m}\right]^n\frac{\left(-\frac{cz}{4}\right)^m}{m!}\right| \\
&\leq \left[\frac{2(1+\lambda)}{2-\gamma}\right]^n\frac{|c|}{2\kappa}\left\{{}_{2n}F_{2n+1}\left[\begin{matrix}((2+\frac{1}{\lambda}))^n, ((3))^n \\ \kappa+1, ((1+\frac{1}{\lambda}))^n, ((3-\gamma))^n\end{matrix}\,\bigg|\,\frac{|c|}{4}\right]\right. \\
&\quad+\frac{|c|[3(1+2\lambda)]^n}{8(\kappa+1)[(1+\lambda)(3-\gamma)]^n} \\
&\quad\left.\times{}_{2n}F_{2n+1}\left[\begin{matrix}((3+\frac{1}{\lambda}))^n, ((4))^n \\ \kappa+2, ((2+\frac{1}{\lambda}))^n, ((4-\gamma))^n\end{matrix}\,\bigg|\,\frac{|c|}{4}\right]\right\}.
\end{aligned}$$

Thus the proof is completed. □

Similarly to preceding part of the running section, by means of Luke's bounds we should evaluate the hypergeometric expressions in the bounds of Theorem 4.

To do this, we choose a sufficient number of denominator Pochhammer expressions and set $(s)_m \geq s^m$ with appropriate choices of s bounding them by either ${}_{p+1}F_p$ for rational expressions, or ${}_pF_p$ for the exponential power terms.

Namely, without easily handleable bounds, it is highly inconvenient to describe precisely the parameter space and constraints which give - at least sufficient - conditions to secure the earned results' validity.

As to the rational simplification of the upper bound (27), we have

Theorem 5. *Let $\nu, b \in \mathbb{R}$ so constrained that $\kappa > 0$. Let $\gamma \in (0,2)$, $\lambda > 0$ and $c \in \mathbb{C}$ and*

$$\min\{\kappa, \lambda^{-1}, 2-\gamma\} \geq 1 + \max\{1, \lambda^{-1}\} > 0.$$

Then for all $z \in \mathbb{U}$ we have

$$\left|(D_\lambda^{n,\gamma}\varphi_{\nu,b,c}(z))' - z^{-1}\,D_\lambda^{n,\gamma}\varphi_{\nu,b,c}(z)\right| \leq \min\{R_6, R_7\}, \tag{32}$$

where

$$\begin{aligned}
R_6 &= \frac{[2(1+\lambda)]^n|c|}{4\kappa(2-\gamma)^n}\left\{1-\psi+\psi\left(1-\frac{|c|\lambda}{4(1+\lambda)(3-\gamma)}\right)^{-\sigma}\right\} \\
R_7 &= \frac{[2(1+\lambda)]^n|c|}{4\kappa(2-\gamma)^n}\left\{1+\frac{\lambda\psi(1+\sigma)|c|}{8(1+\lambda)(3-\gamma)}\left(2-\psi_1\right.\right. \\
&\quad\left.\left.+\psi_1\left(1-\frac{|c|\lambda}{4(1+\lambda)(3-\gamma)}\right)^{-\sigma-1}\right)\right\};
\end{aligned}$$

while $\sigma = 2 + \max\{1, \lambda^{-1}\}$ *and*

$$\psi = \frac{\sigma}{1 + \min\{\kappa, \lambda^{-1}, 2-\gamma\}}, \quad \psi_1 = \frac{1+\sigma}{2 + \min\{\kappa, \lambda^{-1}, 2-\gamma\}}.$$

Further, we have

$$\left| \frac{z\,(D_\lambda^{n,\gamma} \varphi_{\nu,b,c}(z))'}{D_\lambda^{n,\gamma} \varphi_{\nu,b,c}(z)} - 1 \right| \le \frac{1}{L_1}\, \min\{R_6, R_7\}, \tag{33}$$

and L_1 *is described in* Theorem 1*.*

Proof. Choosing in (31) e.g. $s = 1+\lambda^{-1}, 2-\gamma$ (see also the comments in Remark 3), we get

$$C_4 \le \frac{\Lambda_n'}{2}\, {}_{2n}F_{2n-1}\Big[\begin{matrix} ((2+\frac{1}{\lambda}))^n, ((3))^n \\ \kappa+1, ((1+\frac{1}{\lambda}))^{n-1}, ((3-\gamma))^{n-1} \end{matrix} \Big| \frac{|c|\lambda}{4(1+\lambda)(3-\gamma)} \Big] =: C_5,$$

which is ready to be estimated by (17) in evaluating C_4. Bearing in mind the notations (16), in our present setting equal to θ, ψ_1, we conclude

$$C_5 \le \frac{[2(1+\lambda)]^n |c|}{4\kappa(2-\gamma)^n} \left\{ 1 - \theta + \theta \left(1 - \frac{|c|\lambda}{4(1+\lambda)(3-\gamma)} \right)^{-\sigma} \right\};$$

also by (18) we derive a more sophisticated counterpart of this result *viz.*

$$C_5 \le \frac{\Lambda_n'}{2} \left\{ 1 + \frac{\lambda\theta(1+\sigma)|c|}{8(1+\lambda)(3-\gamma)} \left(2 - \psi_1 + \psi_1 \left(1 - \frac{|c|\lambda}{4(1+\lambda)(3-\gamma)} \right)^{-\sigma-1} \right) \right\}.$$

So bound (32).

Next, considering the right-hand side of (27) we arrive at (33). □

Remark 3. *For bound* (32) *one combines* $(s)_m \ge s^m$ *in the denominator concerning the hypergeometric term in* (31)*, we can additionally choose between the* **five** *possible couples* $(s_1, s_2) \in \{(1+\lambda^{-1}, 2-\gamma), (\kappa+1, 1+\lambda^{-1}), (\kappa+1, 3-\gamma), (1+\lambda^{-1}, 1+\lambda^{-1}), (3-\gamma, 3-\gamma)\}$.

Another four resulting bounds coming from (32) *are built similarly to the ones listed earlier. Moreover, because bound* (32) *consists of the minimal expression of* R_6, R_7*, and the above presented Pochhammer symbol minimization* $(s)_m \ge s^m$ *could be applied* separately *to both cases, this approach results in exactly* **25** *different upper bounds* (*including* (32))*, whose derivation is too complex to be presented here.*

However, appropriate changes of the constraints defining the parameter space are necessary, too. By these considerations we exhaust a whole family of related bounding inequalities.

The same remark holds for (33)*.*

To end this section we treat (30) as above. Therefore, obviously

$$\left| z^2 \left(D_\lambda^{n,\gamma} \varphi_{\nu,b,c}(z)\right)'' \right| \le \Lambda_n' \left\{ C_5 + \left[\frac{3(1+2\lambda)}{(1+\lambda)(3-\gamma)} \right]^n \frac{|c|}{8(\kappa+1)} C_6 \right\}, \tag{34}$$

where

$$C_6 := {}_{2n}F_{2n-1}\left[\begin{matrix} ((3+\frac{1}{\lambda}))^n, ((4))^n \\ \kappa+2, ((2+\frac{1}{\lambda}))^{n-1}, ((4-\gamma))^{n-1} \end{matrix} \,\Big|\, \frac{\lambda|c|}{4(1+2\lambda)(4-\gamma)} \right].$$

Here we achieve the bound C_6 by intervention $s = 2+\lambda^{-1}$, $4-\gamma$.

Both C_5 and C_6 are prepared now for the use of Luke's rational bounding inequalities (17) the use of which results in

$$C_6 \le 1 - \theta + \theta \left(1 - \frac{|c|\lambda}{4(1+2\lambda)(4-\gamma)} \right)^{-\sigma} =: R_8, \tag{35}$$

and (18), respectively:

$$C_6 \le 1 + \frac{\lambda\theta(1+\sigma)|c|}{8(1+\lambda)(3-\gamma)} \left(2 - \psi_1 + \psi_1 \left(1 - \frac{|c|\lambda}{4(1+2\lambda)(4-\gamma)} \right)^{-\sigma-1} \right) =: R_9. \tag{36}$$

The upper bound for C_5 remains the same as in the proof of Theorem 5; in turn, the parameters σ, θ, ψ_1 for C_6, that is, in R_8, R_9 become

$$\sigma' = 3 + \max\{1, \tfrac{1}{\lambda}\}; \quad \theta = \frac{\sigma'}{2+\min\{\kappa, \frac{1}{\lambda}, 2-\gamma\}}; \quad \psi_1 = \frac{1+\sigma'}{3+\min\{\kappa, \frac{1}{\lambda}, 2-\gamma\}},$$

pointing out that all the parameters involved are positive, in turn $c \in \mathbb{C}$. By this we proved

Theorem 6. *Let $\nu, b \in \mathbb{R}$ so constrained that $\kappa > 0$. Let $\gamma \in (0,3)$, $\lambda > 0$ and $c \in \mathbb{C}$ and θ, $\psi_1 < 1$. Then for all $z \in \mathbb{U}$ we have*

$$\left| z^2 \left(D_\lambda^{n,\gamma} \varphi_{\nu,b,c}(z)\right)'' \right| \le \Lambda_n' \left\{ \min\{R_6, R_7\} + \left[\frac{3(1+2\lambda)}{(1+\lambda)(3-\gamma)} \right]^n \frac{|c| \, \min\{R_8, R_9\}}{8(\kappa+1)} \right\},$$

where σ', θ, ψ_1 have the same meaning as above and R_8 and R_9 are given by (35) *and* (36)*, respectively.*

Remark 4. *Other choices of the minimization $(s)_m \ge s^m$ in evaluating the denominator Pochhammer symbols in both addends of the right-hand side expression in* (34)*, namely five couples $(s_1, s_2) \in \{(1+\lambda^{-1}, 3-\gamma), (\kappa+1, 1+\lambda^{-1}), (\kappa+1, 3-\gamma), (1+\lambda^{-1}, 1+\lambda^{-1}), (3-\gamma, 3-\gamma)\}$ for C_5 and five couples $(s_1, s_2) \in \{(2+\lambda^{-1}, 4-\gamma)(\kappa+2, 2+\lambda^{-1}), (\kappa+2, 4-\gamma), (2+\lambda^{-1}, 2+\lambda^{-1}), (4-\gamma, 4-\gamma)\}$ applicable to the second hypergeometric term C_6 form* **25** *possible bounding inequalities together with the one reported in* Theorem 6.

We point out that there are further possibilities to build hypergeometric type bounds for the convolution operators

$$(D_\lambda^{n,\gamma}\varphi_{\nu,b,c}(z))' - z^{-1}D_\lambda^{n,\gamma}\varphi_{\nu,b,c}(z); \quad z\,(D_\lambda^{n,\gamma}\varphi_{\nu,b,c}(z))'\,; \quad z^2\,(D_\lambda^{n,\gamma}\varphi_{\nu,b,c}(z))''\,,$$

either by the Gaussian ${}_2F_1$ *minimizing* $2n-2$ *denominator Pochhammer expressions by virtue of* $(s)_m \geq s^m$*, and maximizing numerator ones by* $(s)_m \leq (s+m-1)^m$*; by* ${}_3F_2$*, etc. However, we exposed here an optimal minimal number of modifications to achieve the sharpest possible bounds.*

The adequate exponential type inequalities by Luke (23), (24) *were not applied to differential operators due to the similar conclusions as above.*

3. Univalence criteria

In our considerations we need the next two univalence criteria.

Lemma 1 (see [18]). *Let* $\eta, \Re(\eta) > 0$ *and* $c \in \mathbb{C}$ *be such, that* $|c| \leq 1, c \neq -1$. *If the function* $f \in \mathscr{A}$ *satisfies*

$$\left| c|z|^{2\eta} + \left(1 + |z|^{2\eta}\right)\frac{zf''(z)}{\eta f'(z)} \right| \leq 1, \qquad z \in \mathbb{U},$$

then the function F_η *defined by*

$$F_\eta(z) = \left(\eta \int_0^z t^{\eta-1} f'(t) \mathrm{d}t \right)^{1/\eta}$$

is in the class $\mathscr{S}$ *of normalized univalent functions in* $\mathbb{U}$.

Lemma 2 (see [17]). *If for some* $f \in \mathscr{A}$ *there holds*

$$\left(1 - |z|^{2\Re(\mu)}\right) \left| \frac{zf''(z)}{f'(z)} \right| \leq \Re(\mu), \qquad z \in \mathbb{U};\, \Re(\mu) > 0,$$

then for all $\eta \in \mathbb{C}$ *such that* $Re(\eta) \geq \Re(\mu)$ *the function* $F_\eta \in \mathscr{S}$.

The next result follows by Becker's univalence criterion [19] and the Schwarz lemma.

Lemma 3 (see [19]). *Let* $\zeta \in \mathbb{C}$, $\Re(\zeta) \geq 1$ *and* $\theta > 1$ *be so constrained that* $2\theta|\zeta| \leq 3\sqrt{3}$. *When for* $q \in \mathscr{A}$ *it is fulfilled* $|zq'(z)| \leq \theta, z \in \mathbb{U}$, *then the function* $\mathcal{G}_\zeta : \mathbb{U} \to \mathbb{C}$, *defined by*

$$\mathcal{G}_\zeta(z) = \left[\zeta \int_0^z t^{\zeta-1} \mathrm{e}^{\zeta q(t)} \mathrm{d}t \right]^{1/\zeta},$$

belongs to the class $\mathscr{S}$.

In the past two decades, several authors have obtained sufficient conditions of different kind for the univalence of general families of integral operators, see among others [7, 11, 13, 19, 20, 22, 23, 24] and the references therein. In this paper, we will focus on the integral operators of types (7), (8), choosing $h_j = f_j = D_\lambda^{n,\gamma}\varphi_{\nu_j,b,c}, j = \overline{1,m}$; and specifying in (9) $g = D_\lambda^{n,\gamma}\varphi_{\nu,b,c}$, involving by this choice the normalized forms of generalized Bessel functions of the first kind, that is,

$$\mathcal{H}_{\nu_1,\dots,\nu_m,b,c,\mu_1,\dots,\mu_m,\eta}(z) = \left\{\eta\int_0^z t^{\eta-1}\prod_{j=1}^m\left[\frac{D_\lambda^{n,\gamma}\varphi_{\nu_j,b,c}(t)}{t}\right]^{1/\mu_j}\mathrm{d}t\right\}^{1/\eta}, \tag{37}$$

$$\mathcal{F}_{\nu_1,\dots,\nu_m,b,c,\mu}(z) = \left\{(m\mu+1)\int_0^z\prod_{j=1}^m\left[D_\lambda^{n,\gamma}\varphi_{\nu_j,b,c}(t)\right]^\mu\mathrm{d}t\right\}^{1/(m\mu+1)}, \tag{38}$$

$$\mathcal{G}_{\nu,b,c,\zeta}(z) = \left\{\zeta\int_0^z t^{\zeta-1}\mathrm{e}^{\zeta D_\lambda^{n,\gamma}\varphi_{\nu,b,c}(t)}\mathrm{d}t\right\}^{1/\zeta}. \tag{39}$$

Theorem 7. *Let the parameters $\nu_1,\cdots,\nu_m$, $b\in\mathbb{R}$ and $c\in\mathbb{C}$ be so constrained that*

$$\kappa_j = \nu_j + \frac{b+1}{2} > 0, \qquad j=\overline{1,m}.$$

Consider the functions $D_\lambda^{n,\gamma}\varphi_{\nu_j,b,c}\colon \mathbb{U}\to\mathbb{C}$ defined by (5)*. Also, let $\Re(\eta)>0$; $c\in\mathbb{U}$; $\mu_j\in\mathbb{C}\setminus\{0\}, j=\overline{1,m}$ be constrained so that*

$$|c| + \frac{\min\{R_6,R_7\}}{L_1|\eta|}\sum_{j=1}^m\frac{1}{|\mu_j|}\le 1. \tag{40}$$

Then the function $\mathcal{H}_{\nu_1,\dots,\nu_m,b,c,\mu_1,\dots,\mu_m,\eta}\colon\mathbb{U}\to\mathbb{C}$ belongs to the normalized univalent functions class $\mathscr{S}$.

Proof. Without loss of generality, we consider $\mathcal{H}_{\nu_1,\dots,\nu_m,b,c,\mu_1,\dots,\mu_m,1}$. First of all, we point out that since $D_\lambda^{n,\gamma}\varphi_{\nu_j,b,c}\in\mathscr{A}$, that is,

$$D_\lambda^{n,\gamma}\varphi_{\nu_j,b,c}(0) = (D_\lambda^{n,\gamma}\varphi_{\nu_j,b,c})'(0)-1=0,$$

it is obvious that $\mathcal{H}_{\nu_1,\dots,\nu_m,b,c,\mu_1,\dots,\mu_m,1}\in\mathscr{A}$ as well. On the other hand,

$$\mathcal{H}'_{\nu_1,\dots,\nu_m,b,c,\mu_1,\dots,\mu_m,1}(z) = \prod_{j=1}^m\left(\frac{D_\lambda^{n,\gamma}\varphi_{\nu_j,b,c}(z)}{z}\right)^{1/\mu_j}.$$

We thus find

$$\frac{z\mathcal{H}''_{\nu_1,\dots,\nu_m,b,c,\mu_1,\dots,\mu_m,1}(z)}{\mathcal{H}'_{\nu_1,\dots,\nu_m,b,c,\mu_1,\dots,\mu_m,1}(z)} = \sum_{j=1}^m\frac{1}{\mu_j}\left(\frac{z\left(D_\lambda^{n,\gamma}\varphi_{\nu_j,b,c}(z)\right)'}{D_\lambda^{n,\gamma}\varphi_{\nu_j,b,c}(z)}-1\right).$$

Now, applying inequality (33) of Theorem 5 to each $\nu_j, j=\overline{1,m}$, we obtain

$$\begin{aligned}\left|\frac{z\,\mathcal{H}''_{\nu_1,\dots,\nu_m,b,c,\mu_1,\dots,\mu_m,1}(z)}{\mathcal{H}'_{\nu_1,\dots,\nu_m,b,c,\mu_1,\dots,\mu_m,1}(z)}\right| &\le \sum_{j=1}^m\frac{1}{|\mu_j|}\left|\frac{z\left(D_\lambda^{n,\gamma}\varphi_{\nu_j,b,c}(z)\right)'}{D_\lambda^{n,\gamma}\varphi_{\nu_j,b,c}(z)}-1\right| \\ &\le \sum_{j=1}^m\frac{1}{|\mu_j|}\frac{1}{L_1}\min\{R_6,R_7\}.\end{aligned}$$

Finally, by assuming that all $z \in \mathbb{U}$ we conclude

$$\left| c|z|^{2\eta} + \left(1 - |z|^{2\eta}\right) \frac{z\, \mathcal{H}''_{\nu_1,\ldots,\nu_m,b,c,\mu_1,\ldots,\mu_m,1}(z)}{\mathcal{H}'_{\nu_1,\ldots,\nu_m,b,c,\mu_1,\ldots,\mu_m,1}(z)} \right| \le |c| + \frac{\min\{R_6, R_7\}}{L_1|\eta|} \sum_{j=1}^{m} \frac{1}{|\mu_j|},$$

which, in view of Lemma 1, implies that $\mathcal{H}_{\nu_1,\ldots,\nu_m,b,c,\mu_1,\ldots,\mu_m,\eta} \in \mathscr{S}$. $\square$

Remark 5. *Discussing constraint qualification* (40)*, we see that* $|c| > 1$ *implies*

$$A = \frac{\min\{R_6, R_7\}}{L_1|\eta|} \sum_{j=1}^{m} \frac{1}{|\mu_j|} < 0,$$

which is not possible, with the involved building parameters positive. The only common sense not mentioned is $|c| = 1$*. Then we have* $A = 0$*, which occurs when* $\mu_* = \min_{1\le j\le m} \mu_j \to +\infty$*, that is,*

$$\lim_{\mu_*\to\infty} \mathcal{H}_{\nu_1,\cdots,\nu_m,b,0,\mu_1,\cdots,\mu_m,\eta}(z) = z\,;$$

this result is the expected one, compare (5) et seq.

Upon setting $\mu_1 = \ldots = \mu_m = \mu$ above in Theorem 7, we immediately arrive at the following result.

Corollary 1. *Let the parameters* $\nu_1, \ldots, \nu_m, b, c, \eta$ *and* $\kappa_j, j = \overline{1,m}$ *be prescribed as in* Theorem 7 *and suppose that there holds*

$$|c| + \frac{m \min\{R_6, R_7\}}{L_1|\eta\mu|} \le 1.$$

Then $\mathcal{H}_{\nu_1,\ldots,\nu_m,b,c,((\mu))_m,\eta}(z) \in \mathscr{S}$*.*

Our second result provides sufficient conditions for the integral operator $\mathcal{F}$ described in (38). The key tools in the proof are Lemma 2 and Theorem 4, that is, (33) from Theorem 5.

Theorem 8. *Let the parameters* $\nu_1, \cdots, \nu_m$*,* $b \in \mathbb{R}$*,* $c \in \mathbb{C}$ *be so constrained that*

$$\kappa_j = \nu_j + \frac{b+1}{2} > 0, \qquad j = \overline{1,m}.$$

Also, let $\Re(\mu) > 0$ *and*

$$\frac{|\mu|}{\Re(\mu)} \le \frac{L_1}{m \min\{R_6, R_7\}}. \tag{41}$$

Then $\mathcal{F}_{\nu_1,\ldots,\nu_m,b,c,\mu}(z) \in \mathscr{S}$*.*

Proof. Consider the auxiliary function

$$\widetilde{\mathcal{F}}_{\nu_1,\ldots,\nu_m,b,c,\mu}(z) = (m\mu+1)^{-1}[\mathcal{F}_{\nu_1,\ldots,\nu_m,b,c,\mu}(z)]^{m\mu+1}.$$

Remarking that, $\mathcal{F}_{\nu_1,\dots,\nu_m,b,c,\mu} \in \mathscr{A}$, that is, that

$$\widetilde{\mathcal{F}}_{\nu_1,\dots,\nu_m,b,c,\mu}(0) = \widetilde{\mathcal{F}}'_{\nu_1,\dots,\nu_m,b,c,\mu}(0) - 1 = 0,$$

by virtue of inequality (33), Theorem 5 and by (41), we deduce

$$\begin{aligned}\frac{1-|z|^{2\Re(\mu)}}{\Re(\mu)} \left| \frac{z\widetilde{\mathcal{F}}''_{\nu_1,\dots,\nu_m,b,c,\mu}(z)}{\widetilde{\mathcal{F}}'_{\nu_1,\dots,\nu_m,b,c,\mu}(z)} \right| &\le \frac{|\mu|}{\Re(\mu)} \sum_{j=1}^m \left| \frac{z\left(D_\lambda^{n,\gamma}\varphi_{\nu_j,b,c}(z)\right)'}{D_\lambda^{n,\gamma}\varphi_{\nu_j,b,c}(z)} - 1 \right| \\ &\le \frac{m|\mu|}{\Re(\mu)} \frac{1}{L_1} \min\{R_6, R_7\} \le 1.\end{aligned}$$

Now since $\mathcal{F}_{\nu_1,\dots,\nu_m,b,c,\mu}(z)$ can be rewritten into

$$\mathcal{F}_{\nu_1,\dots,\nu_m,b,c,\mu}(z) = \left\{ (m\mu+1) \int_0^z t^{m\mu} \prod_{j=1}^m \left(\frac{D_\lambda^{n,\gamma}\varphi_{\nu_j,b,c}(t)}{t} \right)^\mu \mathrm{d}t \right\}^{1/(m\mu+1)},$$

in view of Lemma 2, these imply that $\mathcal{F}_{\nu_1,\dots,\nu_m,b,c,\mu} \in \mathscr{S}$. □

Choosing $m = 1$ in Theorem 8 we have obtained

Corollary 2. *Let the parameters $\nu_1, b \in \mathbb{R}$; $c, \mu \in \mathbb{C}, \Re(\mu) > 0$ satisfy (41). Then $\mathcal{F}_{\nu_1,b,c,\mu} \in \mathscr{S}$.*

Finally, applying Lemma 3 and Theorem 4 we get the following result.

Theorem 9. *Let the parameters $\nu, b \in \mathbb{R}$ and $c, \zeta \in \mathbb{C}$ be so constrained that $2\kappa = 2\nu+b+1 > 0$. When $\Re(\zeta) \ge 1$ and $2\max\{0, 2-R_5\}\,|\zeta| \le 3\sqrt{3}$, then $\mathcal{G}_{\nu,b,c,\zeta} \colon \mathbb{U} \to \mathbb{C}$ defined by (39) belongs to $\mathscr{S}$.*

Remark 6. *Taking $n = 0$ in previous results, we recover the same results of* [13].

Forced by the facts exposed in Remark 5, we see that our parameter space does not contain $c = 1$ when the operator $\mathcal{H}_{\nu_1,\dots,\nu_m,b,c,\mu_1,\dots,\mu_m,\eta}(z)$ is in our focus of interest. Therefore, the results achieved here are not completely comparable to the results by Baricz and Frasin [7]. Namely, they proved that the general integral operators $\mathcal{H}_{\nu_1,\dots,\nu_m,1,1,\mu_1,\dots,\mu_m,\eta}(z)$, $\mathcal{F}_{\nu_1,\dots,\nu_m,1,1,\mu}(z)$, $\mathcal{G}_{\nu,1,1,\zeta}(z)$ defined by (37), (38) and (39), respectively, are univalent for all $\min_{1\le j\le m} \nu_j \simeq -0.69098$.

However, this is not the situation with another integral operators $\mathcal{F}, G$, which do not suffer from this insufficiency, since then $c \in \mathbb{C}$, so $c = 1$ can also be used. Further exhaustive comparison analysis will be postponed for some other time.

Finally, we can conclude that a significant extension of parameter spaces for integral operators $\mathcal{H}, \mathcal{F}, \mathcal{G}$ by our derivation method in establishing hypergeometric type bilateral inequalities in Theorems 1, 4 and their bounding inequalities given in Theorems 2, 3, 5 and 6 leaving in mind the appropriate remarks cannot be fully compared to the earlier results for similar generalized integral operators considered by Baricz and Frasin [7] and the authors mentioned in our introductory section.

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