MATHEMATICAL COMMUNICATIONS
Math. Commun. **21**(2016), 239–249

Linear codes with complementary duals from some strongly regular subgraphs of the McLaughlin graph

Dimitri Leemans[1] and Bernardo G. Rodrigues[2,*]

[1] *University of Auckland, Department of Mathematics, Private Bag 92 019, Auckland, New Zealand*
[2] *School of Mathematics, Statistics and Computer Science, University of KwaZulu-Natal, Durban 4 041, South Africa*

Received October 27, 2015; accepted March 22, 2016

Abstract. We examine a number of properties of the ternary linear codes defined by the adjacency matrices of some strongly regular graphs that occur as induced subgraphs of the McLaughlin graph, namely the graphs with parameters $(105, 72, 51, 45), (120, 77, 52, 44)$, $(176, 105, 68, 54)$, and $(253, 140, 87, 65)$, respectively. We show that the codes with parameters $[120, 21, 30]_3$,$[120, 99, 6]_3$, $[176, 21, 56]_3$, $[176, 155, 6]_3$, $[253, 22, 97]_3$ and $[253, 231, 8]_3$ obtained from these graphs are linear codes with complementary duals and thus meet the asymptotic Gilbert-Varshamov bound.

AMS subject classifications: 05B05, 20D45, 94B05

Key words: linear codes, strongly regular graphs, symmetric designs, automorphism groups

1. Introduction

While studying the binary codes defined by the row span of the adjacency matrices of the strongly regular graphs with parameters $(105, 32, 4, 12), (120, 42, 8, 18)$ and $(253, 112, 36, 60)$ it was observed in [17] that the ternary codes of the adjacency matrices of the complements of these graphs possess interesting properties. This paper has a two-fold purpose: Firstly, to examine the linear codes defined by the ternary row span of the adjacency matrices of the strongly regular graphs $\overline{\Lambda}_n$ with $n \in \{105, 120, 176, 253\}$ and parameters $(105, 72, 51, 45), (120, 77, 52, 44)$,$(176, 105, 68, 54)$, and $(253, 140, 87, 65)$. It turns out that some codes from these graphs, in particular those with parameters $[120, 21, 30]_3, [120, 99, 6]_3, [176, 21, 56]_3, [176, 155, 6]_3$ and $[253, 22, 97], [253, 231, 8]_3$ respectively, belong to a class of optimal codes known as linear codes with complementary duals, see [18]. As described by Sendrier in [19], these codes meet the asymptotic Gilbert-Varshamov bound.

Secondly, it was shown in [9] (see also [12]) that for $n \equiv 1 \pmod 4$, the binary code of the complementary graph $\overline{T(n)}$ of the triangular graph $T(n)$ equals the dual code of $T(n)$, and so the question was raised as to whether there exists any other graph for which the code obtained from the graph equals the dual code of

*Corresponding author. *Email addresses:* `d.leemans@auckland.ac.nz` (D. Leemans), `rodrigues@ukzn.ac.za` (B. G. Rodrigues)

its complementary graph; and if any such graph exists, what its defining properties are. We answer this question in the affirmative by showing that the ternary codes of the strongly regular graphs with parameters $(120, 77, 52, 44)$ and $(176, 105, 68, 54)$, respectively, satisfy this property.

Hence, we deduce the following main result summarized in Theorem 1. In the theorem, we collect the parameters and some properties of the codes defined by the ternary row span of the graphs $\overline{\Lambda}_n$, where $n \in \{105, 120, 176, 253\}$ and $\overline{\Lambda}_n$ denotes the complementary graph of an induced subgraph of the McLaughlin graph on n vertices.

Theorem 1. *Let* Γ *respectively* $\overline{\Lambda}_n$ *denote the McLaughlin graph and a complementary graph of a strongly regular graph on* n *vertices with* $n \in \{105, 120, 176, 253\}$, *which occurs as an induced subgraph of* Γ. *Further, let* $C_{\overline{\Lambda}_n}$ *denote the code defined by the ternary row span of the adjacency matrix of* $\overline{\Lambda}_n$. *Then*

(*a*) $C_{\overline{\Lambda}_{105}}$ *is a* $[105, 20, 33]_3$ *self-orthogonal code, and* $\mathrm{Aut}(\overline{\Lambda}_{105}) = \mathrm{Aut}(C_{\overline{\Lambda}_{105}}) \cong L_3(4){:}D_{12}$.

Furthermore, $C_{\overline{\Lambda}_{105}} = \langle C, \mathbf{1} \rangle = C \oplus \mathbf{1}$, *where* $C = [105, 19, 36]_3$ *is a self-orthogonal subcode of co-dimension* 1.

(*b*) $C_{\overline{\Lambda}_{120}} = [120, 21, 30]_3 = C^{\perp}_{\Lambda_{120}}$ *and* ${C_{\overline{\Lambda}_{120}}}^{\perp} = [120, 99, 6]_3 = C_{\Lambda_{120}}$ *are linear codes with complementary duals. Moreover,* $\mathrm{Soc}({\mathbb{F}_3}^{120}) = N$, *where* $N = \langle K, \mathbf{1} \rangle = K \oplus \mathbf{1}$ *and* K *is one of three non-isomorphic 15-dimensional irreducible* $\mathbb{F}_3$*-modules invariant under* $L_3(4)$. *The codes* N *and* K *are self-orthogonal with parameters* $[120, 16, 48]_3$ *and* $[120, 15, 48]_3$, *respectively. Moreover,* $N^{\perp} = [120, 104, 4]_3$ *and* $K^{\perp} = [120, 105, 4]_3$, *and* $\mathrm{Aut}(N) = \mathrm{Aut}(K) \cong L_3(4){:}2^2$.

(*c*) $C_{\overline{\Lambda}_{176}} = [176, 21, 56]_3 = C^{\perp}_{\Lambda_{176}}$ *is a linear code with complementary dual, and* ${C_{\overline{\Lambda}_{176}}}^{\perp} = [176, 155, 6]_3 = C_{\Lambda_{176}}$. *Moreover,* $\mathrm{Aut}(C_{\overline{\Lambda}_{176}}) = \mathrm{Aut}(\overline{\Lambda}_{176}) \cong \mathrm{M}_{22}$, *and* $C_{\overline{\Lambda}_{176}}$ *is the unique 21-dimensional irreducible* $\mathbb{F}_3$*-module invariant under* M_{22}.

(*d*) $C_{\overline{\Lambda}_{253}}$ *is a* $[253, 23, 77]_3$ *code and its dual* $C^{\perp}_{\overline{\Lambda}_{253}}$ *is a* $[253, 230, 6]_3$ *code, and* $\mathrm{Aut}(\overline{\Lambda}_{253}) = \mathrm{Aut}(C_{\overline{\Lambda}_{253}}) \cong M_{23}$. *Further,* $C_{\overline{\Lambda}_{253}} = \langle \tilde{C}, \mathbf{1} \rangle = \tilde{C} \oplus \mathbf{1}$, *where* $\tilde{C} = [253, 22, 97]_3$ *and* M_{23} *acts irreducibly on* $\tilde{C}$ *as an* $\mathbb{F}_3$*-module. Moreover,* $\tilde{C}$ *and* $\tilde{C}^{\perp}$ *are linear codes with complementary duals.*

The proof of Theorem 1 follows from a series of lemmas and propositions in Sections 5 and 6. The paper is organized as follows: after a brief description of our terminology and some background in Sections 2 and 3, Section 4 outlines the construction of the graphs and in Sections 5 and 6 we present our results.

2. Terminology

We assume that the reader is familiar with some basic notions and elementary facts from design and coding theory. Our notation for designs and codes follows that of

[1] and for groups we follow the Atlas [5]. The groups $G.H$, $G:H$, and $G^{\cdot}H$ denote a general extension, a split extension and a non-split extension, respectively. For G a finite group acting on a finite set Ω, the set $\mathbb{F}_p\Omega$, that is, the vector space over $\mathbb{F}_p$ with basis Ω is called an $\mathbb{F}_pG$ *permutation module* if the action of G is extended linearly on Ω. The socle of a module M denoted as $\mathrm{Soc}(M)$ is the sum of minimal submodules of M. Terminology for *graphs* is standard: the *graphs* $\Gamma = (V, E)$ with vertex set V and edge set E, discussed here are undirected and simple, that is, with no loops or multiple edges. A graph is *regular* if all its vertices have the same valency. An *adjacency matrix* A of a graph of order $n := |V|$ is an $n \times n$ matrix with entries a_{ij} such that $a_{ij} = 1$ if vertices v_i and v_j are adjacent, and $a_{ij} = 0$ otherwise. A regular graph is *strongly regular* of type (n, k, λ, μ) if it has n vertices, each of degree k, and if any two adjacent vertices are together adjacent to λ vertices, while any two non-adjacent vertices are together adjacent to μ vertices. If x is a vertex of Γ, then the *neighbourhood graph* $\Gamma(x)$ with respect to x is the subgraph of Γ which is induced by all vertices that are adjacent to x. The neighbourhood graph of a vertex x of a strongly regular graph Γ is also called the *first subconstituent* of Γ. The subgraph of Γ induced on all vertices of Γ which are not adjacent to (and different from) x, is called a *second subconstituent*. The *neighbourhood design* of a regular graph is the 1-design formed by taking the points to be the vertices of the graph and the blocks to be the sets of neighbours of a vertex, for each vertex. The complementary graph of a strongly regular graph with parameters (n, k, λ, μ) is a strongly regular graph with parameters $(n, n-k-1, n-2k+\mu-2, n-2k+\lambda)$.

The codes here are linear codes, and the notation $[n, k, d]_q$ will be used for a q-ary code C of length n, dimension k, and minimum weight d, where the weight, $\mathrm{wt}(v)$, of a vector v is the number of non-zero coordinate entries. A generator matrix for C is a $k \times n$ matrix made up of a basis for C, and the *dual* code $C^{\perp}$ is the orthogonal complement under the standard inner product $(\cdot, \cdot)$, i.e., $C^{\perp} = \{v \in F^n \,|\, (v, c) = 0 \text{ for all } c \in C\}$. The *hull* of C is $\mathrm{Hull}(C) = C \cap C^{\perp}$. A code C is *self-orthogonal* if $C \subseteq C^{\perp}$. The all-one vector will be denoted by $\mathbf{1}$, and it is the constant vector of weight the length of the code whose coordinate entries consist entirely of 1's. An $[n, k]$ linear code C is said to be a *best known linear* $[n, k]$ *code* if C has the highest minimum weight among all known $[n, k]$ linear codes. An $[n, k]$ linear code C is said to be an *optimal linear* $[n, k]$ code if the minimum weight of C achieves the theoretical upper bound on the minimum weight of $[n, k]$ linear codes, and *near-optimal* if its minimum distance is at most 1 less than the largest possible value. The weight enumerator of C is defined as $W_C(x) = \sum_{i=0}^{n} A_i x^i$, where A_i denotes the number of codewords of weight i in C. If C_1 is an $[n_1, k_1]$-code, and C_2 is an $[n_2, k_2]$-code, then we say that C is the *direct sum* of C_1 and C_2 if (up to reordering of coordinates) $C = \{(x, y) \,|\, x \in C_1, y \in C_2\}$. We denote this by $C = C_1 \oplus C_2$. If moreover C_1 and C_2 are nonzero, then we say that C decomposes into C_1 and C_2. A code C is said to be *decomposable* if and only if it is equivalent to a code which is the direct sum of two or more non-zero linear codes. Otherwise it is called indecomposable. An *automorphism* of a code is any permutation of the coordinate positions that maps codewords to codewords. The *code* of a graph Γ over a finite field F is the row span of an adjacency matrix A over the field F, denoted by $C_F(\Gamma)$ or $C_F(A)$. The dimension of the code is the rank of the matrix over F, also written

$\mathrm{rank}_p(A)$ if $F = \mathbb{F}_p$, in which case we speak of the *p-rank* of A or Γ, and write $C_p(A)$ or $C_p(\Gamma)$ for the code. Throughout the paper we adopt the notation C_Γ for the code of the graph.

3. Preliminary results and construction

We obtain our codes through the use of the following method of construction of symmetric 1-designs and regular graphs which was described in [14, Proposition 1], corrected in [15] and used in [16]:

Result 1. *Let G be a finite primitive permutation group acting on the set Ω of size n. Further, let $\alpha \in \Omega$, and let $\Delta \neq \{\alpha\}$ be an orbit of the stabilizer G_α of α. If*

$$\mathcal{B} = \{\Delta g : g \in G\}$$

and, given $\delta \in \Delta$,

$$\mathcal{E} = \{\{\alpha, \delta\}g : g \in G\},$$

then $\mathcal{D} = (\Omega, \mathcal{B})$ is a symmetric $1-(n, |\Delta|, |\Delta|)$ design. Further, if Δ is a self-paired orbit of G_α, then $\Gamma(\Omega, \mathcal{E})$ is a regular connected graph of valency $|\Delta|$, $\mathcal{D}$ is self-dual, and G acts as an automorphism group on each of these structures, primitive on vertices of the graph, and on points and blocks of the design.

4. The graphs

Result 1 outlines a construction of codes using the row span of the adjacency matrices of regular graphs. As an extension of the study in [17], this paper addresses the two questions posed in Section 1 for the ternary codes obtained from the row span of the adjacency matrices of the strongly regular graphs with parameters $(105, 72, 51, 45)$, $(120, 77, 52, 44)$, $(176, 105, 68, 54)$ and $(253, 140, 87, 65)$, respectively. We now give a brief description of the construction of these graphs, omitting detail. Constructions of these graphs can be found for example in [6, 3]. The uniqueness of a graph with parameters $(105, 72, 51, 45)$ follows from that of its complement which was shown in [6, Theorem 3], (see also [7, Theorem 1]). A graph with these parameters can be constructed from the second subconstituent of the McLaughlin graph, that is, the unique strongly regular graph with parameters $(162, 105, 72, 60)$, as follows: the 162 vertices form a single orbit. The vertex stabilizer in that graph is $L_3(4){:}2^2$ with vertex orbit sizes $1 + 56 + 105$. The orbit of size 105 corresponds to the flags (i.e., the pairs (p, L), where p is a point of $PG_2(4)$, L is a line and p is a point of L) of $PG_2(4)$, where two flags (x, R) and (y, S) are adjacent when x, y are equal or R, S are equal, and x is on S or y on R. The induced subgraph is strongly regular with parameters $(105, 72, 51, 45)$ and spectrum $\{[72]^1, [9]^{20}, [-3]^{84}\}$.

Constructions of strongly regular graphs with parameters $(120, 77, 52, 44)$, $(176, 105, 68, 54)$ and $(253, 140, 87, 65)$, respectively, can be found for example in [3]. For the sake of completeness we give a brief outline of the mentioned constructions as described in [3, pp. 343]. Take $X = \{\alpha_1, \alpha_2, \ldots, \alpha_{23}\}$ to be a set of size 23, and let $\mathcal{D} = (X, \Delta)$ be a Steiner system $S(4, 7, 23)$ on X. Define a graph Λ

on Δ such that two vertices are adjacent whenever the cardinality of their intersection equals 1, and for $j = 0, 1, 2$, let $\mathcal{D}^j = (X^{(j)}, \Delta^{(j)})$ be the design formed from $\mathcal{D}$ by deleting the symbols $\alpha_1, \dots, \alpha_j$ and all blocks containing at least one of those symbols. Define $\Lambda^{(j)}$ as the subgraph of Λ induced by $\Delta^{(j)}$. Then $\Lambda^{(0)}, \Lambda^{(1)}$, and $\Lambda^{(2)}$ are strongly regular with parameters $(253, 112, 36, 60)$, $(176, 70, 18, 34)$ and $(120, 42, 8, 18)$, respectively, and spectra $\{[112]^1[2]^{230}[-26]^{22}\}$, $\{[70]^1[2]^{154}[-18]^{21}\}$ and $\{[42]^1[2]^{99}[-12]^{20}\}$. Form this construction we obtain the respective complementary graphs with parameters $(253, 140, 87, 65)$, $(176, 105, 68, 54)$ and $(120, 77, 52, 44)$ and spectra $\{[140]^1[25]^{22}[-3]^{230}\}$, $\{[105]^1[17]^{21}[-3]^{154}\}$ and $\{[77]^1[11]^{20}[-3]^{99}\}$. We make extensive use of the spectra of the graphs to study some pertinent properties of the ternary codes discussed in the sequel.

5. The ternary codes of the graphs

We denote the complementary graphs discussed above by $\overline{\Lambda}_{105}$, $\overline{\Lambda}_{120}$, $\overline{\Lambda}_{176}$ and $\overline{\Lambda}_{253}$, respectively, and their corresponding ternary codes by $C_{\overline{\Lambda}_{105}}$, $C_{\overline{\Lambda}_{120}}$, $C_{\overline{\Lambda}_{176}}$ and $C_{\overline{\Lambda}_{253}}$. Using only the parameters of the graphs we can show that the p-ary codes of these graphs, for $p \neq 3$, are either the ambient space or a code of codimension 1 in the ambient space. The only exception is the case $p = 5$ for $\overline{\Lambda}_{253}$ where the 5-rank equals 230, which is the multiplicity of the negative eigenvalue. The lemma that follows gives the p-ranks for the codes defined by the row span of the adjacency matrix of the graph $\overline{\Lambda}_{105}$. Similar results can be obtained for the codes of the other graphs examined in the paper.

5.1. Ternary codes of $\overline{\Lambda}_{105}$

Lemma 1. *The adjacency matrix $\overline{A}$ of the graph $\overline{\Lambda}_{105}$ has 2-rank 104, 3-rank 20, and p-rank 105 for $p \neq 2, 3$.*

Proof. It is easy to see that if $\det(\overline{A}) \equiv 0 \pmod p$, then $\overline{A}$ does not have full rank over $\mathbb{F}_p$. This is the case for $p = 2$ or $p = 3$. Now, the 2-rank of $\overline{A}$ equals 104 since $\det(J - \overline{A}) \not\equiv 0 \pmod 2$, where J is the all-one matrix.

The 3-rank of $\overline{\Lambda}_{105}$ and thus the dimension of $C_{\overline{\Lambda}_{105}}$ can be deduced readily by using the spectrum of the graph. It follows from Section 4 that the eigenvalues of an adjacency matrix $\overline{A}$ of $\overline{\Lambda}_{105}$ are $\theta_0 = 72, \theta_1 = 9$, and $\theta_2 = -3$ with corresponding multiplicities $f_0 = 1, f_1 = 20$ and $f_2 = 84$. Now, using results of [3, Section 3] we obtain an upper bound on the 3-rank of $\overline{\Lambda}_{105}$, namely that $\operatorname{rank}_3(\overline{\Lambda}_{105}) \leq \min(f_1 + 1, f_2 + 1) = 21$.

In addition, since $\overline{\Lambda}_{105}$ is an induced subgraph of the unique strongly regular $(162, 105, 72, 60)$ graph and the 3-rank of an adjacency matrix for this graph is 21 (see [3] or [8, Proposition 5.13]) we conclude that $\operatorname{rank}_3(A) \neq 21$. Furthermore, from [3, Section 3] and by using minimal idempotents we deduce that $\operatorname{rank}_3(\overline{\Lambda}_{105}) = f_1 = 20$ since $\theta_0 - \theta_2$ and $n = 105$ are both divisible by 3 and $\theta_0 - \theta_2/n$ can be interpreted in $\mathbb{F}_3$. $\square$

Proposition 1. *$C_{\overline{\Lambda}_{105}}$ is a $[105,20,33]_3$ self-orthogonal code, and its dual ${C_{\overline{\Lambda}_{105}}}^{\perp}$ is a $[105,85,6]_3$ code with 15680 words of weight 6. Moreover,* $\mathrm{Aut}(\overline{\Lambda}_{105}) = \mathrm{Aut}(C_{\overline{\Lambda}_{105}}) \cong L_3(4){:}D_{12}$ *and $C_{\overline{\Lambda}_{105}}$ is a decomposable $\mathbb{F}_3$-module invariant under $L_3(4){:}D_{12}$.*

Proof. Since in $\overline{\Lambda}_{105}$ we have $k = 72, \lambda = 51$ and $\mu = 45$, and these three numbers are divisible by 3, self-orthogonality of $C_{\overline{\Lambda}_{105}}$ follows. Now, since the adjacency matrix of $\overline{\Lambda}_{105}$ can be regarded as the incidence matrix of the 1-$(105,72,72)$ self-dual symmetric design, then from $72 \equiv 0 \pmod 3$ we deduce that $\mathbf{1} \in {C_{\overline{\Lambda}_{105}}}^{\perp}$. Through computations with Magma [2] we obtain that the sum (modulo 3) of all rows of the generator matrix $\mathcal{G}$ of $C_{\overline{\Lambda}_{105}}$ is the all-ones vector, so $\mathbf{1} \in C_{\overline{\Lambda}_{105}}$. The dimension of $C_{\overline{\Lambda}_{105}}$ follows from Lemma 1, and the minimum distance 33 can be deduced from the weight distribution of $C_{\overline{\Lambda}_{105}}$ which is given in Table 1. In this table, m represents the weight of a codeword w_m in $C_{\overline{\Lambda}_{105}}$ and A_m denotes the number of codewords of weight m.

m	A_m	m	A_m	m	A_m
0	1	57	21993720	81	58508240
33	770	60	108522288	84	9106500
36	3360	63	303515520	87	1230880
42	29640	66	606412800	90	127008
45	38304	69	818230000	93	128520
48	141120	72	804875190	96	15680
51	987840	75	526684032	99	3360
54	3649800	78	222579280	105	548

Table 1: *The weight distribution of $C_{\overline{\Lambda}_{105}}$*

From [20] we have $\mathrm{Aut}(\overline{\Lambda}_{105}) = L_3(4){:}D_{12}$ acting transitively as a rank-4 group. Moreover, by construction we have $\mathrm{Aut}(\overline{\Lambda}_{105}) \subseteq \mathrm{Aut}(C_{\overline{\Lambda}_{105}})$; and since $|\mathrm{Aut}(\overline{\Lambda}_{105})| = |\mathrm{Aut}(C_{\overline{\Lambda}_{105}})|$, we deduce that $\mathrm{Aut}(C_{\overline{\Lambda}_{105}}) = \mathrm{Aut}(\overline{\Lambda}_{105}) \cong L_3(4){:}D_{12}$. Taking the images of the supports of the codewords of weight 36 in $C_{\overline{\Lambda}_{105}}$ we form a set $W_{36} = \{w \in C_{\overline{\Lambda}_{105}} \mid wt(w) = 36\}$. We can show that W_{36} spans a subcode C of codimension 1 in $C_{\overline{\Lambda}_{105}}$. Now, let $\rho \in \mathrm{Aut}(C)$. Since $\rho(\mathbf{1}) = \mathbf{1}$ and $C_{\overline{\Lambda}_{105}} = \langle C, \mathbf{1}\rangle$, we have $\rho \in \mathrm{Aut}(C_{\overline{\Lambda}_{105}})$ so that $\mathrm{Aut}(C) \subseteq \mathrm{Aut}(C_{\overline{\Lambda}_{105}})$. From $L_3(4){:}D_{12} = \mathrm{Aut}(\overline{\Lambda}_{105}) \leq \mathrm{Aut}(C) \leq \mathrm{Aut}(C_{\overline{\Lambda}_{105}}) = L_3(4){:}D_{12}$, we obtain that $\mathrm{Aut}(C) = L_3(4){:}D_{12}$. Since $C_{\overline{\Lambda}_{105}} = \langle C, \mathbf{1}\rangle = C \oplus \mathbf{1}$ and C and $\mathbf{1}$ are irreducible $\mathbb{F}_3$-modules, we deduce that $C_{\overline{\Lambda}_{105}}$ is a decomposable $\mathbb{F}_3$-module under the action of $L_3(4){:}D_{12}$.

That ${C_{\overline{\Lambda}_{105}}}^{\perp}$ has minimum weight 6 was found using Magma [2]. Finally, from [10, 11] we obtain that ${C_{\overline{\Lambda}_{105}}}^{\perp}$ is a distance 2 less than the recorded distance. □

We now have the following immediate consequence of Proposition 1.

Corollary 1. *The codewords of weight 36 in $C_{\overline{\Lambda}_{105}}$ span a subcode C of codimension 1. C is a $[105,19,36]_3$ self-orthogonal code. The dual code $C^{\perp}$ of C is a* $[105,86,5]_3$

with 84 *codewords of weight* 5. *Further, C is a non-trivial $\mathbb{F}_3$-module of the smallest dimension on which $L_3(4){:}D_{12}$ acts irreducibly.*

Proof. The dimension and self-orthogonality of C follow readily from Proposition 1. The assertion on the minimum distance of C follows from the weight distribution of $C_{\overline{\Lambda}_{105}}$. In fact, the weight distribution for C is given in Table 2 below.

m	A_m	m	A_m	m	A_m
0	1	57	6552000	78	74177040
36	3360	60	36809136	81	20054720
42	9480	63	101499720	84	2743020
45	336	66	202063680	87	544320
48	100800	69	269951640	90	54432
51	144480	72	272891220	96	15120
54	1659840	75	172986912	105	210

Table 2: *The weight distribution of C*

Using Table 2 we readily see that C does not contain an invariant subspace of dimension 1. Suppose for a contradiction that C contains an invariant subspace U of dimension 18. Then $C = U \oplus \overline{U}$, where $\overline{U} \cap U = \{0\}$ and $\dim(\overline{U}) = 1$, and $\overline{U}$ is invariant, but this is not possible. Hence, 19 is the smallest possible dimension for any non-trivial $\mathbb{F}_3$-module on which $L_3(4){:}D_{12}$ acts irreducibly. Finally, $C^{\perp}$ is a distance 2 less than the recorded lower bound on the minimum distance for the given length and dimension (see [10, 11]). $\square$

Remark 1. *The set of codewords of minimum weight* 33 *in $C_{\overline{\Lambda}_{105}}$ splits into four obits of lengths* 105, 105, 280, *and* 280*, respectively, under the action of* $\mathrm{Aut}(C_{\Gamma_{105}})$.

Remark 2. *The code C is the only submodule of its dimension in the representation of degree* 105. *In this representation we obtained three reducible and non-isomorphic* 20*-dimensional modules, which we denote by C_1, C_2, and C_3, respectively. We found that C_3 is isomorphic to $C_{\overline{\Lambda}_{105}}$. The code C_1 is a* $[105, 20, 30]_3$ *code with* 672 *codewords of weight* 30 *and C_2 is a* $[105, 20, 28]_3$ *with* 720 *codewords of weight* 28.

6. Linear codes with complementary dual

Massey [18] defined a linear code with complementary dual as a linear code C whose dual $C^{\perp}$ satisfies $C \cap C^{\perp} = \{0\}$, and gave the algebraic characterization of these codes. He showed further that linear codes with complementary dual are asymptotically good codes, but stopping short of showing whether these codes attain the Gilbert-Varshamov bound. A while later, Sendrier [19] showed that linear codes with complementary dual meet the asymptotic Gilbert-Varshamov bound. In the sections which follow we use results of [19] to show that the codes with parameters $[120, 21, 30]_3$, $[120, 99, 6]_3$, $[176, 21, 56]_3$, $[176, 155, 6]_3$ and $[253, 22, 97]$, $[253, 231, 8]_3$

are linear codes with complementary duals and thus meet the Gilbert-Varshamov bound. In addition, this section addresses the second main aim of the paper which is to deal with the question on the existence of graphs other than the triangular graphs $T(n)$ for which the code obtained from the graph equals the dual code of its complementary graph. In particular, if any such graph exists, what are its defining properties? We now give an affirmative answer to the existence part of this question.

While exploring the extent of the question on the defining properties of such graphs, we have found that a more appropriate question is: Given a graph Γ and its complement $\overline{\Gamma}$, let C_Γ and $C_{\overline{\Gamma}}^\perp$ be the code of Γ and the dual code of $\overline{\Gamma}$, respectively. If $C_\Gamma = C_{\overline{\Gamma}}^\perp$, does it follow necessarily that $C_\Gamma \cap {C_\Gamma}^\perp = \{0\}$, and conversely?

We start by examining the codes of the graph $\overline{\Lambda}_{120}$.

6.1. The graph $\overline{\Lambda}_{120}$ and related codes

Proposition 2. *Let $C_{\overline{\Lambda}_{120}}$ and $C_{\Lambda 120}$ denote the codes of the graphs $\overline{\Lambda}_{120}$ and Λ_{120}, respectively. Then $C_{\overline{\Lambda}_{120}} = [120, 21, 30]_3 = C_{\Lambda_{120}}^\perp$ is a linear code with complementary dual and* 112 *codewords of weight* 30. *The code ${C_{\overline{\Lambda}_{120}}}^\perp = [120, 99, 6]_3 = C_{\Lambda_{120}}$ is a code with* 33600 *words of weight* 6. *Moreover, $\mathbf{1} \in C_{\overline{\Lambda}_{120}}$ and $\mathrm{Aut}(C_{\overline{\Lambda}_{120}}) = \mathrm{Aut}(\overline{\Lambda}_{120}) \cong \mathrm{L}_3(4){:}2^2$.*

Proof. Arguing as in Lemma 1, we can show that $\mathrm{rk}_3(\overline{\Lambda}_{120}) = 21$. Recall that the eigenvalues of an adjacency matrix $\overline{A}$ of $\overline{\Lambda}_{120}$ are $\theta_0 = 77, \theta_1 = 11$, and $\theta_2 = -3$ with algebraic multiplicities $m_0 = 1, m_1 = 20$ and $m_2 = 99$, respectively. Observe that $\mathrm{rank}_3(\overline{A}) \geq \sum\{m_i \,|\, \theta_i \not\equiv 0 \pmod 3\} = 1{+}20$. In addition, observe that precisely one eigenvalue, namely $\theta_2 \equiv 0 \pmod 3$. So, we deduce from [4, Proposition 13.7.1(ii)] that $\mathrm{rk}_3(\overline{\Lambda}_{120}) = n - m_2 = 120 - 99 = 21$, and the claim holds. Since $3 \,|\, 120$ and $3 \nmid 77$, we have that all the row sums are non-zero modulo 3, and so $\mathbf{1} \in C_{\overline{\Lambda}_{120}}$ (see [3, Section 3]). Furthermore, since the dimension of the hull is zero, we have $\mathrm{Hull}(C_{\overline{\Lambda}_{120}}) = \{0\}$, and from this we infer $C_{\overline{\Lambda}_{120}} \oplus {C_{\overline{\Lambda}_{120}}}^\perp = \mathbb{F}_3^{120}$ as claimed. Now, [12, Proposition 3.2 (iv)] adapted to the ternary case teaches us that $C_{\overline{\Lambda}_{120}} = C_{\Lambda_{120}}^\perp$ and ${C_{\overline{\Lambda}_{120}}}^\perp = C_{\Lambda_{120}}$, since $\theta_1 \not\equiv \theta_2 \pmod 3$ and θ_0 is odd. In fact, in this case we also have $A \cdot \overline{A} \equiv 0 \pmod 3$, where A is the adjacency matrix of Λ_{120}. Further, we have that $C_{\overline{\Lambda}_{120}}$ contains a subcode S of codimension 1 with parameters $[120, 20, 30]_3$. The dual $S^\perp$ of S has parameters $[120, 100, 6]_3$ and 43680 words of weight 6, and it can be shown that $S^\perp = C_{\overline{\Lambda}_{120}+I}$. Further, we have $\mathrm{Aut}(S) \cong L_3(4) : 2^2$ and $\mathrm{Hull}(S) = \langle \mathbf{1} \rangle$. Finally, the reader can verify from [11] that the minimum distance 6 of $C_{\Gamma_{120}}^\perp$ is 2 less than the theoretical upper bound on the minimum distance for a $[120, 99]_3$ code. $\square$

Remark 3. *It is known that given a finite group acting primitively on a set Ω of degree n, using* Meat-Axe *one is able to determine all irreducible faithful representations invariant under the group. However, no general method is known to appropriately locate the degree of the representations which contain such irreducible representations. This seems to be an intricate problem in computational modular representation theory. This is more apparent when n is very large. We will illustrate this using the group $L_3(4)$. The reader will notice for example from [13] that*

$L_3(4)$ has three non-isomorphic faithfully irreducible 15-dimensional representations over $\mathbb{F}_3$. There is however no immediate technique available to ascertain the primitive representations of $L_3(4)$ that contain such irreducible modules. In what follows, and using a search through the primitive representations of $L_3(4)$ we are able to find that one of these irreducible 15-dimensional modules occurs as a submodule of a decomposable 16-dimensional module realizable as the socle of the 120-dimensional permutation module over $\mathbb{F}_3$ defined by the action of $L_3(4)$ on the cosets of $L_3(2)$. In this way, we locate the degree of the representation which contains one of the faithful irreducible 15-dimensional modules. However, we cannot immediately localize the remaining irreducible modules of this dimension. It is likely that these submodules result from tensoring of modules or that they are found as constituents in other representations of $L_3(4)$.

Hence, viewing the modules as codes invariant under $L_3(4)$ we show in Proposition 3 that these 15 and 16-dimensional codes are self-orthogonal.

Proposition 3. *Let M be the 120-dimensional permutation module over $\mathbb{F}_3$ induced by the primitive action of $L_3(4)$ on the cosets of $L_3(2)$. Then $Soc(M) = N$ where N is a 16-dimensional decomposable module invariant under $L_3(4)$. Moreover, $N = \langle K, \mathbf{1}\rangle = K \oplus \mathbf{1}$ and K is one of three non-isomorphic 15-dimensional irreducible $\mathbb{F}_3$-modules invariant under $L_3(4)$. The submodules N and K are respectively $[120, 16, 48]_3$ and $[120, 15, 48]_3$ self-orthogonal codes with 1260 codewords of weight 48. Further, $N^\perp = [120, 104, 4]_3$ and $K^\perp = [120, 105, 4]_3$ and $\mathrm{Aut}(N) \cong \mathrm{Aut}(K) = L_3(4){:}2^2$.*

Proof. The proof is essentially based on a combination of constructive arguments using MEAT-AXE and those given in Proposition 1 and Corollary 1, thus we omit them here. We refer the reader to [13] for the irreducibility of K and the existence of two non-isomorphic other 15-dimensional modules invariant under $L_3(4)$. Moreover, from [11] we deduce that $K^\perp$ and $N^\perp$ are near-optimal codes (both codes are a distance 1 less than optimal). $\square$

6.2. The ternary code of the graph $\overline{\Lambda}_{176}$

Proposition 4. *$C_{\overline{\Lambda}_{176}} = [176, 21, 56]_3 = C^\perp_{\Lambda_{176}}$ is a code with 924 codewords of weight 56, and ${C_{\overline{\Lambda}_{176}}}^\perp = [176, 155, 6]_3 = C_{\Lambda_{176}}$ is a code with 184800 words of weight 6. Moreover, $\mathbf{1} \in {C_{\overline{\Lambda}_{176}}}^\perp$, $\mathrm{Aut}(C_{\overline{\Lambda}_{176}}) = \mathrm{Aut}(\overline{\Lambda}_{176}) \cong \mathrm{M}_{22}$, and $C_{\overline{\Lambda}_{176}} \oplus {C_{\overline{\Lambda}_{176}}}^\perp = \mathbb{F}_3^{176}$. Further, $C_{\overline{\Lambda}_{176}}$ is the unique $\mathbb{F}_3$-module on which M_{22} acts irreducibly.*

Proof. By the divisibility of the valency of $\overline{\Lambda}_{176}$ we obtain $\mathbf{1} \in {C_{\overline{\Lambda}_{176}}}^\perp$. Notice that the 3-modular character table of M_{22} is completely known (see [13]). We can deduce from it that the irreducible 21-dimensional $\mathbb{F}_3$-representation is unique. It also follows from it that 21 is the smallest possible dimension for any non-trivial irreducible $\mathbb{F}_3$-module invariant under M_{22}. Further, ${C_{\overline{\Lambda}_{176}}}^\perp$ is a distance 2 less than the theoretical lower bound for a $[176, 155]_3$ code. Arguing as in Proposition 2, we can show that $C_{\overline{\Lambda}_{176}} = C^\perp_{\Lambda_{176}}$ and ${C_{\overline{\Lambda}_{176}}}^\perp = C_{\Lambda_{176}}$ since for $\overline{\Lambda}_{176}$ the eigenvalues

$\theta_1 = 17$ and $\theta_2 = -3$ are such that $\theta_1 \not\equiv \theta_2 \pmod 3$ and θ_0 is odd. This fact can also be verified with MAGMA [2]. □

6.3. Ternary codes from $\overline{\Lambda}_{253}$

Recall that the uniqueness of a graph with parameters $(253, 140, 87, 65)$ remains unknown. In Proposition 5 below, we summarize the properties of the codes related to the graph $\overline{\Lambda}_{253}$.

Lemma 2. *The adjacency matrix* $\overline{A}$ *of the graph* $\overline{\Lambda}_{253}$ *has* 2 *and* 7*-rank* 252*,* 3*-rank* 23*,* 5*-rank* 230 *and* p*-rank* 253 *for* $p = 11, 23$*.*

Proof. The arguments are similar to those used in the proof of Lemma 1 or Proposition 2, so we leave them to the reader. □

Proposition 5. *The code* $C_{\overline{\Lambda}_{253}}$ *is a* $[253, 23, 77]_3$ *code with* 46 *words of weight* 77*. Its dual* $C_{\overline{\Lambda}_{253}}^{\perp} = [253, 230, 6]_3$ *contains* 850080 *words of minimum weight* 6 *and* $\mathbf{1} \in C_{\overline{\Lambda}_{253}}$*. The code* $C_{\overline{\Lambda}_{253}}$ *contains a subcode* $\tilde{C}$ *of codimension* 1*, and* $\tilde{C}$ *and* $\tilde{C}^{\perp}$ *have parameters* $[253, 22, 97]_3$ *and* $[253, 231, 8]_3$*. Moreover,* $C_{\overline{\Lambda}_{253}} \oplus C_{\overline{\Lambda}_{253}}{}^{\perp} = \langle \mathbf{1} \rangle \oplus \tilde{C} \oplus C_{\overline{\Lambda}_{253}}^{\perp} = \mathbb{F}_3^{253}$ *and* $\tilde{C} \oplus \tilde{C}^{\perp} = \mathbb{F}_3^{253}$*. Further,* $\tilde{C}$ *and* $\tilde{C}^{\perp}$ *are irreducible modules invariant under* M_{23} *and* $\tilde{C}$ *is the unique and smallest irreducible* $\mathbb{F}_3$*-module of* M_{23}*.* $\mathrm{Aut}(\overline{\Lambda}_{253}) = \mathrm{Aut}(C_{\overline{\Lambda}_{253}}) \cong M_{23}$*.*

We omit the details of the proof as these follow virtually similar arguments to those used in the proofs of the earlier propositions.

Acknowledgement

Part of this work was done while visiting the Department of Mathematics at the University of Auckland. B. G. Rodrigues wishes to express his sincere thanks to the Mathematics Department for splendid hospitality. This work is based on the research supported by the National Research Foundation of South Africa (Grant Numbers 87470 and 91495).

References

[1] E. F. ASSMUS, JR., J. D. KEY, *Designs and their codes*, Cambridge University Press, Cambridge, 1992.
[2] W. BOSMA, J. CANNON, C. PLAYOUST, *The Magma algebra system I: The user language*, J. Symbolic Comput. **24**(1997), 235–265.
[3] A. E. BROUWER, C. J. VAN EIJL, *On the p-rank of the adjacency matrices of strongly regular graphs*, J. Algebraic Combin. **1**(1992), 329–346.
[4] A. E. BROUWER, W. H. HAEMERS, *Spectra of graphs*, Springer, New York, 2012.
[5] J. H. CONWAY, R. T. CURTIS, S. P. NORTON, R. A. PARKER, R. A. WILSON, *An atlas of finite groups*, Oxford University Press, Oxford, 1985.
[6] K. COOLSAET, *The uniqueness of the strongly regular graph* srg $(105, 32, 4, 12)$, Bull. Belg. Math. Soc. Simon Stevin **12**(2005), 707–718.

[7] K. Coolsaet, J. Degraer, *Classification of some strongly regular subgraphs of the McLaughlin graph*, Discrete Math. **308**(2008), 395–400.

[8] D. Crnković, V. Mikulić, B. G. Rodrigues, *Some strongly regular graphs and self-orthogonal codes from the unitary group* $U_4(3)$, Glas. Mat. Ser. III **45**(2010), 307–323.

[9] W. Fish, R. Fray, E. Mwambene, *Binary codes from the complements of the triangular graphs*, Quaest. Math. **33**(2010), 399–408.

[10] M. Grassl, *Searching for linear codes with large minimum distance*, in: *Discovering mathematics with magma*, (W. Bosma, J. Cannon, Eds.), Springer, 2006.

[11] M. Grassl, *Bounds on the minimum distance of linear codes and quantum codes*, available at `http://www.codetables.de`.

[12] W. H. Haemers, R. Peeters, J. M. van Rijckevorsel, *Binary codes of strongly regular graphs*, Des. Codes Cryptogr. **17**(1999), 187–209.

[13] C. Jansen, K. Lux, R. Parker, R. Wilson, *An atlas of brauer characters*, Oxford Scientific Publications, Clarendon Press, Oxford, 1995.

[14] J. D. Key, J. Moori, *Designs, codes and graphs from the Janko groups* J_1 *and* J_2, J. Combin. Math. and Combin. Comput. **40**(2002), 143–159.

[15] J. D. Key, J. Moori, *Correction to: "Codes, designs and graphs from the Janko groups* J_1 *and* J_2*" [J. Combin. Math. Combin. Comput.* ***40****(2002), 143–159]*, J. Combin. Math. Combin. Comput. **64**(2008), 153.

[16] J. D. Key, J. Moori, B. G. Rodrigues, *On some designs and codes from primitive representations of some finite simple groups*, J. Combin. Math. and Combin. Comput. **45**(2003), 3–19.

[17] D. Leemans, B. G. Rodrigues, *Binary codes of some strongly regular subgraphs of the McLaughlin graph*, Des. Codes and Cryptogr. **67** (2013), 93–109.

[18] J. Massey, *Linear codes with complementary duals*, Disc. Math. **106/107**(1992), 337–342.

[19] N. Sendrier, *Linear codes with complementary duals meet the Gilbert-Varshamov bound*, Discrete Math. **285**(2004), 345–347.

[20] L. H. Soicher, *Three new distance-regular graphs*, European J. Combin. **14**(1993), 501–505.