MATHEMATICAL COMMUNICATIONS
Math. Commun. **21**(2016), 219–225

Two transformations for a binomial sum

Jian Cao[1] and Wenchang Chu[2,*]

[1] *Department of Mathematics, Hangzhou Normal University, Hangzhuo 310 036, P. R. China*
[2] *Dipartimento di Matematica e Fisica "Ennio de Giorgi", Università del Salento, Lecce-Arnesano P. O. Box 193, Lecce 73 100, Italy*

Received June 8, 2015; accepted February 22, 2016

Abstract. By means of finite differences and partial fraction decomposition, we establish two binomial transformations, that extend, with four free parameters, the recent results due to Prodinger (2010) and Dahlberg-Ferdinands-Tefera (2010).

AMS subject classifications: Primary 05A10; Secondary 05A19
Key words: binomial sum, finite difference, partial fraction decomposition

1. Introduction and motivation

Let $\mathbb{N}$ and $\mathbb{N}_0$ be the sets of natural numbers and nonnegative integers, respectively. For an indeterminate x, denote the Pochhammer symbol by

$$(x)_0 = 1 \quad \text{and} \quad (x)_n = x(x+1)\cdots(x+n-1), \quad \text{with} \quad n \in \mathbb{N}.$$

The following binomial sum has recently attracted much attention

$$\sum_{k=0}^{n} \binom{n}{k} \frac{(-1/2)^k}{(m+k)_\varepsilon}, \quad \text{where} \quad m, n, \varepsilon \in \mathbb{N}.$$

By utilizing the Gauss hypergeometric ${}_2F_1$-series, Choi-Zörnig-Rathie [2] deduced the following two closed formulae:

$$\sum_{k=0}^{n} \binom{n}{k} \frac{k(-1/2)^k}{(n+k)(n+k+1)} = \frac{n!n!2^{n-1}}{(2n+1)!} - \frac{1}{2^{n+1}}, \tag{1}$$

$$\sum_{k=0}^{n-2} \binom{n-2}{k} \frac{k(-1/2)^k}{(n+k)(n+k+1)} = \frac{3n!n!2^n}{(n-1)(2n)!} - \frac{n+2}{(n-1)2^{n-1}}. \tag{2}$$

Prodinger [7] extended it slightly to the following binomial transformation

$$\sum_{k=0}^{n} \binom{n}{k} \frac{(-1/2)^k}{m+k} = \frac{n!(m-1)!}{2^n(n+m)!} \sum_{k=0}^{n} \binom{m+n}{k}, \quad \text{where} \quad m, n \in \mathbb{N}. \tag{3}$$

*Corresponding author. *Email addresses:* chu.wenchang@unisalento.it (W. Chu), 21caojian@163.com (J. Cao)

Dahlberg-Ferdinands-Tefera [3] give a WZ-style proof and find further

$$\sum_{k=0}^{n}\binom{n}{k}\frac{k(-1/2)^k}{(n+k)_{m+1}}=\sum_{k=0}^{m}\frac{(-1)^{k+1}}{2^{n+1}m!}\frac{\binom{m}{k}}{\binom{2n+k}{n}}\Big\{2^{2n+k}-\sum_{j=n+1}^{n+k-1}\binom{2n+k}{j}\Big\}. \tag{4}$$

Different approaches and variants of these sums can be found in [5, 6, 7, 8, 9].

The purpose of this short paper is to examine the generalized binomial sum with four free paramters given below

$$\Omega_n(\lambda,\mu;a,c,x):=\sum_{k=0}^{n}x^k\binom{n}{k}\frac{(a+k)_\lambda}{(c+k)_{\mu+1}}, \tag{5}$$

where $n,\lambda,\mu\in\mathbb{N}_0$ and $a,c\in\mathbb{C}$, the set of complex numbers. The following two interesting transformation formulae will be shown.

Theorem 1. *For $\lambda,\mu,n\in\mathbb{N}_0$ with $\lambda\le\mu$, the following binomial transformation holds*

$$\begin{aligned}\Omega_n(\lambda,\mu;a,c,x)=\;&\frac{(-1)^\lambda n!\lambda!}{(c)_{n+\mu+1}}\sum_{k=0}^{n}\frac{(c)_k}{k!}(1+x)^k\\&\times\sum_{i=0}^{\lambda}\binom{-c-k}{i}\binom{c-a}{\lambda-i}\binom{n+\mu-k-i}{\mu-i}.\end{aligned}$$

Theorem 2. *For $\lambda,\mu,n\in\mathbb{N}_0$ with $\lambda\le\mu$, the following binomial transformation holds*

$$\begin{aligned}\Omega_n(\lambda,\mu;a,c,x)=\;&\frac{(-1)^\lambda n!\lambda!}{(c)_{n+\mu+1}}(1+x)^n\sum_{k=0}^{n}\binom{c+n+\mu}{n-k}\Big\{\frac{-x}{1+x}\Big\}^k\\&\times\sum_{j=0}^{\mu}\binom{-c}{j}\binom{c-a+j}{\lambda}\binom{c+\mu+k}{\mu-j}.\end{aligned}$$

As Prodinger [7] observed, equality (3) is implied by Pfaff's reflection law (cf. Bailey [1, §1.2] and Graham-Knuth-Patashnik [4, §5.6]). However, the two relations obtained in this paper cannot be deduced directly in this manner. By combining the finite differences with partial fraction decomposition, we shall prove these two transformations in the next section. Then the paper will end up with a discussion about how the binomial identities due to Prodinger [7] and Dahlberg-Ferdinands-Tefera [3] can be recovered from Theorems 1 and 2.

2. Proofs of the main results

In order to prove the theorem, let us recall finite differences and their properties. For any given function $f(\tau)$, denote by Δ the usual difference operator with the unit increment $\Delta f(\tau)=f(\tau+1)-f(\tau)$. Then the Newton-Gregory formula reads

$$\Delta^n f(\tau)=\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}f(\tau+k),\qquad\text{where}\qquad n\in\mathbb{N}_0.$$

In particular, we shall briefly write $\Delta_0^n f(\tau) := \Delta^n f(\tau)\big|_{\tau=0}$. The higher differences also satisfy the following useful Leibniz rule

$$\Delta^n\big\{f(\tau)g(\tau)\big\} = \sum_{k=0}^{n}\binom{n}{k}\Delta^k f(\tau)\Delta^{n-k}g(\tau+k), \quad \text{where} \quad n\in\mathbb{N}_0.$$

For $\lambda \le \mu$, we first expand the rational function $R(k)$ into partial fractions

$$R(k) := \frac{(a+k)_\lambda}{(c+k)_{\mu+1}} = \sum_{i=0}^{\mu}\frac{w_i}{c+k+i},$$

where the coefficients $\{w_0.w_1,\cdots,w_\mu\}$ are independent of the variable k and can be determined by the following limiting process

$$w_i = \lim_{k\to -c-i}(c+k+i)R(k) = \frac{(a-c-i)_\lambda}{(\mu-i)!(-i)_i} = (-1)^i\binom{\mu}{i}\frac{(a-c-i)_\lambda}{\mu!}.$$

We have therefore established the following identity

$$\frac{(a+k)_\lambda}{(c+k)_{\mu+1}} = \sum_{i=0}^{\mu}\frac{(-1)^i}{\mu!}\binom{\mu}{i}\frac{(a-c-i)_\lambda}{c+k+i}.$$

Substituting the last relation into (5) and then interchanging the summation order, we get the following double sum expressions

$$\begin{aligned}\Omega_n(\lambda,\mu;a,c) &= \sum_{k=0}^{n}x^k\binom{n}{k}\sum_{i=0}^{\mu}\frac{(-1)^i}{\mu!}\binom{\mu}{i}\frac{(a-c-i)_\lambda}{c+k+i}\\ &= \sum_{i=0}^{\mu}\frac{(-1)^i}{\mu!}\binom{\mu}{i}(a-c-i)_\lambda\sum_{k=0}^{n}\binom{n}{k}\frac{x^k}{c+k+i}.\end{aligned}$$

Writing the inner sum in terms of finite differences and then invoking the Leibniz rule, we have

$$\begin{aligned}\sum_{k=0}^{n}\binom{n}{k}\frac{x^k}{c+k+i} &= (-1)^n\Delta_0^n\frac{(-x)^\tau}{c+\tau+i}\\ &= (-1)^n\sum_{k=0}^{n}\binom{n}{k}\Delta_0^k(-x)^\tau\Delta_0^{n-k}\frac{1}{c+\tau+k+i}.\end{aligned}$$

Observe that

$$\Delta_0^k(-x)^\tau = \sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}(-x)^j = (-1-x)^k$$

and

$$\Delta_0^{n-k}\frac{1}{c+\tau+k+i} = \sum_{j=0}^{n-k}\binom{n-k}{j}\frac{(-1)^{n-k-j}}{c+k+i+j} = \frac{(-1)^{n-k}(n-k)!}{(c+k+i)_{n-k+1}},$$

which have been justified by the binomial theorem and the induction principle, respectively. We find consequently another double sum expression

$$\begin{aligned}\Omega_n(\lambda,\mu;a,c) &= \sum_{i=0}^{\mu}\frac{(-1)^i}{\mu!}\binom{\mu}{i}(a-c-i)_\lambda\sum_{k=0}^{n}\binom{n}{k}\frac{(n-k)!(1+x)^k}{(c+k+i)_{n-k+1}}\\ &= \frac{n!}{\mu!}\sum_{k=0}^{n}\frac{(1+x)^k}{k!}\sum_{i=0}^{\mu}(-1)^i\binom{\mu}{i}\frac{(a-c-i)_\lambda}{(c+k+i)_{n-k+1}}.\end{aligned}$$

The last sum with respect to i can again be stated as finite differences

$$\begin{aligned}&\lambda!\sum_{i=0}^{\mu}(-1)^{\lambda-i}\binom{\mu}{i}\frac{\binom{c-a+i}{\lambda}}{(c+k+i)_{n-k+1}}\\ &\quad= (-1)^{\lambda+\mu}\lambda!\Delta_0^{\mu}\frac{\binom{c-a+\tau}{\lambda}}{(c+k+\tau)_{n-k+1}}\\ &\quad= (-1)^{\lambda+\mu}\lambda!\sum_{i=0}^{\mu}\binom{\mu}{i}\Delta_0^{i}\binom{c-a+\tau}{\lambda}\Delta_0^{\mu-i}\frac{1}{(c+k+\tau+i)_{n-k+1}}.\end{aligned}$$

By means of the induction principle, we can compute both finite differences

$$\Delta_0^{i}\binom{c-a+\tau}{\lambda}=\binom{c-a}{\lambda-i}$$

and

$$\Delta_0^{\mu-i}\frac{1}{(c+k+\tau+i)_{n-k+1}}=\frac{(-1)^{\mu-i}(n-k+1)_{\mu-i}}{(c+k+i)_{n+\mu-k-i+1}}.$$

This surprisingly yields the following expression

$$\begin{aligned}&\sum_{i=0}^{\mu}(-1)^i\binom{\mu}{i}\frac{(a-c-i)_\lambda}{(c+k+i)_{n-k+1}}\\ &\quad= \lambda!\sum_{i=0}^{\lambda}(-1)^{\lambda-i}\binom{\mu}{i}\binom{c-a}{\lambda-i}\frac{(n-k+1)_{\mu-i}}{(c+k+i)_{n+\mu-k-i+1}}\\ &\quad= \frac{(-1)^\lambda\lambda!\mu!}{(c+k)_{n+\mu-k+1}}\sum_{i=0}^{\lambda}\binom{-c-k}{i}\binom{c-a}{\lambda-i}\binom{n+\mu-k-i}{\mu-i}.\end{aligned}$$

Summing up, we have found the following transformation formula

$$\begin{aligned}\Omega_n(\lambda,\mu;a,c) = \sum_{k=0}^{n}&\frac{(1+x)^k}{k!}\frac{(-1)^\lambda\lambda!n!}{(c+k)_{n-k+\mu+1}}\\ &\times\sum_{i=0}^{\lambda}\binom{-c-k}{i}\binom{c-a}{\lambda-i}\binom{n+\mu-k-i}{\mu-i},\end{aligned}$$

which is clearly equivalent to that displayed in Theorem 1.

Now we turn to prove Theorem 2. According to the Chu-Vandermonde convolution formula, we have the following equality

$$\binom{n+\mu-k-i}{\mu-i} = \sum_{j=i}^{\mu}\binom{-c-k-i}{j-i}\binom{c+n+\mu}{\mu-j}.$$

Then we can reformulate the following binomial sum

$$\begin{aligned}
&\sum_{i=0}^{\lambda}\binom{-c-k}{i}\binom{c-a}{\lambda-i}\binom{n+\mu-k-i}{\mu-i}\\
&\quad=\sum_{i=0}^{\mu}\binom{-c-k}{i}\binom{c-a}{\lambda-i}\sum_{j=i}^{\mu}\binom{-c-k-i}{j-i}\binom{c+n+\mu}{\mu-j}\\
&\quad=\sum_{j=0}^{\mu}\binom{-c-k}{j}\binom{c+n+\mu}{\mu-j}\sum_{i=0}^{j}\binom{j}{i}\binom{c-a}{\lambda-i}\\
&\quad=\sum_{j=0}^{\mu}\binom{-c-k}{j}\binom{c-a+j}{\lambda}\binom{c+n+\mu}{\mu-j},
\end{aligned}$$

where the last passage has been justified again by the Chu-Vandermonde convolution formula. Therefore, we can express the double sum displayed in Theorem 1 as follows:

$$\begin{aligned}
&\sum_{k=0}^{n}\frac{(c)_k}{k!}(1+x)^k\sum_{i=0}^{\mu}\binom{-c-k}{i}\binom{c-a}{\lambda-i}\binom{n+\mu-k-i}{\mu-i}\\
&\quad=\sum_{j=0}^{\mu}\binom{c-a+j}{\lambda}\binom{c+n+\mu}{\mu-j}\sum_{k=0}^{n}\binom{-c-k}{j}\frac{(c)_k}{k!}(1+x)^k\\
&\quad=\sum_{j=0}^{\mu}\binom{-c}{j}\binom{c-a+j}{\lambda}\binom{c+n+\mu}{\mu-j}\sum_{k=0}^{n}\binom{c+k+j-1}{k}(1+x)^k.
\end{aligned}$$

Let $[\tau^n]f(\tau)$ stand for the coefficient of τ^n in the formal power series $f(\tau)$. We can further rewrite the last sum with respect to k as

$$\begin{aligned}
\sum_{k=0}^{n}\binom{c+k+j-1}{k}(1+x)^k &= \sum_{k=0}^{n}[\tau^k]\big\{1-\tau(1+x)\big\}^{-c-j}\\
&= [\tau^n]\frac{\big\{1-\tau(1+x)\big\}^{-c-j}}{1-\tau}\\
&= [\tau^n]\frac{\big\{1-\tau(1+x)\big\}^{-c-j}}{1-\tau(1+x)+\tau x}\\
&= \sum_{k=0}^{n}[\tau^n]\frac{(-\tau x)^k}{\big\{1-\tau(1+x)\big\}^{c+k+j+1}}\\
&= (1+x)^n\sum_{k=0}^{n}\binom{c+n+j}{n-k}\Big\{\frac{-x}{1+x}\Big\}^k.
\end{aligned}$$

This leads to the double sum transformation

$$\begin{aligned}
&\sum_{k=0}^{n}\frac{(c)_k}{k!}(1+x)^k\sum_{i=0}^{\mu}\binom{-c-k}{i}\binom{c-a}{\lambda-i}\binom{n+\mu-k-i}{\mu-i}\\
&\quad=\sum_{j=0}^{\mu}\binom{-c}{j}\binom{c-a+j}{\lambda}\binom{c+n+\mu}{\mu-j}(1+x)^n\sum_{k=0}^{n}\binom{c+n+j}{n-k}\Big\{\frac{-x}{1+x}\Big\}^k\\
&\quad=(1+x)^n\sum_{k=0}^{n}\Big\{\frac{-x}{1+x}\Big\}^k\sum_{j=0}^{\mu}\binom{-c}{j}\binom{c-a+j}{\lambda}\binom{c+n+j}{n-k}\binom{c+n+\mu}{\mu-j}.
\end{aligned}$$

According to Theorem 1, this equation confirms the transformation formula displayed in Theorem 2.

3. Further discussion

Now we are going to show that four identities (1), (2), (3) and (4) are very particular cases of Theorems 1 and 2.

Firstly, it is not hard to check that (1) follows from Theorem 2.

$$\begin{aligned}
\Omega_n(1,1;0,n,-\tfrac12)=&\frac{n!}{2^n(n)_{n+2}}\sum_{k=0}^{n}\binom{2n+1}{n-k}\big\{n(n+1)-n(n+k+1)\big\}\\
=&\frac{n!n!}{2^n(2n+1)!}\Big\{(n+1)\sum_{k=0}^{n}\binom{2n+1}{n-k}-(2n+1)\sum_{k=0}^{n}\binom{2n}{n-k}\Big\}\\
=&\frac{n!n!}{2^n(2n+1)!}\Big\{2^{2n}(n+1)-(2n+1)2^{2n-1}-\frac{2n+1}{2}\binom{2n}{n}\Big\}\\
=&\frac{n!n!2^{n-1}}{(2n+1)!}-\frac{1}{2^{n+1}}.
\end{aligned}$$

Analogously for (2), we can show it below.

$$\begin{aligned}
\Omega_{n-2}(1,1;0,n,-\tfrac12)=&\frac{(n-2)!}{2^{n-2}(n)_n}\sum_{k=0}^{n-2}\binom{2n-1}{n-k-2}\big\{n(n+1)-n(n+k+1)\big\}\\
=&\frac{n!(n-2)!}{2^{n-2}(2n-1)!}\sum_{k=0}^{n-2}\Big\{(n+1)\binom{2n-1}{n-k-2}-(2n-1)\binom{2n-2}{n-k-2}\Big\}\\
=&\frac{n!(n-2)!}{2^{n-2}(2n-1)!}\Big\{3\times 2^{2n-3}-\frac{n(n+2)}{2}\binom{2n-1}{n}\Big\}\\
=&\frac{3n!n!2^n}{(2n)!(n-1)}-\frac{n+2}{2^{n-1}(n-1)}.
\end{aligned}$$

Instead, equation (3) is almost a trivial instance of Theorem 2:

$$\Omega_n(0,0;a,m,-\tfrac12)=\frac{n!}{2^n(m)_{n+1}}\sum_{k=0}^{n}\binom{m+n}{n-k}=\frac{n!(m-1)!}{2^n(m+n)!}\sum_{k=0}^{n}\binom{m+n}{k}.$$

Finally, for equation (4), we need to further manipulate the double sum displayed in Theorem 2 as follows

$$\begin{aligned}\Omega_n(1,m;0,n,-\tfrac{1}{2}) =&\frac{-n!}{2^n(n)_{m+n+1}}\sum_{k=0}^{n}\binom{2n+m}{n-k}\sum_{j=0}^{m}(n+j)\binom{-n}{j}\binom{m+n+k}{m-j}\\ =&\sum_{j=0}^{m}\frac{n!(-1)^{j+1}}{2^n(n+1)_{m+n}}\binom{n+j}{j}\binom{m+2n}{m-j}\sum_{k=0}^{n}\binom{2n+j}{n+k+j}\\ =&\sum_{j=0}^{m}\frac{(-1)^{j+1}}{2^n m!}\frac{\binom{m}{j}}{\binom{2n+j}{n}}\sum_{k=0}^{n}\binom{2n+j}{n+k+j}.\end{aligned}$$

This becomes the binomial sum on the right hand side of (4) thanks to the binomial relation

$$2\sum_{k=0}^{n}\binom{2n+j}{n+k+j} = 2^{2n+j}-\sum_{k=n+1}^{n+j-1}\binom{2n+j}{k}.$$

In addition, via WZ-method Dahlberg-Ferdinands-Tefera [3] also find the following identity

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{m+k}{m+k+\nu} = \frac{-n!\nu}{(m+\nu)_{n+1}}.$$

This can also be verified through Theorem 2. In fact, we have

$$\Omega_n(1,0;m,m+\nu,-1) = \frac{-n!\nu}{(m+\nu)_{n+1}},$$

because there is only one surviving term with the summation index $k = n$ on the right-hand side of the corresponding equation displayed in Theorem 2.

References

[1] W. N. Bailey, *Generalized hypergeometric series*, Cambridge University Press, Cambridge, 1935.

[2] J. Choi, P. Zörnig, A. K. Rathie, *Sums of certain classes of series*, Comm. Korean Math. Soc. **14**(1999), 641–647.

[3] S. Dahlberg, T. Ferdinands, A. Tefera, *A Wilf-Zeilberger approach to sums of Choi, Zörnig and Rathie*, Quaest. Math. **33**(2010), 341–346.

[4] R. L. Graham, D. E. Knuth, O. Patashnik, *Concrete mathematics*, Addison Wesley, Boston, 1994.

[5] A. Knopfmacher, H. Prodinger, *A simple card guessing game revisited*, Electron. J. Combin. **8**(2001), Article 13, available at `http://www.combinatorics.org/ojs/index.php/eljc/article/view/v8i2r13`.

[6] J. L. Lavoie, F. Grondin, A. K. Rathie, *Generalizations of Whipple's theorem on the sum of a $_3F_2$*, J. Comput. Appl. Math. **72**(1996), 293–300.

[7] H. Prodinger, *Sums of Choi, Zörnig and Rathie-an elementary approach*, Quaest. Math. **32**(2009), 265–267.

[8] H. M. Srivastava, *Sums of a certain family of series*, Elem. Math. **43**(1988), 54–58.

[9] M. Vowe, H. J. Seiffert, *Aufgabe 946*, Elem. Math. **42**(1987), 111–112.