Math. Commun. **21**(2016), 199–218

Generalized Euler–Lagrange equations for fuzzy fractional variational calculus

Javad Soolaki*, Omid Solaymani Fard and Akbar Hashemi Borzabadi

School of Mathematics and Computer Science, Damghan University, Damghan 36 715–364, Iran

Received October 4, 2015; accepted February 15, 2016

Abstract. This paper presents the necessary optimality conditions of Euler–Lagrange type for variational problems with natural boundary conditions and problems with holonomic constraints where the fuzzy fractional derivative is described in the combined Caputo sense. The new results are illustrated by computing the extremals of two fuzzy variational problems.

AMS subject classifications: 65D10, 92C45

Key words: fuzzy fractional variational problem, fuzzy fractional holonomic problem, fuzzy fractional Euler–Lagrange equations

1. Introduction

Fuzzy calculus of variations extends classical variational calculus by considering fuzzy variables and their derivatives into variational integrals to be extremized. This may occur naturally in many problems in physics and mechanics. Although the notion of fuzzy set is widely spread to various research areas such as linear programming, optimization, differential equations and even fractional differential equations, very few papers have been written on fuzzy calculus of variations and fuzzy optimal control [6, 7, 8, 9, 10, 18, 19]. Farhadinia [9] studied necessary optimality conditions for fuzzy variational problems by using the fuzzy differentiability concept due to Buckley and Feuring [5]. In [8], by using α–differentiability concept Fard and Zadeh obtained extended fuzzy Euler–Lagrange conditions. Fard et al. [6] presented fuzzy Euler–Lagrange conditions for fuzzy constrained and unconstrained variational problems under the generalized Hukuhara differentiability.

Fractional derivatives play an increasing role in mathematics, physics and engineering [12, 13, 17, 21]. The theory of calculus of variations and optimal control has been extended in order to deal with more general systems containing noninteger order derivatives [1, 14, 20]. The most famous fuzzy fractional derivatives are Riemann–Liouville and Caputo. Salahshour et al. [22] proposed the concept of Riemann–Liouville fuzzy fractional differentiability. The Caputo–type fuzzy fractional derivative is based on Hukuhara difference and strongly generalized fuzzy differentiability was introduced by Mazandarani and Kamyad [16].

*Corresponding author. *Email addresses:* javad.soolaki@gmail.com (J. Soolaki), omidsfard@gmail.com (O. S. Fard), borzabadi@du.ac.ir (A. H. Borzabadi)

In this paper, we propose a combined Caputo–type fuzzy fractional derivative. To this end, a direct procedure is adopted to derive such concept which is constructed based on the combination of strongly generalized differentiability [3] and a combined Caputo derivative [15]. The combined Caputo derivative operator (${}^{C}D_{\gamma}^{\alpha,\beta}$) is a convex combination of the left Caputo fractional derivative of order α and the right Caputo fractional derivative of order β. The advantage of the fractional combined Caputo–type derivative (${}^{C}D_{\gamma}^{\alpha,\beta}$) lies in the fact thatby using this derivative we can describe a more general class of variational problems [15]. It is also worth pointing out that the fractional derivative (${}^{C}D_{\gamma}^{\alpha,\beta}$) allows us to generalize the results presented in [7].

The paper is organized as follows. Section 2 presents some preliminaries needed in the sequel. The notion of the combined Caputo fuzzy fractional derivative is defined in Section 3. In Section 4, fuzzy fractional Euler-Lagrange conditions for fractional variational problems with natural boundary conditions are obtained. In Section 5, we obtain the necessary conditions for optimization problems with holonomic constraints. Finally, we give a conclusion in Section 6.

2. Preliminaries

This section presents some definitions and basic concepts which will be used in this paper. By $\mathbb{R}$, we denote the set of all real numbers and by $\mathbb{R}_{\tilde{\mathcal{F}}}$, the space of n-dimensional fuzzy numbers $\tilde{u}(x) : \mathbb{R}^n \to [0,1]$, satisfying the following requirements:

(i) $\tilde{u}(x)$, is normal, *i.e.* $\exists x_0 \in \mathbb{R}^n$, for which $\tilde{u}(x_0) = 1$,

(ii) $\tilde{u}(x)$, is fuzzy convex, *i.e.* $\forall x_1, x_2 \in \mathbb{R}^n, \lambda \in [0,1], \quad \tilde{u}(\lambda x_1 + (1-\lambda)x_2) \geq \min\{\tilde{u}(x_1), \tilde{u}(x_2)\}$,

(iii) $\mathrm{supp}\tilde{u}(x) = \{x \in \mathbb{R}^n | \tilde{u}(x) \geq 0\}$ is the support of the $\tilde{u}(x)$ and its closure $cl(\mathrm{supp}\tilde{u}(x))$ is compact,

(iv) $\tilde{u}(x)$ is upper semi-continuous.

For $0 < r \leq 1$, let $[\tilde{u}]^r = \{x \in \mathbb{R}; \tilde{u}(x) \geq r\}$ and $[\tilde{u}]^0 = \overline{\{x \in \mathbb{R}; \tilde{u}(x) \geq 0\}}$. Then, it is well known that $[\tilde{u}]^r$ is a bounded closed interval for any $r \in [0,1]$.

Lemma 1 (see Theorem 1.1 of [11] and Lemma 2.1 of [24]). *If* $\underline{a}^r : [0,1] \to \mathbb{R}$ *and* $\overline{a}^r : [0,1] \to \mathbb{R}$ *satisfy the conditions*

(i) $\underline{a}^r : [0,1] \to \mathbb{R}$ *is a bounded nondecreasing function,*

(ii) $\overline{a}^r : [0,1] \to \mathbb{R}$ *is a bounded nonincreasing function,*

(iii) $\underline{a}^1 \leq \overline{a}^1$,

(iv) for $0 < k \leq 1$, $\lim_{r\to k^-} \underline{a}^r = \underline{a}^k$ *and* $\lim_{r\to k^-} \overline{a}^r = \overline{a}^k$,

(v) $\lim_{r\to 0^+} \underline{a}^r = \underline{a}^0$ *and* $\lim_{r\to 0^+} \overline{a}^r = \overline{a}^0$,

then $\tilde{a} : \mathbb{R} \to [0,1]$, characterized by $\tilde{a}(t) = \sup\{r | \underline{a}^r \leq t \leq \overline{a}^r\}$, is a fuzzy number with $[\tilde{a}]^r = [\underline{a}^r, \overline{a}^r]$. The converse is also true: if $\tilde{a}(t) = \sup\{r | \underline{a}^r \leq t \leq \overline{a}^r\}$ is a fuzzy number with parametrization given by $[\tilde{a}]^r = [\underline{a}^r, \overline{a}^r]$, then functions $\underline{a}^r$ and $\overline{a}^r$ satisfy conditions (i)–(v).

Addition $\tilde{u} + \tilde{v}$ and $\tilde{u} \odot \tilde{v}$ and scalar multiplication by λ are defined as

$$
\begin{aligned}
[\tilde{u} + \tilde{v}]^r &= [\tilde{u}]^r + [\tilde{v}]^r, \\
[\lambda \tilde{u}]^r &= \lambda [\tilde{u}]^r, \\
[\tilde{u} \odot \tilde{v}]^r &= [\min\{\underline{u}^r \underline{v}^r, \underline{u}^r \overline{v}^r, \overline{u}^r \underline{v}^r, \overline{u}^r \overline{v}^r\}, \max\{\underline{u}^r \underline{v}^r, \underline{u}^r \overline{v}^r, \overline{u}^r \underline{v}^r, \overline{u}^r \overline{v}^r\}],
\end{aligned}
$$

for all $r \in [0,1]$, where $[\tilde{u}]^r + [\tilde{v}]^r$ means a usual addition of two intervals (subsets) of $\mathbb{R}$ and $\lambda[\tilde{u}]^r$ means the usual product between a scalar and a subset of $\mathbb{R}$. The metric structure is given by the Hausdorff distance $D : \mathbb{R}_{\tilde{\mathcal{F}}} \times \mathbb{R}_{\tilde{\mathcal{F}}} \to \mathbb{R}_+ \cup \{0\}$, $D(\tilde{u}, \tilde{v}) = \sup_{r \in [0,1]} \max\{|\underline{u}^r - \underline{v}^r|, |\overline{u}^r - \overline{v}^r|\}$.

We say that the fuzzy number $\tilde{u}$ is triangular if $\underline{u}^1 = \overline{u}^1$, $\underline{u}^r = \underline{u}^1 - (1-r)(\underline{u}^1 - \underline{u}^0)$ and $\overline{u}^r = \underline{u}^1 - (1-r)(\overline{u}^0 - \underline{u}^1)$. The triangular fuzzy number u is generally denoted by $\tilde{u} =< \underline{u}^0, \underline{u}^1, \overline{u}^0 >$. We define the fuzzy zero $\tilde{0}_x$ as

$$
\tilde{0}_x = \begin{cases} 1 & \text{if } x = 0, \\ 0 & \text{if } x \neq 0. \end{cases}
$$

Definition 1 (see [9]). *We say that fuzzy function $\tilde{f} : [a,b] \to \mathbb{R}_{\tilde{\mathcal{F}}}$ with r-level set $[\tilde{f}(.)]^r = [\underline{f}^r(.), \overline{f}^r(.)]$ is continuous at $x \in [a,b]$, if crisp functions $\underline{f}^r(.)$ and $\overline{f}^r(.)$ are continuous functions at $x \in [a,b]$ for all $r \in [0,1]$.*

Definition 2. *Let $\tilde{f} : [a,b] \to \mathbb{R}_{\tilde{\mathcal{F}}}$ and $x \in [a,b]$. We say that fuzzy function $\tilde{f}(.)$ with r-level set $[\tilde{f}(.)]^r = [\underline{f}^r(.), \overline{f}^r(.)]$ is a fuzzy smooth function if crisp functions $\underline{f}^r(.)$ and $\overline{f}^r(.)$ are smooth functions (in the usual sense).*

Definition 3 (see [3]). *Let $\tilde{u}, \tilde{v} \in \mathbb{R}_{\tilde{\mathcal{F}}}$. If there exists $\tilde{w} \in \mathbb{R}_{\tilde{\mathcal{F}}}$ such that $\tilde{u} = \tilde{v} + \tilde{w}$, then $\tilde{w}$ is called the Hukuhara difference (H-difference for short) of $\tilde{u}, \tilde{v}$, and it is denoted by $\tilde{u} \ominus \tilde{v}$. Note that $\tilde{u} \ominus \tilde{v} \neq \tilde{u} + (-1)\tilde{v}$.*

Definition 4 (see [16]). *Let $\tilde{f} : [a,b] \to \mathbb{R}_{\tilde{\mathcal{F}}}$ and $x_0 \in (a,b)$, then:*

$\tilde{f}(.)$ is differentiable at x_0, in the first form, if for $h > 0$ sufficiently near 0, there exist the H-differences $\tilde{f}(x_0 + h) \ominus \tilde{f}(x_0)$, $\tilde{f}(x_0) \ominus \tilde{f}(x_0 - h)$ and the limits

$$
\dot{\tilde{f}}(x_0) = \lim_{h \to 0^+} \frac{\tilde{f}(x_0 + h) \ominus \tilde{f}(x_0)}{h} = \lim_{h \to 0^+} \frac{\tilde{f}(x_0) \ominus \tilde{f}(x_0 - h)}{h} \tag{1}
$$

or

$\tilde{f}(.)$ is differentiable at x_0, in the second form, if for $h > 0$ sufficiently near 0, there exist the H-differences $\tilde{f}(x_0) \ominus \tilde{f}(x_0 + h)$, $\tilde{f}(x_0 - h) \ominus \tilde{f}(x_0)$ and the limits

$$
\dot{\tilde{f}}(x_0) = \lim_{h \to 0^+} \frac{\tilde{f}(x_0) \ominus \tilde{f}(x_0 + h)}{(-h)} = \lim_{h \to 0^+} \frac{\tilde{f}(x_0 - h) \ominus \tilde{f}(x_0)}{(-h)}. \tag{2}
$$

Remark 1. *Please notice that the subject of switching points of fuzzy-valued functions in order to determine types of strongly generalized differentiability are out of the scope of this paper and will be studied carefully in the future.*

If the fuzzy function $\tilde{f}(.)$ is continuous in the metric D, then its definite integral exists. Furthermore,

$$\left(\underline{\int_a^b \tilde{f}(x)dx}\right)^r = \int_a^b \underline{f}^r(x)dx, \quad \left(\overline{\int_a^b \tilde{f}(x)dx}\right)^r = \int_a^b \overline{f}^r(x)dx.$$

Definition 5 (see [9]). *Let $\tilde{a}, \tilde{b} \in \mathbb{R}_{\tilde{\mathcal{F}}}$. We write $\tilde{a} \preceq \tilde{b}$, if $\underline{a}^r \leq \underline{b}^r$ and $\overline{a}^r \leq \overline{b}^r$ for all $r \in [0,1]$. We also write $\tilde{a} \prec \tilde{b}$, if $\tilde{a} \preceq \tilde{b}$ and there exists an $r' \in [0,1]$ so that $\underline{a}^{r'} < \underline{b}^{r'}$ or $\overline{a}^{r'} < \overline{b}^{r'}$. Moreover, $\tilde{a} \approx \tilde{b}$ if $\tilde{a} \preceq \tilde{b}$ and $\tilde{a} \succeq \tilde{b}$, that is, $[\tilde{a}]^r = [\tilde{b}]^r$ for all $r \in [0,1]$.*

We say that $\tilde{a}, \tilde{b} \in \mathbb{R}_{\tilde{\mathcal{F}}}$ are comparable if either $\tilde{a} \preceq \tilde{b}$ or $\tilde{a} \succeq \tilde{b}$; and noncomparable otherwise.

Definition 6. *Let T is an open subset of $\mathbb{R}$ and $\tilde{f} : T \to \mathbb{R}_{\tilde{\mathcal{F}}}$ and $x_0 \in T$. We say $\tilde{f}(x_0)$ is a locally minimum(maximum) of $\tilde{f}(.)$ if there exists some $\epsilon > 0$ such that $\tilde{f}(x_0) \preceq (\succeq)\tilde{f}(x)$ when $x \in N_\epsilon(x_0)$.*

Theorem 1 (see [4]). *Let f be a real–valued function differentiable on the open interval I. If f has a local extremum at $x \in I$, then $\frac{d}{dx}f(x) = 0$.*

Lemma 2. *Let $\tilde{f} : [a,b] \to \mathbb{R}_{\tilde{\mathcal{F}}}$ be a fuzzy function. If the local minimum of $\tilde{f}(.)$ is attended in the point $x^* \in \mathbb{R}$, then the local minimum of real–valued crisp functions $\underline{f}^r(.)$ and $\overline{f}^r(.)$ is attended in x^* for all $r \in [0,1]$. So we have $\frac{d\underline{f}^r}{dx}(x^*) = \frac{d\overline{f}^r}{dx}(x^*) = 0$.*

Proof. In the neighborhood $N_\epsilon(x^*)$ we have $\tilde{f}(x^*) \preceq \tilde{f}(x)$ for all $x \in N_\epsilon(x^*)$. Using Definition 5 we get

$$\underline{f}^r(x^*) \leq \underline{f}^r(x), \ \overline{f}^r(x^*) \leq \overline{f}^r(x)$$

for all $r \in [0,1]$ and $x \in N_\epsilon(x^*)$. So $\underline{f}^r(x^*)$ and $\overline{f}^r(x^*)$ are local minimum of real–valued functions $\underline{f}^r(.)$ and $\overline{f}^r(.)$, respectively, for all $r \in [0,1]$. We can consider r to be aconstant and by Theorem 1 we arrive at $\frac{d\underline{f}^r}{dx}(x^*) = \frac{d\overline{f}^r}{dx}(x^*) = 0$. □

3. Fuzzy fractional calculus

Following [23], we denote the space of all continuous fuzzy valued functions on $[a,b] \in \mathbb{R}$ by $C^F[a,b]$; the class of fuzzy functions with continuous first derivatives on $[a,b] \in \mathbb{R}$ by $C^{F1}[a,b]$; and the space of all Lebesgue integrable fuzzy valued functions on the bounded interval $[a,b]$ is indicated by $L^F[a,b]$.

Definition 7 (see [2]). *Let $\tilde{f} \in C^F[a,b] \cap L^F[a,b]$ be a fuzzy valued function and $\alpha > 0$. Then the Riemann–Liouville fractional integral of order α is defined by*

$${}_aI_x^\alpha \tilde{f}(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \tilde{f}(t)(x-t)^{\alpha-1}dt,$$

where $\Gamma(\alpha)$ is the Gamma function and $x > a$.

Definition 8 (see [2]). *Let $\tilde{f} \in C^F[a,b] \cap L^F[a,b]$ be a fuzzy valued function. The fuzzy (left) Riemann–Liouville integral of $\tilde{f}(.)$, based on its r-level representation, can be expressed as follows:*

$$[{}_aI_x^\alpha \tilde{f}(x)]^r = [{}_aI_x^\alpha \underline{f}^r(x), {}_aI_x^\alpha \overline{f}^r(x)], \quad 0 \le r \le 1,$$

where

$${}_aI_x^\alpha \underline{f}^r(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \underline{f}^r(t)(x-t)^{\alpha-1}dt, \quad {}_aI_x^\alpha \overline{f}^r(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \overline{f}^r(t)(x-t)^{\alpha-1}dt.$$

The definition of the right fuzzy fractional operator ${}_xI_b^\alpha$ of order α is completely analogous.

Now we introduce our definition of the fuzzy combined Caputo fractional derivatives of order $\alpha, \beta \in (0,1)$ and $\gamma \in [0,1]$. The definition is similar to the concept of the combined Caputo derivative in crisp case and is a direct extension of strongly generalized differentiability (Bede and Gal [3]) to the fractional context.

Lemma 3 (see [16]). *Let $f(.)$ be a crisp continuous function and differentiable in the independent variable x over the interval of differentiation (integration) $[a,b]$. Then the following relations hold for $0 < \alpha < 1, 0 < \beta < 1$:*

$${}_a^CD_t^\alpha f(x) = {}_aD_t^\alpha(f(x) - f(a)), \quad {}_t^CD_b^\beta f(x) = {}_tD_b^\beta(f(x) - f(b)),$$

where, ${}_a^CD_t^\alpha$, ${}_t^CD_b^\beta$ are the Caputo derivative operators and ${}_aD_t^\alpha$, ${}_tD_b^\beta$ are the Riemann-Liouville derivative operators which can be defined as follows:

$${}_a^CD_x^\alpha f(x) = \frac{1}{\Gamma(1-\alpha)} \int_a^x (x-t)^{-\alpha} \frac{d}{dt} f(t)dt, \qquad {}_x^CD_b^\beta f(x) = \frac{-1}{\Gamma(1-\alpha)} \int_x^b (t-x)^{-\beta} \frac{d}{dt} f(t)dt,$$

$${}_aD_x^\alpha f(x) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_a^x \frac{f(t)}{(x-t)^\alpha} dt, \qquad {}_xD_b^\beta f(x) = \frac{-1}{\Gamma(1-\beta)} \frac{d}{dx} \int_x^b \frac{f(t)}{(t-x)^\beta} dt.$$

Definition 9. *Let $\tilde{f} \in C^F[a,b] \cap L^F[a,b]$ and consider $\tilde{\Phi}(.)$ as follows*

$$\tilde{\Phi}(x) = \gamma \left(\frac{1}{\Gamma(1-\alpha)} \int_a^x \frac{\tilde{f}(t) \ominus \tilde{f}(a)}{(x-t)^\alpha} dt \right) + (1-\gamma) \left(\frac{1}{\Gamma(1-\beta)} \int_x^b \frac{\tilde{f}(t) \ominus \tilde{f}(b)}{(t-x)^\beta} dt \right).$$

We say that $\tilde{f}(x)$ is a combined Caputo fuzzy fractional differentiable function of order $\alpha, \beta \in (0,1)$ and $\gamma \in [0,1]$, at $x_0 \in (a,b)$, if there exists an element ${}^CD_\gamma^{\alpha,\beta} \tilde{f}(x_0) \in C^F[a,b]$ such that for $h > 0$ sufficiently near zero, either

$$(a)\ {}^CD_\gamma^{\alpha,\beta} \tilde{f}(x_0) = \lim_{h \to 0^+} \frac{\tilde{\Phi}(x_0+h) \ominus \tilde{\Phi}(x_0)}{h} = \lim_{h \to 0^+} \frac{\tilde{\Phi}(x_0) \ominus \tilde{\Phi}(x_0-h)}{h}, \quad or$$

$$(b)\ {}^CD_\gamma^{\alpha,\beta} \tilde{f}(x_0) = \lim_{h \to 0^+} \frac{\tilde{\Phi}(x_0) \ominus \tilde{\Phi}(x_0+h)}{(-h)} = \lim_{h \to 0^+} \frac{\tilde{\Phi}(x_0-h) \ominus \tilde{\Phi}(x_0)}{(-h)}.$$

If the fuzzy valued function $\tilde{f}(.)$ is differentiable as in Definition 9 cases (a), it is combined Caputo-type differentiable in the first form and denoted by ${}^CD_{\gamma,1}^{\alpha,\beta} \tilde{f}(.)$. If $\tilde{f}(.)$ is differentiable as in Definition 9 case (b), it is the combined Caputo-type differentiable in the second form and denoted by ${}^CD_{\gamma,2}^{\alpha,\beta} \tilde{f}(.)$.

Theorem 2. *Let $\tilde{f} \in C^F[a,b] \cap L^F[a,b]$ with an r-level set $[\tilde{f}(.)]^r = [\underline{f}^r(.), \overline{f}^r(.)]$ be a fuzzy function and $x_0 \in (a,b)$. Then*

(a) if $\tilde{f}(.)$ is a combined Caputo fuzzy fractional differentiable function in the first form, then for $\alpha, \beta \in (0,1)$ and $\gamma \in [0,1]$,

$$[{}^C\!D^{\alpha,\beta}_{\gamma,1} \tilde{f}(x_0)]^r = [{}^C\!D^{\alpha,\beta}_{\gamma} \underline{f}^r(x_0),\ {}^C\!D^{\alpha,\beta}_{\gamma} \overline{f}^r(x_0)],$$

(b) if $\tilde{f}(.)$ is a combined Caputo fuzzy fractional differentiable function in the second form, then for $\alpha, \beta \in (0,1)$ and $\gamma \in [0,1]$,

$$[{}^C\!D^{\alpha,\beta}_{\gamma,2} \tilde{f}(x_0)]^r = [{}^C\!D^{\alpha,\beta}_{\gamma} \overline{f}^r(x_0),\ {}^C\!D^{\alpha,\beta}_{\gamma} \underline{f}^r(x_0)],$$

where

$$\begin{aligned}
{}^C\!D^{\alpha,\beta}_{\gamma} \underline{f}^r(x_0) &= \gamma\, {}^C_a D^{\alpha}_x \underline{f}^r(x_0) + (1-\gamma)\, {}^C_x D^{\beta}_b \underline{f}^r(x_0),\\
{}^C\!D^{\alpha,\beta}_{\gamma} \overline{f}^r(x_0) &= \gamma\, {}^C_a D^{\alpha}_x \overline{f}^r(x_0) + (1-\gamma)\, {}^C_x D^{\beta}_b \overline{f}^r(x_0),\\
{}^C_a D^{\alpha}_x f(x_0) &= \left[\frac{1}{\Gamma(1-\alpha)} \int_a^x (x-t)^{-\alpha} \frac{d}{dt} f(t)dt\right]_{x=x_0},\\
{}^C_x D^{\beta}_b f(x_0) &= \left[\frac{-1}{\Gamma(1-\alpha)} \int_x^b (t-x)^{-\beta} \frac{d}{dt} f(t)dt\right]_{x=x_0}.
\end{aligned}$$

Proof. Let us consider $\tilde{f}(.)$ as a combined Caputo-type fuzzy fractional differentiable function in the first form and $x_0 \in (a,b)$, then we have the following:

$$\begin{aligned}
[\tilde{\Phi}(x_0+h) \ominus \tilde{\Phi}(x_0)]^r &= [\underline{\Phi}^r(x_0+h) - \underline{\Phi}^r(x_0), \overline{\Phi}^r(x_0+h) - \overline{\Phi}^r(x_0)]\\
[\tilde{\Phi}(x_0) \ominus \tilde{\Phi}(x_0-h)]^r &= [\underline{\Phi}^r(x_0) - \underline{\Phi}^r(x_0-h), \overline{\Phi}^r(x_0) - \overline{\Phi}^r(x_0-h)].
\end{aligned}$$

Multiplying both sides by $\frac{1}{h} > 0$

$$\begin{aligned}
\left[\frac{\tilde{\Phi}(x_0+h) \ominus \tilde{\Phi}(x_0)}{h}\right]^r &= \left[\frac{\underline{\Phi}^r(x_0+h) - \underline{\Phi}^r(x_0)}{h}, \frac{\overline{\Phi}^r(x_0+h) - \overline{\Phi}^r(x_0)}{h}\right],\\
\left[\frac{\tilde{\Phi}(x_0) \ominus \tilde{\Phi}(x_0-h)}{h}\right]^r &= \left[\frac{\underline{\Phi}^r(x_0) - \underline{\Phi}^r(x_0-h)}{h}, \frac{\overline{\Phi}^r(x_0) - \overline{\Phi}^r(x_0-h)}{h}\right].
\end{aligned}$$

Passing to the limit we obtain:

$$\begin{aligned}
&\left[\lim_{h\to 0^+} \frac{\tilde{\Phi}(x_0) \ominus \tilde{\Phi}(x_0-h)}{h}\right]^r\\
&\quad= \Big[\gamma_a D^{\alpha}_x(\underline{f}^r(x_0) - \underline{f}^r(a)) + (1-\gamma)_x D^{\beta}_b(\underline{f}^r(x_0) - \underline{f}^r(a)),\\
&\qquad \gamma_a D^{\alpha}_x(\overline{f}^r(x_0) - \overline{f}^r(a)) + (1-\gamma)_x D^{\beta}_b(\overline{f}^r(x_0) - \overline{f}^r(a))\Big].
\end{aligned}$$

Using Lemma 3 and Definition 9 leads to

$$[{}^{C}D_{\gamma}^{\alpha,\beta}\tilde{f}(x_0)]^r = [{}^{C}D_{\gamma}^{\alpha,\beta}\underline{f}^r(x_0),\ {}^{C}D_{\gamma}^{\alpha,\beta}\overline{f}^r(x_0)].$$

We proved the theorem for case (a). For the other cases, the proof is similar to the previous one and hence omitted. $\square$

Note that for crisp function f, ${}^{C}D_{0}^{\alpha,\beta}f(x) = {}_{x}^{C}D_{b}^{\beta}f(x)$ and ${}^{C}D_{1}^{\alpha,\beta}f(x) = {}_{a}^{C}D_{x}^{\alpha}f(x)$. In the discussion to follow, we will also need the following formula for fractional integrations by parts (see [15]):

$$\begin{aligned}\int_a^b g(x){}^{C}D_{\gamma}^{\alpha,\beta}f(x)dx =& \gamma[f(x)_xI_b^{1-\alpha}g(x)]_{x=a}^{x=b} \\ &+ (1-\gamma)[-f(x)_aI_x^{1-\beta}g(x)]_{x=a}^{x=b} + \int_a^b f(x)D_{1-\gamma}^{\beta,\alpha}g(x)dx, \quad (3)\end{aligned}$$

where $D_{1-\gamma}^{\beta,\alpha} := (1-\gamma)_aD_x^{\beta} + \gamma_xD_b^{\alpha}$.

Let $N \in \mathbb{N}$ and $\tilde{\mathbf{f}} = (\tilde{f}_1, ..., \tilde{f}_N) : [a,b] \to \mathbb{R}_{\tilde{\mathcal{F}}}^N$ and $\alpha, \beta, \gamma \in \mathbb{R}^N$ with $\alpha_i, \beta_i \in (0,1)$ and $\gamma_i \in [0,1], i = 1, ..., N$. Then,

$${}^{C}D_{\gamma}^{\alpha,\beta}\tilde{\mathbf{f}}(x) := ({}^{C}D_{\gamma_1}^{\alpha_1,\beta_1}\tilde{f}_1(x), ..., {}^{C}D_{\gamma_N}^{\alpha_N,\beta_N}\tilde{f}_N(x)).$$

Throughout the paper we denote by $\partial_i K, \quad i = 1, ..., M (M \in \mathbb{N})$ the partial derivative of function $K : \mathbb{R}^M \to \mathbb{R}$ with respect to its ith argument.

4. Fuzzy fractional natural boundary conditions

Definition 10 (Admissible curve)**.** *We say that $\tilde{\boldsymbol{y}} = \tilde{\boldsymbol{y}}(.)$ is admissible if it satisfies the end-conditions and has a continuous combined Caputo fuzzy fractional derivative of order $\alpha, \beta \in (0,1)$ and $\gamma \in [0,1]$. We denote the set of all admissible curves by $\tilde{X}_{ad}$.*

Let us consider the following problem:

$$\begin{gathered}\tilde{J}(\tilde{\mathbf{y}}) = \int_a^b \tilde{L}(x, \tilde{\mathbf{y}}, {}^{C}D_{\gamma}^{\alpha,\beta}\tilde{\mathbf{y}}(x))dx \longrightarrow \min, \\ (\tilde{\mathbf{y}}(a) = \tilde{\mathbf{y}}^a), \quad (\tilde{\mathbf{y}}(b) = \tilde{\mathbf{y}}^b),\end{gathered} \tag{4}$$

where $x \in [a,b]$ is the independent variable; $\tilde{\mathbf{y}}(x) \in \mathbb{R}_{\tilde{\mathcal{F}}}^N$ is a fuzzy vector variable and the Lagrange function $\tilde{L}$ is assumed to be of class C^{F1} on all its arguments. To develop the necessary conditions for the extremum for (4), assume that $\tilde{\mathbf{y}}(.)$ is the desired function. Let $\epsilon \in \mathbb{R}$, and define a family of curves $\tilde{\mathbf{y}}(x) + \epsilon\tilde{\mathbf{h}}(x)$, where $\tilde{\mathbf{h}}(.)$ is an arbitrary admissible variation. We do not require $\tilde{\mathbf{h}}(a) = \tilde{\mathbf{0}}$ or $\tilde{\mathbf{h}}(b) = \tilde{\mathbf{0}}$ in the case when $\tilde{\mathbf{y}}(a)$ or $\tilde{\mathbf{y}}(b)$, respectively, is free (it is possible that both are free). Let

$$\tilde{J}(\epsilon) = \int_a^b \tilde{L}\left(x, \tilde{\mathbf{y}} + \epsilon\tilde{\mathbf{h}}, {}^{C}D_{\gamma}^{\alpha,\beta}\left(\tilde{\mathbf{y}}(x) + \epsilon\tilde{\mathbf{h}}(x)\right)\right)dx.$$

The lower bound and upper bounds of $\tilde{J}$ are

$$\underline{J}^r(\epsilon)=\int_a^b\left\{\underline{L}^r\left[x,\tilde{\mathbf{y}}+\epsilon\tilde{\mathbf{h}},{}^C D_{\gamma}^{\alpha,\beta}\left(\tilde{\mathbf{y}}(x)+\epsilon\tilde{\mathbf{h}}(x)\right)\right]^r\right\}dx$$

and

$$\overline{J}^r(\epsilon)=\int_a^b\left\{\overline{L}^r\left[x,\tilde{\mathbf{y}}+\epsilon\tilde{\mathbf{h}},{}^C D_{\gamma}^{\alpha,\beta}\left(\tilde{\mathbf{y}}(x)+\epsilon\tilde{\mathbf{h}}(x)\right)\right]^r\right\}dx,$$

respectively, where

$$\begin{aligned}\left[x,\tilde{\mathbf{y}}+\epsilon\tilde{\mathbf{h}},\ {}^C D_{\gamma}^{\alpha,\beta}\left(\tilde{\mathbf{y}}(x)+\epsilon\tilde{\mathbf{h}}(x)\right)\right]^r &:= (x,\ \underline{y}_1^r(x)+\epsilon\underline{h}_1^r(x),\dots,\underline{y}_N^r(x)+\epsilon\underline{h}_N^r(x),\\ &\overline{y}_1^r(x)+\epsilon\overline{h}_1^r(x),\dots,\overline{y}_N^r(x)+\epsilon\overline{h}_N^r(x),{}^C D_{\gamma_1}^{\alpha_1,\beta_1}\left(\underline{y}_1^r(x)+\epsilon\underline{h}_1^r(x)\right),\dots,\\ {}^C D_{\gamma_N}^{\alpha_N,\beta_N}\left(\underline{y}_N^r(x)+\epsilon\underline{h}_N^r(x)\right),{}^C D_{\gamma_1}^{\alpha_1,\beta_1}&\left(\overline{y}_1^r(x)+\epsilon\overline{h}_1^r(x)\right),\dots,{}^C D_{\gamma_N}^{\alpha_N,\beta_N}\left(\overline{y}_N^r(x)+\epsilon\overline{h}_N^r(x)\right).\end{aligned}$$

By Lemma 2, $\tilde{J}(\epsilon)$ is extremum at $\epsilon=0$, therefore necessary conditions for $\tilde{y}$ to be an extremizer are given by set $\frac{d\underline{J}^r}{d\epsilon}|_{\epsilon=0}=0$, $\frac{d\overline{J}^r}{d\epsilon}|_{\epsilon=0}=0$,
$\frac{d\underline{J}^r}{d\epsilon}|_{\epsilon=0}=0\longrightarrow$

$$\begin{aligned}&\int_a^b\left[\sum_{i=2}^{N+1}\left(\partial_i\underline{L}^r\cdot\underline{h}_{i-1}^r(x)+\partial_{2N+i}\underline{L}^r\cdot{}^C D_{\gamma_{i-1}}^{\alpha_{i-1},\beta_{i-1}}\underline{h}_{i-1}^r(x)\right)\right.\\ &\left.+\sum_{i=N+2}^{2N+1}\left(\partial_i\underline{L}^r\cdot\overline{h}_{i-N-1}^r(x)\ +\partial_{2N+i}\underline{L}^r\cdot{}^C D_{\gamma_{i-N-1}}^{\alpha_{i-N-1},\beta_{i-N-1}}\overline{h}_{i-N-1}^r(x)\right)\right]dx=0\end{aligned}\tag{5}$$

and $\frac{d\overline{J}^r}{d\epsilon}|_{\epsilon=0}=0\longrightarrow$

$$\begin{aligned}&\int_a^b\left[\sum_{i=2}^{N+1}\left(\partial_i\overline{L}^r\cdot\underline{h}_{i-1}^r(x)+\partial_{2N+i}\overline{L}^r\cdot{}^C D_{\gamma_{i-1}}^{\alpha_{i-1},\beta_{i-1}}\underline{h}_{i-1}^r(x)\right)\right.\\ &\left.+\sum_{i=N+2}^{2N+1}\left(\partial_i\overline{L}^r\cdot\overline{h}_{i-N-1}^r(x)\ +\partial_{2N+i}\overline{L}^r\cdot{}^C D_{\gamma_{i-N-1}}^{\alpha_{i-N-1},\beta_{i-N-1}}\overline{h}_{i-N-1}^r(x)\right)\right]dx=0.\end{aligned}\tag{6}$$

For the moment, we consider only equation (5). Using (3) for integration by parts, we get

$$\int_a^b \left\{ \left(\sum_{i=2}^{N+1} \partial_i \underline{L}^r + D_{1-\gamma_{i-1}}^{\beta_{i-1},\alpha_{i-1}} \partial_{2N+i} \underline{L}^r \right) \cdot \underline{h}_{i-1}^r(x) \right.$$
$$\left. + \left(\sum_{i=N+2}^{2N+1} \partial_i \underline{L}^r \ + D_{1-\gamma_{i-1}}^{\beta_{i-1},\alpha_{i-1}} \partial_{2N+i} \underline{L}^r \right) \cdot \overline{h}_{i-N-1}^r(x) \right\} dx$$
$$+ \gamma \left(\sum_{i=2}^{N+1} \underline{h}_{i-1}^r(x) \left({}_x I_b^{1-\alpha_{i-1}} \partial_{2N+i} \underline{L}^r \right) \right) |_{x=a}^{x=b} + \gamma \left(\sum_{i=N+2}^{2N+1} \overline{h}_{i-N-1}^r(x) \right.$$
$$\left. \cdot \left({}_x I_b^{1-\alpha_{i-1}} \partial_{2N+i} \underline{L}^r \right) \right) |_{x=a}^{x=b} - (1-\gamma) \cdot \left(\sum_{i=2}^{N+1} \underline{h}_{i-1}^r(x) \cdot \left({}_a I_x^{1-\beta_{i-1}} \partial_{2N+i} \underline{L}^r \right) \right) |_{x=a}^{x=b}$$
$$-(1-\gamma) \cdot \left(\sum_{i=N+2}^{2N+1} \overline{h}_{i-N-1}^r(x) \cdot \left({}_a I_x^{1-\beta_{i-1}} \partial_{2N+i} \underline{L}^r \right) \right) |_{x=a}^{x=b} = 0. \tag{7}$$

Let $\tilde{\mathbf{y}}(a) = \tilde{\mathbf{y}}^a$ and $\tilde{\mathbf{y}}(b) = \tilde{\mathbf{y}}^b$. Since $\underline{h}_i^r(a) = \overline{h}_i^r(a) = \underline{h}_i^r(b) = \overline{h}_i^r(b) = 0$, for $i = 1, ..., N$, and by the fundamental lemma of the calculus of variations we deduce that

$$\left(\partial_i \underline{L}^r + D_{1-\gamma_{i-1}}^{\beta_{i-1},\alpha_{i-1}} \partial_{2N+i} \underline{L}^r \right) \left[x, \tilde{\mathbf{y}}, \ {}^C D_\gamma^{\alpha,\beta} \left(\tilde{\mathbf{y}}(x) \right) \right]^r = 0, \quad i = 2, ..., 2N+1, \tag{8}$$

for all $x \in [a, b]$. Following the scheme of obtaining (8) and adapting it to the case under consideration involving (6), one can show that

$$\int_a^b \left\{ \left(\sum_{i=2}^{N+1} \partial_i \overline{L}^r \ + D_{1-\gamma_{i-1}}^{\beta_{i-1},\alpha_{i-1}} \partial_{2N+i} \overline{L}^r \right) \cdot \underline{h}_{i-1}^r(x) \right.$$
$$\left. + \left(\sum_{i=N+2}^{2N+1} \partial_i \overline{L}^r \ + D_{1-\gamma_{i-1}}^{\beta_{i-1},\alpha_{i-1}} \partial_{2N+i} \overline{L}^r \right) \cdot \overline{h}_{i-N-1}^r(x) \right\} dx$$
$$+ \gamma \left(\sum_{i=2}^{N+1} \underline{h}_{i-1}^r(x) \left({}_x I_b^{1-\alpha_{i-1}} \partial_{2N+i} \overline{L}^r \right) \right) |_{x=a}^{x=b} + \gamma \left(\sum_{i=N+2}^{2N+1} \overline{h}_{i-N-1}^r(x) \right.$$
$$\left. \cdot \left({}_x I_b^{1-\alpha_{i-1}} \partial_{2N+i} \overline{L}^r \right) \right) |_{x=a}^{x=b} - (1-\gamma) \cdot \left(\sum_{i=2}^{N+1} \underline{h}_{i-1}^r(x) \cdot \left({}_a I_x^{1-\beta_{i-1}} \partial_{2N+i} \overline{L}^r \right) \right) |_{x=a}^{x=b}$$
$$-(1-\gamma) \cdot \left(\sum_{i=N+2}^{2N+1} \overline{h}_{i-N-1}^r(x) \cdot \left({}_x I_b^{1-\alpha_{i-1}} \partial_{2N+i} \overline{L}^r \right) \right) |_{x=a}^{x=b} = 0 \tag{9}$$

and

$$\left(\partial_i \overline{L}^r + D_{1-\gamma_{i-1}}^{\beta_{i-1},\alpha_{i-1}} \partial_{2N+i} \overline{L}^r \right) \left[x, \tilde{\mathbf{y}}, \ {}^C D_\gamma^{\alpha,\beta} \left(\tilde{\mathbf{y}}(x) \right) \right]^r = 0, \quad i = 2, ..., 2N+1. \tag{10}$$

Let $l \in \{1, ..., N\}$. Assume that $\tilde{\mathbf{y}}(a) = \tilde{\mathbf{y}}^a, \tilde{y}_i(b) = \tilde{y}_i^b, i = 1, ..., N, i \neq l$, but $\tilde{y}_l(b)$

is free. Then, $\tilde{h}_l(b)$ is free and by equations (7) and (9) we obtain

$$\left[\left(\gamma\ {}_xI_b^{1-\alpha_l}\partial_{2N+l+1}\underline{L}^r-(1-\gamma)\ {}_aI_x^{1-\beta_l}\partial_{2N+l+1}\underline{L}^r\right)\left[x,\tilde{\mathbf{y}},\ {}^C D_\gamma^{\alpha,\beta}\left(\tilde{\mathbf{y}}(x)\right)\right]^r\right]|_{x=b}=0,$$
$$\left[\left(\gamma\ {}_xI_b^{1-\alpha_l}\partial_{3N+l+1}\underline{L}^r-(1-\gamma)\ {}_aI_x^{1-\beta_l}\partial_{3N+l+1}\underline{L}^r\right)\left[x,\tilde{\mathbf{y}},\ {}^C D_\gamma^{\alpha,\beta}\left(\tilde{\mathbf{y}}(x)\right)\right]^r\right]|_{x=b}=0,$$
$$\left[\left(\gamma\ {}_xI_b^{1-\alpha_l}\partial_{2N+l+1}\overline{L}^r-(1-\gamma)\ {}_aI_x^{1-\beta_l}\partial_{2N+l+1}\overline{L}^r\right)\left[x,\tilde{\mathbf{y}},\ {}^C D_\gamma^{\alpha,\beta}\left(\tilde{\mathbf{y}}(x)\right)\right]^r\right]|_{x=b}=0,$$
$$\left[\left(\gamma\ {}_xI_b^{1-\alpha_l}\partial_{3N+l+1}\overline{L}^r-(1-\gamma)\ {}_aI_x^{1-\beta_l}\partial_{3N+l+1}\overline{L}^r\right)\left[x,\tilde{\mathbf{y}},\ {}^C D_\gamma^{\alpha,\beta}\left(\tilde{\mathbf{y}}(x)\right)\right]^r\right]|_{x=b}=0,$$

where

$$\left[x,\tilde{\mathbf{y}},\ {}^C D_\gamma^{\alpha,\beta}\left(\tilde{\mathbf{y}}(x)\right)\right]^r=\left(x,\ \underline{y}_1^r(x),\dots,\underline{y}_N^r(x),\overline{y}_1^r(x),\dots,\overline{y}_N^r(x),{}^C D_{\gamma_1}^{\alpha_1,\beta_1}\left(\underline{y}_1^r(x)\right),\right.$$
$$\left.\dots,{}^C D_{\gamma_N}^{\alpha_N,\beta_N}\left(\underline{y}_N^r(x)\right),{}^C D_{\gamma_1}^{\alpha_1,\beta_1}\left(\overline{y}_1^r(x)\right),\dots,{}^C D_{\gamma_N}^{\alpha_N,\beta_N}\left(\overline{y}_N^r(x)\right)\right).$$

Now we are in a position to state the necessary conditions for a relative (local) minimum of problem (4) as follows:

Theorem 3. *Let $\tilde{\boldsymbol{y}}=(\tilde{y}_1,...,\tilde{y}_N)$ be a local minimizer to problem* (4)*. Then, $\tilde{\boldsymbol{y}}$ satisfies the following system of fractional Euler–Lagrange equations:*

$$\begin{aligned}&\left(\partial_i\underline{L}^r+D_{1-\gamma_{i-1}}^{\beta_{i-1},\alpha_{i-1}}\partial_{2N+i}\underline{L}^r\right)\left[x,\tilde{\boldsymbol{y}},\ {}^C D_\gamma^{\alpha,\beta}\left(\tilde{\boldsymbol{y}}(x)\right)\right]^r=0,\quad i=2,...,2N+1,\\&\left(\partial_i\overline{L}^r+D_{1-\gamma_{i-1}}^{\beta_{i-1},\alpha_{i-1}}\partial_{2N+i}\overline{L}^r\right)\left[x,\tilde{\boldsymbol{y}},\ {}^C D_\gamma^{\alpha,\beta}\left(\tilde{\boldsymbol{y}}(x)\right)\right]^r=0,\quad i=2,...,2N+1.\end{aligned}\tag{11}$$

Moreover, let $l\in\{1,...,N\}$ and $\tilde{\boldsymbol{y}}(a)=\tilde{\boldsymbol{y}}^a,\tilde{y}_i(b)=\tilde{y}_i^b,i=1,...,N,i\neq l$, but $\tilde{y}_l(b)$ is free. Then

$$\left[\left(\gamma\ {}_xI_b^{1-\alpha_l}\partial_{2N+l+1}\underline{L}^r-(1-\gamma)\ {}_aI_x^{1-\beta_l}\partial_{2N+l+1}\underline{L}^r\right)\left[x,\tilde{\boldsymbol{y}},\ {}^C D_\gamma^{\alpha,\beta}\left(\tilde{\boldsymbol{y}}(x)\right)\right]^r\right]|_{x=b}=0,$$
$$\left[\left(\gamma\ {}_xI_b^{1-\alpha_l}\partial_{3N+l+1}\underline{L}^r-(1-\gamma)\ {}_aI_x^{1-\beta_l}\partial_{3N+l+1}\underline{L}^r\right)\left[x,\tilde{\boldsymbol{y}},\ {}^C D_\gamma^{\alpha,\beta}\left(\tilde{\boldsymbol{y}}(x)\right)\right]^r\right]|_{x=b}=0,$$
$$\left[\left(\gamma\ {}_xI_b^{1-\alpha_l}\partial_{2N+l+1}\overline{L}^r-(1-\gamma)\ {}_aI_x^{1-\beta_l}\partial_{2N+l+1}\overline{L}^r\right)\left[x,\tilde{\boldsymbol{y}},\ {}^C D_\gamma^{\alpha,\beta}\left(\tilde{\boldsymbol{y}}(x)\right)\right]^r\right]|_{x=b}=0,$$
$$\left[\left(\gamma\ {}_xI_b^{1-\alpha_l}\partial_{3N+l+1}\overline{L}^r-(1-\gamma)\ {}_aI_x^{1-\beta_l}\partial_{3N+l+1}\overline{L}^r\right)\left[x,\tilde{\boldsymbol{y}},\ {}^C D_\gamma^{\alpha,\beta}\left(\tilde{\boldsymbol{y}}(x)\right)\right]^r\right]|_{x=b}=0,$$

for all $x\in[a,b]$.

Example 1. *Find the extremal of the following problem:*

$$\tilde{J}(\tilde{y})=\int_1^2 x^2\left({}^C D_\gamma^{\alpha,\beta}\tilde{y}(x)\right)^2+\left({}^C D_\gamma^{\alpha,\beta}\tilde{y}(x)\right)dx\longrightarrow\min,\tag{12}$$

where $\tilde{y}(1)=<-1,0,1>$ and $\tilde{y}(2)$ is free.

Solution. According to the strongly generalized fuzzy combined Caputo differentiability of $\tilde{y}$ and the product of two fuzzy numbers, the following four cases may occur:

Case (1): First, we suppose that $\tilde{y}$ is a combined Caputo differentiable function in the first form of Definition 9 and

$$0 \leq^C D_\gamma^{\alpha,\beta}\left(\underline{y}^r(x)\right) \leq 1, \quad 0 \leq^C D_\gamma^{\alpha,\beta}\left(\overline{y}^r(x)\right) \leq 1 \quad \left(0 \preceq^C D_\gamma^{\alpha,\beta}\left(\tilde{y}(x)\right) \preceq 1\right)$$

or

$$^C D_\gamma^{\alpha,\beta}\left(\underline{y}^r(x)\right) \leq -1, \quad ^C D_\gamma^{\alpha,\beta}\left(\overline{y}^r(x)\right) \leq -1 \quad \left(^C D_\gamma^{\alpha,\beta}\left(\tilde{y}(x)\right) \preceq -1\right)$$

for all $r \in [0,1]$ and $x \in [1,2]$. Then the r–level set of $\tilde{J}$ is:

$$\left[\tilde{J}(\tilde{y})\right]^r = \left[\int_1^2 x^2 \left(^C D_\gamma^{\alpha,\beta}\left(\overline{y}^r(x)\right)\right)^2 + {}^C D_\gamma^{\alpha,\beta}\left(\underline{y}^r(x)\right) dx\,, \right.$$
$$\left. \int_1^2 x^2 \left(^C D_\gamma^{\alpha,\beta}\left(\underline{y}^r(x)\right)\right)^2 + {}^C D_\gamma^{\alpha,\beta}\left(\overline{y}^r(x)\right) dx\right].$$

Case (2): If $\tilde{y}$ is a combined Caputo differentiable function in the first form of Definition 9 and

$$-1 \leq^C D_\gamma^{\alpha,\beta}\left(\underline{y}^r(x)\right) \leq 0, \quad -1 \leq^C D_\gamma^{\alpha,\beta}\left(\overline{y}^r(x)\right) \leq 0 \quad \left(-1 \preceq^C D_\gamma^{\alpha,\beta}\left(\tilde{y}(x)\right) \preceq 0\right)$$

or

$$^C D_\gamma^{\alpha,\beta}\left(\underline{y}^r(x)\right) \geq 1, \quad ^C D_\gamma^{\alpha,\beta}\left(\overline{y}^r(x)\right) \geq 1 \quad \left(^C D_\gamma^{\alpha,\beta}\left(\tilde{y}(x)\right) \succeq 1\right)$$

for all $r \in [0,1]$ and $x \in [1,2]$, then the r–level set of $\tilde{J}$ is:

$$\left[\tilde{J}(\tilde{y})\right]^r = \left[\int_1^2 x^2 \left(^C D_\gamma^{\alpha,\beta}\left(\underline{y}^r(x)\right)\right)^2 + {}^C D_\gamma^{\alpha,\beta}\left(\underline{y}^r(x)\right) dx\,, \right.$$
$$\left. \int_1^2 x^2 \left(^C D_\gamma^{\alpha,\beta}\left(\overline{y}^r(x)\right)\right)^2 + {}^C D_\gamma^{\alpha,\beta}\left(\overline{y}^r(x)\right) dx\right].$$

Case (3): If $\tilde{y}$ is a combined Caputo differentiable function in the second form of Definition 9 and

$$0 \leq^C D_\gamma^{\alpha,\beta}\left(\underline{y}^r(x)\right) \leq 1, \quad 0 \leq^C D_\gamma^{\alpha,\beta}\left(\overline{y}^r(x)\right) \leq 1 \quad \left(0 \preceq^C D_\gamma^{\alpha,\beta}\left(\tilde{y}(x)\right) \preceq 1\right)$$

or

$$^C D_\gamma^{\alpha,\beta}\left(\underline{y}^r(x)\right) \leq -1, \; ^C D_\gamma^{\alpha,\beta}\left(\overline{y}^r(x)\right) \leq -1 \left(^C D_\gamma^{\alpha,\beta}\left(\tilde{y}(x)\right) \preceq -1\right)$$

for all $r \in [0,1]$ and $x \in [1,2]$, then the r–level set of $\tilde{J}$ is:

$$\left[\tilde{J}(\tilde{y})\right]^r = \left[\int_1^2 x^2 \left(^C D_\gamma^{\alpha,\beta}\left(\underline{y}^r(x)\right)\right)^2 + {}^C D_\gamma^{\alpha,\beta}\left(\overline{y}^r(x)\right) dx, \right.$$
$$\left. \int_1^2 x^2 \left(^C D_\gamma^{\alpha,\beta}\left(\overline{y}^r(x)\right)\right)^2 + {}^C D_\gamma^{\alpha,\beta}\left(\underline{y}^r(x)\right) dx\right].$$

Case (4): If $\tilde{y}$ is a combined Caputo differentiable function in the second form of Definition 9 and

$$-1 \leq^C D_\gamma^{\alpha,\beta}\left(\underline{y}^r(x)\right) \leq 0, \quad -1 \leq^C D_\gamma^{\alpha,\beta}\left(\overline{y}^r(x)\right) \leq 0 \quad \left(-1 \preceq^C D_\gamma^{\alpha,\beta}\left(\tilde{y}(x)\right) \preceq 0\right)$$

or

$$ {}^{C}D_{\gamma}^{\alpha,\beta}\left(\underline{y}^{r}(x)\right)\geq 1,\quad {}^{C}D_{\gamma}^{\alpha,\beta}\left(\overline{y}^{r}(x)\right)\geq 1\quad\left({}^{C}D_{\gamma}^{\alpha,\beta}\left(\tilde{y}(x)\right)\succeq 1\right) $$

for all $r\in[0,1]$ and $x\in[1,2]$, then the r–level set of $\tilde{J}$ is:

$$ \begin{aligned}\left[\tilde{J}(\tilde{y})\right]^{r}=\Bigg[&\int_{1}^{2}x^{2}\left({}^{C}D_{\gamma}^{\alpha,\beta}\left(\overline{y}^{r}(x)\right)\right)^{2}+{}^{C}D_{\gamma}^{\alpha,\beta}\left(\overline{y}^{r}(x)\right)dx\ ,\\ &\int_{1}^{2}x^{2}\left({}^{C}D_{\gamma}^{\alpha,\beta}\left(\underline{y}^{r}(x)\right)\right)^{2}+{}^{C}D_{\gamma}^{\alpha,\beta}\left(\underline{y}^{r}(x)\right)dx\Bigg].\end{aligned} $$

From fuzzy Euler–Lagrange conditions and natural boundary conditions for cases (2), (4) we get the following equations

$$ \begin{gathered} D_{1-\gamma}^{\beta,\alpha}\left(1+2x^{2}\cdot{}^{C}D_{\gamma}^{\alpha,\beta}\left(\underline{y}^{r}(x)\right)\right)=0,\\ D_{1-\gamma}^{\beta,\alpha}\left(1+2x^{2}\cdot{}^{C}D_{\gamma}^{\alpha,\beta}\left(\overline{y}^{r}(x)\right)\right)=0,\\ \left[\gamma\ {}_{x}I_{b}^{1-\alpha}\left(1+2x^{2}\cdot{}^{C}D_{\gamma}^{\alpha,\beta}\left(\underline{y}^{r}(x)\right)\right)-(1-\gamma)\ {}_{a}I_{x}^{1-\beta}\left(1+2x^{2}\cdot{}^{C}D_{\gamma}^{\alpha,\beta}\left(\underline{y}^{r}(x)\right)\right)\right]|_{x=2}=0,\\ \left[\gamma\ {}_{x}I_{b}^{1-\alpha}\left(1+2x^{2}\cdot{}^{C}D_{\gamma}^{\alpha,\beta}\left(\overline{y}^{r}(x)\right)\right)-(1-\gamma)\ {}_{a}I_{x}^{1-\beta}\left(1+2x^{2}\cdot{}^{C}D_{\gamma}^{\alpha,\beta}\left(\overline{y}^{r}(x)\right)\right)\right]|_{x=2}=0. \end{gathered} \tag{13} $$

Note that it is difficult to solve the above fractional equations to get the extremals. For $0<\alpha<1, 0<\beta<1$ and $0\leq\gamma\leq 1$, a numerical method should be used. When $\alpha\to 1$ and $\gamma\to 1$, problem (12) reduces to

$$ \tilde{J}(\tilde{y})=\int_{1}^{2}\dot{\tilde{y}}(x)+x^{2}\dot{\tilde{y}}^{2}(x)dx\longrightarrow\min, \tag{14} $$

where $\tilde{y}(1)=<-1,0,1>$ and $\tilde{y}(2)$ is free.

The extremals for (14) are obtained from (13) and the initial conditions, considering $\alpha\to 1$ and $\gamma\to 1$:

$$ \begin{aligned} &x^{2}\ddot{\underline{y}}^{r}+2x\dot{\underline{y}}^{r}=0,\\ &x^{2}\ddot{\overline{y}}^{r}+2x\dot{\underline{y}}^{r}=0,\\ &\left(1+2x^{2}\dot{\underline{y}}^{r}=0\right)|_{x=2},\\ &\left(1+2x^{2}\dot{\overline{y}}^{r}=0\right)|_{x=2}. \end{aligned} \tag{15} $$

By solving equations (15) we have

$$ \underline{y}^{r}(x)=\frac{1}{2x}+r-\frac{3}{2},\quad\overline{y}^{r}(x)=\frac{1}{2x}-r+\frac{1}{2}. $$

One can easily show that $\underline{y}^{r}(x)$ and $\overline{y}^{r}(x)$ satisfy Lemma 1. This solution is shown in Figure 1, where the solid lines are $\underline{y}^{1}(x)=\overline{y}^{1}(x)$; the dashed lines are $\overline{y}^{0}(x)$; the doted lines are $\underline{y}^{0}(x)$.

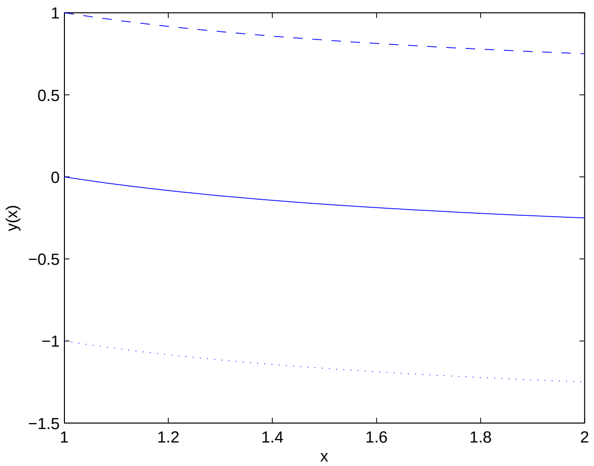

Figure 1: *Fuzzy extremals for fuzzy variational problem* (14) *in Example 1*

5. Fuzzy fractional variational problem with holonomic constraints

In this section, we consider the following problem,

$$\begin{aligned} \tilde{J}(\tilde{\mathbf{y}}) &= \int_a^b \tilde{L}(x,\tilde{\mathbf{y}},{}^C D_\gamma^{\alpha,\beta}\tilde{\mathbf{y}}(x))dx \longrightarrow \min, \\ \tilde{G}_j(x,\tilde{\mathbf{y}}(x)) &= \tilde{0}, \quad j=1,\ldots,m, \quad m<n, \\ \tilde{\mathbf{y}}(a) &= \tilde{\mathbf{y}}^a, \quad \tilde{\mathbf{y}}(b) = \tilde{\mathbf{y}}^b, \end{aligned} \tag{16}$$

where $\tilde{\mathbf{y}}(x) \in \mathbb{R}_{\tilde{\mathcal{F}}}^N$ and $\tilde{L}, \tilde{G}_j, \quad j=1,\ldots,m$, are smooth functions and the equations

$$\begin{aligned} \underline{G}_j^r\left(x,\underline{y}_1^r(x),\ldots,\underline{y}_N^r(x),\overline{y}_1^r(x),\ldots,\overline{y}_N^r(x)\right) &= 0, \quad j=1,\ldots,m, \\ \overline{G}_j^r\left(x,\underline{y}_1^r(x),\ldots,\underline{y}_N^r(x),\overline{y}_1^r(x),\ldots,\overline{y}_N^r(x)\right) &= 0, \quad j=1,\ldots,m, \end{aligned}$$

are independent, i.e., one of the Jacobians of order $2m$ is different from zero, for instance

$$\frac{D\left(\underline{G}_1^r,\ldots,\underline{G}_m^r,\overline{G}_1^r,\ldots,\overline{G}_m^r\right)}{D\left(\underline{y}_1^r,\ldots,\underline{y}_m^r,\overline{y}_1^r,\ldots,\overline{y}_m^r\right)} \neq 0.$$

As will be seen in the next theorem, we will show the necessary conditions for the extremal of problem (16).

Theorem 4. *A function $\tilde{\mathbf{y}}$ which is a solution to problem (16) satisfies, for suitably chosen functions $\lambda_j,\ \mu_j,\ j=1,\ldots,m$, the following system of fractional Euler–*

Lagrange equations

$$\left(\partial_i \underline{F}^r + D^{\beta_{i-1},\alpha_{i-1}}_{1-\gamma_{i-1}} \partial_{2N+i} \underline{F}^r\right) \left[x, \tilde{\boldsymbol{y}},\ {}^C D^{\alpha,\beta}_{\gamma} \left(\tilde{\boldsymbol{y}}(x)\right)\right]^r = 0, \quad i = 2, \ldots, 2N+1,$$

$$\left(\partial_i \overline{F}^r + D^{\beta_{i-1},\alpha_{i-1}}_{1-\gamma_{i-1}} \partial_{2N+i} \overline{F}^r\right) \left[x, \tilde{\boldsymbol{y}},\ {}^C D^{\alpha,\beta}_{\gamma} \left(\tilde{\boldsymbol{y}}(x)\right)\right]^r = 0, \quad i = 2, \ldots, 2N+1,$$

for all $x \in [a, b]$, where

$$\begin{aligned} \underline{F}^r \left[x, \tilde{\boldsymbol{y}},\ {}^C D^{\alpha,\beta}_{\gamma} \left(\tilde{\boldsymbol{y}}(x)\right)\right]^r =& \underline{L}^r \left[x, \tilde{\boldsymbol{y}},\ {}^C D^{\alpha,\beta}_{\gamma} \left(\tilde{\boldsymbol{y}}(x)\right)\right]^r \\ &+ \sum_{j=1}^{m} \lambda_j(x) \underline{G}^r_j \left(x, \underline{y}^r_1, \ldots, \underline{y}^r_N, \overline{y}^r_1, \ldots, \overline{y}^r_N\right), \\ \overline{F}^r \left[x, \tilde{\boldsymbol{y}},\ {}^C D^{\alpha,\beta}_{\gamma} \left(\tilde{\boldsymbol{y}}(x)\right)\right]^r =& \overline{L}^r \left[x, \tilde{\boldsymbol{y}},\ {}^C D^{\alpha,\beta}_{\gamma} \left(\tilde{\boldsymbol{y}}(x)\right)\right]^r \\ &+ \sum_{j=1}^{m} \mu_j(x) \overline{G}^r_j \left(x, \underline{y}^r_1, \ldots, \underline{y}^r_N, \overline{y}^r_1, \ldots, \overline{y}^r_N\right). \end{aligned}$$

Proof. Suppose that $\tilde{\mathbf{y}} = (\tilde{y}_1, \ldots, \tilde{y}_N)$ is the solution to problem (16) and define a family of curves $\tilde{\mathbf{y}} + \epsilon \tilde{\mathbf{h}}$, where $\tilde{\mathbf{h}} = (\tilde{h}_1, \ldots, \tilde{h}_N)$ is an arbitrary admissible variation, *i.e.*, $\tilde{h}_i(a) = \tilde{h}_i(b) = \tilde{0},\ i = 1, \ldots, N$, and $\tilde{G}_j(\tilde{\mathbf{y}} + \epsilon \tilde{\mathbf{h}}) = \tilde{0},\ j = 1, \ldots, m$, where $\epsilon \in \mathbb{R}$ is a small parameter. Under these assumptions, the functionals $\underline{J}^r$ and $\overline{J}^r$ become simple functions of the parameter ϵ and are extremized at $\epsilon = 0$. It follows that

$$\frac{d}{d\epsilon} \underline{J}^r(\epsilon)|_{\epsilon=0} = 0, \quad \frac{d}{d\epsilon} \overline{J}^r(\epsilon)|_{\epsilon=0} = 0.$$

that is,

$$\begin{aligned} \int_a^b \Bigg[&\sum_{i=2}^{N+1} \partial_i \underline{L}^r \cdot \underline{h}^r_{i-1}(x) + \sum_{i=2}^{N+1} \partial_{2N+i} \underline{L}^r \cdot {}^C D^{\alpha_{i-1},\beta_{i-1}}_{\gamma_{i-1}} \underline{h}^r_{i-1}(x) \\ &+ \sum_{i=N+2}^{2N+1} \partial_i \underline{L}^r \cdot \overline{h}^r_{i-N-1}(x) + \sum_{i=N+2}^{2N+1} \partial_{2N+i} \underline{L}^r \cdot {}^C D^{\alpha_{i-1},\beta_{i-1}}_{\gamma_{i-1}} \overline{h}^r_{i-N-1}(x) \Bigg] dx = 0 \end{aligned} \tag{17}$$

and

$$\begin{aligned} \int_a^b \Bigg[&\sum_{i=2}^{N+1} \partial_i \overline{L}^r \cdot \underline{h}^r_{i-1}(x) + \sum_{i=2}^{N+1} \partial_{2N+i} \overline{L}^r \cdot {}^C D^{\alpha_{i-1},\beta_{i-1}}_{\gamma_{i-1}} \underline{h}^r_{i-1}(x) + \sum_{i=N+2}^{2N+1} \partial_i \overline{L}^r \cdot \\ &\overline{h}^r_{i-N-1}(x) + \sum_{i=N+2}^{2N+1} \partial_{2N+i} \overline{L}^r \cdot {}^C D^{\alpha_{i-1},\beta_{i-1}}_{\gamma_{i-1}} \overline{h}^r_{i-N-1}(x) \Bigg] dx = 0 \end{aligned} \tag{18}$$

and for $j = 1, \ldots, m$,

$$\sum_{i=2}^{N+1} \partial_i \underline{G}^r_j \cdot \underline{h}^r_{i-1}(x) + \sum_{i=N+2}^{2N+1} \partial_i \underline{G}^r_j \cdot \overline{h}^r_{i-N-1}(x) = 0, \tag{19}$$

$$\sum_{i=2}^{N+1} \partial_i \overline{G}^r_j \cdot \underline{h}^r_{i-1}(x) + \sum_{i=N+2}^{2N+1} \partial_i \overline{G}^r_j \cdot \overline{h}^r_{i-N-1}(x) = 0. \tag{20}$$

Multiplying the jth equation of systems (19) and (20) by unspecified functions $\lambda_j(.)$ and $\mu_j(.)$, respectively, for all $j = 1, \dots, m$, integrating with respect to x, and adding the left-hand sides to the integrand of (17) and (18), by considering (17) and (19) we obtain

$$\begin{aligned}\int_a^b \{ &\sum_{i=2}^{N+1} \left(\partial_i \underline{L}^r + \sum_{j=1}^m \lambda_j(x) \partial_i \underline{G}_j^r \right) \underline{h}_{i-1}^r(x) + \sum_{i=2}^{N+1} \partial_{2N+i} \underline{L}^r \cdot \left({}^C D_{\gamma_{i-1}}^{\alpha_{i-1},\beta_{i-1}} \underline{h}_{i-1}^r(x) \right) \\ &+ \sum_{i=N+2}^{2N+1} \left(\partial_i \underline{L}^r + \sum_{j=1}^m \lambda_j(x) \partial_i \underline{G}_j^r \right) \overline{h}_{i-N-1}^r(x) \\ &+ \sum_{i=N+2}^{2N+1} \partial_{2N+i} \underline{L}^r \cdot \left({}^C D_{\gamma_{i-1}}^{\alpha_{i-N-1},\beta_{i-1}} \overline{h}_{i-N-1}^r(x) \right) \Bigg\} dx = 0.\end{aligned}$$

Integrating by parts,

$$\begin{aligned}\int_a^b \Bigg[&\left\{ \sum_{i=2}^{N+1} \partial_i \underline{L}^r + D_{1-\gamma_{i-1}}^{\beta_{i-1},\alpha_{i-1}} \partial_{2N+i} \underline{L}^r + \sum_{j=1}^m \lambda_j(x) \partial_i \underline{G}_j^r \right\} \underline{h}_{i-1}^r(x) \\ &+ \left\{ \sum_{i=N+2}^{2N+1} \partial_i \underline{L}^r + D_{1-\gamma_{i-1}}^{\beta_{i-1},\alpha_{i-1}} \partial_{2N+i} \underline{L}^r + \sum_{j=1}^m \lambda_j(x) \partial_i \underline{G}_j^r \right\} \overline{h}_{i-N-1}^r(x) \Bigg] dx = 0.\end{aligned}$$

Or, if we introduce the notation $\underline{F}^r$ in Theorem 4 we get

$$\begin{aligned}\int_a^b \Bigg[&\sum_{i=2}^{N+1} \left\{ \partial_i \underline{F}^r + D_{1-\gamma_{i-1}}^{\beta_{i-1},\alpha_{i-1}} \partial_{2N+i} \underline{F}^r \right\} \underline{h}_{i-1}^r(x) \\ &+ \sum_{i=N+2}^{2N+1} \left\{ \partial_i \underline{F}^r + D_{1-\gamma_{i-1}}^{\beta_{i-1},\alpha_{i-1}} \partial_{2N+i} \underline{F} \right\} \overline{h}_{i-N-1}^r(x) \Bigg] dx = 0.\end{aligned} \tag{21}$$

Following the scheme of obtaining (21) and adapting it to the case under consideration involving (18) and (20), we have

$$\begin{aligned}\int_a^b \Bigg[&\sum_{i=2}^{N+1} \left\{ \partial_i \overline{F}^r + D_{1-\gamma_{i-1}}^{\beta_{i-1},\alpha_{i-1}} \partial_{2N+i} \overline{F}^r \right\} \underline{h}_{i-1}^r(x) \\ &+ \sum_{i=N+2}^{2N+1} \left\{ \partial_i \overline{F}^r + D_{1-\gamma_{i-1}}^{\beta_{i-1},\alpha_{i-1}} \partial_{2N+i} \overline{F} \right\} \overline{h}_{i-N-1}^r(x) \Bigg] dx = 0.\end{aligned} \tag{22}$$

Because of (19) and (20), we cannot regard $2N$ functions $\underline{h}_1^r(.), \dots, \underline{h}_N^r(.)$ and $\overline{h}_1^r(.), \dots, \overline{h}_N^r(.)$ as free for an arbitrary choice. There is a subset of $2m$ of these functions whose assignment is restricted by the assignment of the remaining $2N - 2M$. We can assume, without loss of generality, that $\underline{h}_1^r(.), \dots, \underline{h}_m^r(.)$ and $\overline{h}_1^r(.), \dots, \overline{h}_m^r(.)$ are the functions of the set whose dependence upon the choice of the arbitrary

$\underline{h}^r_{m+1},\ldots,\underline{h}^r_N$ and $\overline{h}^r_{m+1}(.),\ldots,\overline{h}^r_N(.)$ is governed by (19) and (20). We now assign the functions $\lambda_1(.),\ldots,\lambda_m(.)$ and $\mu_1(.),\ldots,\mu_m(.)$ to be the set of $2m$ functions that make (for all x between a and b) the coefficients of $\underline{h}^r_1(.),\ldots,\underline{h}^r_m(.)$ and $\overline{h}^r_1(.),\ldots,\overline{h}^r_m(.)$ in the integrand of (21) and (22) vanish. That is, $\lambda_1(.),\ldots,\lambda_m(.)$ and $\mu_1(.),\ldots,\mu_m(.)$ are chosen so as to satisfy

$$\begin{aligned}
&\left(\partial_i\underline{F}^r + D^{\beta_{i-1},\alpha_{i-1}}_{1-\gamma_{i-1}}\partial_{2N+i}\underline{F}^r\right)\left[x,\tilde{\mathbf{y}},\ {}^C D^{\alpha,\beta}_{\gamma}\left(\tilde{\mathbf{y}}(x)\right)\right]^r = 0, \quad i = 2,\ldots,m+1,\\
&\left(\partial_i\overline{F}^r + D^{\beta_{i-1},\alpha_{i-1}}_{1-\gamma_{i-1}}\partial_{2N+i}\overline{F}^r\right)\left[x,\tilde{\mathbf{y}},\ {}^C D^{\alpha,\beta}_{\gamma}\left(\tilde{\mathbf{y}}(x)\right)\right]^r = 0, \quad i = 2,\ldots,m+1,\\
&\left(\partial_i\underline{F}^r + D^{\beta_{i-1},\alpha_{i-1}}_{1-\gamma_{i-1}}\partial_{2N+i}\underline{F}^r\right)\left[x,\tilde{\mathbf{y}},\ {}^C D^{\alpha,\beta}_{\gamma}\left(\tilde{\mathbf{y}}(x)\right)\right]^r = 0, \quad i = N+2,\ldots,N+m+1,\\
&\left(\partial_i\overline{F}^r + D^{\beta_{i-1},\alpha_{i-1}}_{1-\gamma_{i-1}}\partial_{2N+i}\overline{F}^r\right)\left[x,\tilde{\mathbf{y}},\ {}^C D^{\alpha,\beta}_{\gamma}\left(\tilde{\mathbf{y}}(x)\right)\right]^r = 0, \quad i = N+2,\ldots,N+m+1,
\end{aligned}$$

for all $x \in [a,b]$. With this choice, (21) and (22) give

$$\begin{aligned}
\int_a^b \Bigg[&\sum_{i=m+2}^{N+1}\left\{\partial_i\underline{F}^r + D^{\beta_{i-1},\alpha_{i-1}}_{1-\gamma_{i-1}}\partial_{2N+i}\underline{F}^r\right\}\underline{h}^r_{i-1}(x)\\
&+\sum_{i=N+m+2}^{2N+1}\left\{\partial_i\underline{F}^r + D^{\beta_{i-1},\alpha_{i-1}}_{1-\gamma_{i-1}}\partial_{2N+i}\underline{F}\right\}\overline{h}^r_{i-N-1}(x)\Bigg]\,dx = 0
\end{aligned}$$

and

$$\begin{aligned}
\int_a^b \Bigg[&\sum_{i=m+2}^{N+1}\left\{\partial_i\overline{F}^r + D^{\beta_{i-1},\alpha_{i-1}}_{1-\gamma_{i-1}}\partial_{2N+i}\overline{F}^r\right\}\underline{h}^r_{i-1}(x)\\
&+\sum_{i=N+m+2}^{2N+1}\left\{\partial_i\overline{F}^r + D^{\beta_{i-1},\alpha_{i-1}}_{1-\gamma_{i-1}}\partial_{2N+i}\overline{F}\right\}\overline{h}^r_{i-N-1}(x)\Bigg]\,dx = 0.
\end{aligned}$$

Since the functions $\underline{h}^r_{m+1}(.),\ldots,\underline{h}^r_N(.)$ and $\overline{h}^r_{m+1}(.),\ldots,\overline{h}^r_N(.)$ are arbitrary, we may employ the fundamental lemma of the calculus of variations to conclude that

$$\begin{aligned}
&\left(\partial_i\underline{F}^r + D^{\beta_{i-1},\alpha_{i-1}}_{1-\gamma_{i-1}}\partial_{2N+i}\underline{F}^r\right)\left[x,\tilde{\mathbf{y}},\ {}^C D^{\alpha,\beta}_{\gamma}\left(\tilde{\mathbf{y}}(x)\right)\right]^r = 0, \quad i = m+2,\ldots,N+1,\\
&\left(\partial_i\overline{F}^r + D^{\beta_{i-1},\alpha_{i-1}}_{1-\gamma_{i-1}}\partial_{2N+i}\overline{F}^r\right)\left[x,\tilde{\mathbf{y}},\ {}^C D^{\alpha,\beta}_{\gamma}\left(\tilde{\mathbf{y}}(x)\right)\right]^r = 0, \quad i = m+2,\ldots,N+1,\\
&\left(\partial_i\underline{F}^r + D^{\beta_{i-1},\alpha_{i-1}}_{1-\gamma_{i-1}}\partial_{2N+i}\underline{F}^r\right)\left[x,\tilde{\mathbf{y}},\ {}^C D^{\alpha,\beta}_{\gamma}\left(\tilde{\mathbf{y}}(x)\right)\right]^r = 0, \quad i = N+m+2,\ldots,2N+1,\\
&\left(\partial_i\overline{F}^r + D^{\beta_{i-1},\alpha_{i-1}}_{1-\gamma_{i-1}}\partial_{2N+i}\overline{F}^r\right)\left[x,\tilde{\mathbf{y}},\ {}^C D^{\alpha,\beta}_{\gamma}\left(\tilde{\mathbf{y}}(x)\right)\right]^r = 0, \quad i = N+m+2,\ldots,2N+1,
\end{aligned}$$

for all $x \in [a,b]$. $\square$

Example 2. *Find the extremal of the following problem*

$$\begin{aligned}\tilde{J}(\tilde{y}) := \int_0^1 \left({}^C D_\gamma^{\alpha,\beta}\tilde{y}_1(x)\right)^2 + \left({}^C D_\gamma^{\alpha,\beta}\tilde{y}_2(x)\right)^2 + 1dx \longrightarrow \min,\\ \tilde{y}_1 + \tilde{y}_2 \ominus (\tilde{2}.x^2) = \tilde{0},\\ \tilde{y}_1(0) = \tilde{y}_2(0) = \tilde{0},\\ \tilde{y}_1(1) =< 1, 2, 3 >,\\ \tilde{y}_2(1) =< -1, 0, 1 >,\end{aligned}$$

where $\tilde{2} =< 0, 2, 4 >$.

Solution. For the sake of avoiding the argument described in Example 1, here we only focus on the following case. From the definition of $\underline{F}^r$ and $\overline{F}^r$ in Theorem 4, suppose that

$$\begin{aligned}\underline{F}^r &= \left({}^C D_\gamma^{\alpha,\beta}\underline{y}_1^r(x)\right)^2 + \left({}^C D_\gamma^{\alpha,\beta}\underline{y}_2^r(x)\right)^2 + 1 + \lambda\left(\underline{y}_1^r + \underline{y}_2^r - 2rx^2\right),\\ \overline{F}^r &= \left({}^C D_\gamma^{\alpha,\beta}\overline{y}_1^r(x)\right)^2 + \left({}^C D_\gamma^{\alpha,\beta}\overline{y}_2^r(x)\right)^2 + 1 + \mu\left(\overline{y}_1^r + \overline{y}_2^r + (2r-4)x^2\right).\end{aligned}$$

By considering fuzzy Euler–Lagrange conditions in Theorem 4, we get the following equations

$$\begin{aligned}\lambda + D_{1-\gamma}^{\beta,\alpha}\left(2{}^C D_\gamma^{\alpha,\beta}\underline{y}_1^r(x)\right) = 0,\\ \lambda + D_{1-\gamma}^{\beta,\alpha}\left(2{}^C D_\gamma^{\alpha,\beta}\underline{y}_2^r(x)\right) = 0,\\ \mu + D_{1-\gamma}^{\beta,\alpha}\left(2{}^C D_\gamma^{\alpha,\beta}\overline{y}_1^r(x)\right) = 0,\\ \mu + D_{1-\gamma}^{\beta,\alpha}\left(2{}^C D_\gamma^{\alpha,\beta}\overline{y}_2^r(x)\right) = 0.\end{aligned} \tag{23}$$

Similarly to Example 1, it is difficult to solve the above fractional equations, for $0 < \alpha < 1, 0 < \beta < 1, 0 \le \gamma \le 1$ and in that case, a numerical method should be used. When α, γ tend to one, however, one has the following problem:

$$\begin{aligned}\tilde{J}(\tilde{y}) := \int_0^1 \dot{\tilde{y}}_1^2 + \dot{\tilde{y}}_2^2 + 1dx \longrightarrow \min,\\ \tilde{y}_1 + \tilde{y}_2 \ominus (\tilde{2}.x^2) = \tilde{0},\\ \tilde{y}_1(0) = \tilde{y}_2(0) = \tilde{0},\\ \tilde{y}_1(1) =< 1, 2, 3 >,\\ \tilde{y}_2(1) =< -1, 0, 1 >,\end{aligned} \tag{24}$$

where $\tilde{2} =< 0, 2, 4 >$.
It follows from (23) that

$$\begin{aligned}\underline{\ddot{y}}_1^r = \frac{\lambda}{2}, \quad \ddot{\overline{y}}_1^r = \frac{\mu}{2},\\ \underline{\ddot{y}}_2^r = \frac{\lambda}{2}, \quad \ddot{\overline{y}}_2^r = \frac{\mu}{2},\end{aligned}$$

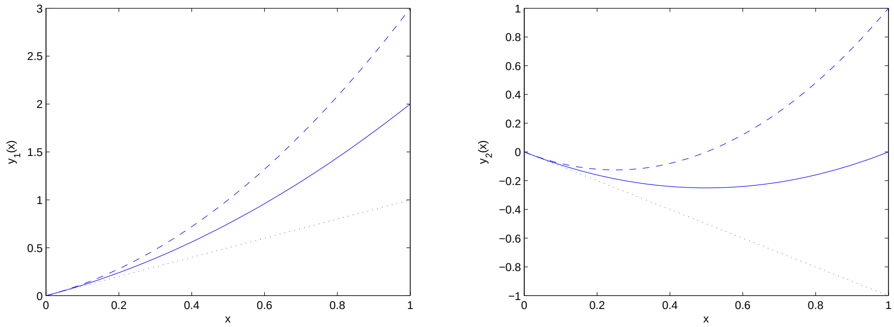

Figure 2: *Fuzzy extremals for fuzzy variational problem* (24) *in Example 2*

which have general solutions

$$\underline{y}_1^r = \frac{\lambda}{4}x^2 + c_1x + c_2, \quad \overline{y}_1^r = \frac{\mu}{4}x^2 + c_3x + c_4$$

and

$$\underline{y}_2^r = \frac{\lambda}{4}x^2 + d_1x + d_2, \quad \overline{y}_2^r = \frac{\mu}{4}x^2 + d_3x + d_4,$$

where c_k and d_k are constants for $k = 1, \ldots, 4$. From the initial conditions and the subsequent conditions

$$\underline{y}_1^r + \underline{y}_2^r - 2rx^2 = 0, \quad \overline{y}_1^r + \overline{y}_2^r + (2r-4)x^2 = 0,$$

we find

$$\lambda = 4r, \quad \mu = 8 - 4r, \quad c_1 = c_3 = 1, \quad c_2 = d_2 = c_4 = d_4 = 0, \quad d_1 = d_3 = -1.$$

Thus, the required extremizing functions are

$$\underline{y}_1^r = rx^2 + x, \quad \overline{y}_1^r = (2-r)x^2 + x$$

and

$$\underline{y}_2^r = rx^2 - x, \ \overline{y}_2^r = (2-r)x^2 - x.$$

One can check that the conditions of Lemma 1 are satisfied, so $\tilde{y}_1(x)$ and $\tilde{y}_2(x)$ are fuzzy functions in [0,1]. This solution is shown in Figure 2, where the solid lines are $\underline{y}_1^1(x) = \overline{y}_1^1(x)$ and $\underline{y}_2^1(x) = \overline{y}_2^1(x)$; the dashed lines are $\overline{y}_1^0(x)$ and $\overline{y}_2^0(x)$; the doted lines are $\underline{y}_1^0(x)$ and $\underline{y}_2^0(x)$.

6. Conclusion

We established fuzzy fractional Euler-Lagrange equations to variational problems with natural boundary conditions, and problems with holonomic constraints. The main features of our optimality conditions were summarized and highlighted with two illustrative examples. The fuzzy fractional optimality conditions are in general difficult to solve and, as future work, we intend to develop numerical methods to address the issue.

References

[1] R. Almeida, S. Pooseh, D. F. M. Torres, *Computational methods in the fractional calculus of variations*, Imp. Coll. Press, London, 2015.

[2] S. Arshad, V. Lupulescu, *On the fractional differential equations with uncertainty*, Nonlinear Anal. **74**(2011), 3685–3693.

[3] B. Bede, I. J. Rudas, A. L. Bencsik, *First order linear fuzzy differential equations under generalized differentiability*, Inform Sci **177**(2007), 1648-1662.

[4] B. V. Brunt, *The Calculus of Variations*, Springer-Verlag, Heidelberg, 2004.

[5] J. J. Buckley, T. Feuring, *Introduction to fuzzy partial differential equations*, Fuzzy Sets and Systems **105**(1999), 241–248.

[6] O. S. Fard, A. H. Borzabadi, M. Heidari, *On fuzzy Euler-Lagrange equations*, Ann. Fuzzy Math. Inform. **7**(2014), 447–461.

[7] O. S. Fard, M. Salehi, *A survey on fuzzy fractional variational problems,* J. Comput. Appl. Math. **271**(2014), 71–82.

[8] O. S. Fard, M. S. Zadeh,*Note on "Necessary optimality conditions for fuzzy variational problems"*, J. Adv. Res. Dyn. Control Syst. **4**(2012), 1–9.

[9] B. Farhadinia, *Necessary optimality conditions for fuzzy variational problems,* Inform. Sci. **181**(2011), 1348–1357.

[10] B. Farhadinia, *Pontryagin's minimum principle for fuzzy optimal control problems,* Iran. J. Fuzzy Syst. **11**(2014), 27–43.

[11] R. Goetschel, W. Voxman, *Elementary fuzzy calculus,* Fuzzy Sets and Systems **18**(1986), 31–43.

[12] R. Hilfer, *Applications of fractional calculus in physics*, World Sci. Publishing, River Edge, NJ, 2000.

[13] A. A. Kilbas, H. M. Srivastava, J. J. Trujillo, *Theory and applications of fractional differential equations*, North-Holland Mathematics Studies, 204, Elsevier, Amsterdam, 2006.

[14] A. B. Malinowska, D. F. M. Torres, *Introduction to the fractional calculus of variations*, Imp. Coll. Press, London, 2012.

[15] A. B. Malinowska, D. F. Torres, *Multiobjective fractional variational calculus in terms of a combined Caputo derivative*, Applied Mathematics and Computation **218**(2012), 5099–5111.

[16] M. Mazandarani, A. V. Kamyad, *Modified fractional Euler method for solving fuzzy fractional initial value problem,* Commun. Nonlinear Sci. Numer. Simul. **18**(2013), 12–21.

[17] K. S. Miller, B. Ross, *An introduction to the fractional calculus and fractional differential equations*, A Wiley-Interscience Publication, Wiley, New York, 1993.

[18] M. Najariyan, M. H. Farahi, *Optimal control of fuzzy linear controlled system with fuzzy initial conditions,* Iran. J. Fuzzy Syst. **10**(2013), 21–35.

[19] M. Najariyan, M. H. Farahi, *A new approach for the optimal fuzzy linear time invariant controlled system with fuzzy coefficients,* J. Comput. Appl. Math. **259**(2014), part B, 682–694.

[20] T. Odzijewicz, A. B. Malinowska, D. F. M. Torres, *Advanced methods in the fractional calculus of variations*, Springer Briefs in Applied Sciences and Technology, Springer, 2015.

[21] I. Podlubny, *Fractional differential equations*, Mathematics in Science and Engineering, 198, Academic Press, San Diego, CA, 1999.

[22] S. Salahshour, T. Allahviranloo, S. Abbasbandy, *Solving fuzzy fractional differential equations by fuzzy Laplace transforms,* Commun. Nonlinear Sci. Numer. Simul.

17(2012), 1372–1381.

[23] S. Salahshour, T. Allahviranloo, S. Abbasbandy, D. Baleanu, *Existence and uniqueness results for fractional differential equations with uncertainty*, Adv. Difference Equ. **2012**(2012), 2012:112, 12pp.

[24] J. Xu, Z. Liao, J. J. Nieto, *A class of linear differential dynamical systems with fuzzy matrices*, J. Math. Anal. Appl. **368**(2010), 54–68.