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FREE-VIBRATION OF BERNOULLI-EULER BEAM USING THE SPECTRAL ELEMENT METHOD

SLOBODNA VIBRACIJA BERNOULLI-EULEROVE ZRAKE KORIŠTENJEM METODE SPEKTRALNOG ELEMENTA

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Preliminary communication

Abstract: *The present work describes the spectral finite element formulation and the solution of Bernoulli-Euler free vibrations beams. The formulation including the partial differential equations of motion, the spectral displacement field and the dynamic stiffness matrix, is established in the scope of the spectral element method. The development is recognized for the general case without considering boundary conditions. The work describes the solution of the problem using three distinct methods: (1) the finite element method, (2) the analytical method and (3) the spectral element method. Particularly, natural frequencies for clamped-free vibration of Bernoulli-Euler beams are established. In the finite element approach, the element number varied in order to improve the accurate solution. Contrary, the spectral element method requires only one to two elements. Results using this method are compared with the finite element method and analytical procedure ones. The spectral element method shows notable advantages compared to the finite element method reducing the number of elements as well as increasing the accuracy.*

Keywords: *Analytical solution, Bernoulli- Euler beam, finite element method, free vibration, Spectral element method, spectral stiffness matrix.*

Prethodno priopćenje

Sažetak: *Rad opisuje formulaciju konačnog spektralnog elementa i rješenje slobodnih vibracija Bernoulli-Eulerovih zraka. Uključujući i parcijalne diferencijalne jednadžbe gibanja, spektralna polja pomaka i matricu dinamičke krutosti, uspostavljena je formulacija u okviru metode spektralnog elementa. Razvoj se promatra za općenite slučajeve, bez uzimanja rubnih uvjeta u obzir. Rad opisuje rješenje problema pomoću tri različite metode: (1) metoda konačnih elemenata, (2) analitička metoda i (3) metoda spektralnog elementa. Ustanovljene su prirodne slobodne vibracije Bernoulli-Eulerovih zraka. U metodi konačnog elementa, broj elementa mijenjao se s ciljem poboljšanja ispravnog rješenja. Nasuprot tome, metoda spektralnog elementa zahtijeva samo jedna do dva elementa. Rezultati koji koriste ovu metodu uspoređeni su s metodom konačnog elementa i metodom analitičkog pristupa. Metoda spektralnog elementa pokazala je značajnu prednost u odnosu na metodu konačnog elementa u smanjenju broja elemenata i povećanju točnosti.*

Ključne riječi: *analitičko rješenje, Bernoulli-Eulerova traka, metoda konačnog elementa, slobodna vibracija, metoda spektralnog elementa, matrica spektralne krutosti*

1. INTRODUCTION

Beam members are the basic structural components widely used in mechanical, aeronautical, automobile and civil engineering fields. For these reasons, important researchers have made efforts to deal with the static and dynamic analyses. In the Bernoulli-Euler's theory, the shear force is neglected in the formulation of the spectral and finite element methods, indifferently.

In structural mechanics, the finite element method (FEM) becomes a popular tool largely used in many areas of engineering and science. In this scope, the FE technique is chosen to analyze a crane under a moving load [1]. The FEM and the boundary element method (BEM) are combined in order to analyze a 2D domain [2] and the Euler-Bernoulli nano-beam responses are studied by using the FEM [3]. However, the solutions using FEM become inaccurate in the higher frequency range. Vibration shapes of a structure vary with the frequencies. The FEM subdivides the structure into finite elements to accurate solutions and a large number of finite elements should be used to obtain reliable solutions.

Recently, the spectral finite element method (SFEM) based on the fast Fourier transform (FFT) has widely been used in the structural dynamic analysis. SFEM is an efficient tool for the treatment of vibration problems in the higher frequency domain. SFEM leads to the exact solution of partial differential equations in the frequency-domain by using just one to two elements. The dynamic stiffness matrix (DSM) is formulated from the frequency-dependent shape functions that are obtained from the exact solutions of governed partial differential equations. In addition to that, the method using DSM does not require a subdivision of the structure into a larger number of elements. It can be able to predict eigen-solutions by using a minimum number of the degrees of freedom.

In this field, the basic concepts of SFEM are developed in [4]. During the last decades, SFEM is applied to the wave propagation in structures [5]. Doyle [6] and Doyle and Farris later [7] formulated SFEM for elementary isotropic waveguides. Furthermore, SFEM is developed for higher-order waveguides [8]. In general, many researchers applied SFEM to analyze the dynamic problems of beams, plates, trusses and other complex structures [9-11]. In the same context, the spectral element models are applied to the study of free vibrations and wave propagation analysis of uniform and tapered rotating beams [12].

In the present work, the SFEM formulation is developed to analyze the free vibration of the Euler-Bernoulli beams. By using dynamic shape functions and the dynamic stiffness method of beams, the natural frequencies of clamped-free boundary conditions are evaluated. The study can be easily applied to other boundary conditions. The SFEM results compared with the FEM ones show a notable accuracy by using a reduced finite element number. Practically, SFEM leads to the exact solution and needs less computational effort and time computing.

2. MATHEMATICAL FORMULATIONS

This section illustrates the formulation of clamped-free vibration beams with the finite element method, the spectral element method, and finally with the conventional analytical solution.

2.1. Finite element method

A prismatic beam having the length L, a cross section Ω, a moment of inertia I, Young's modulus E and a density ρ, is selected (Figure 1).

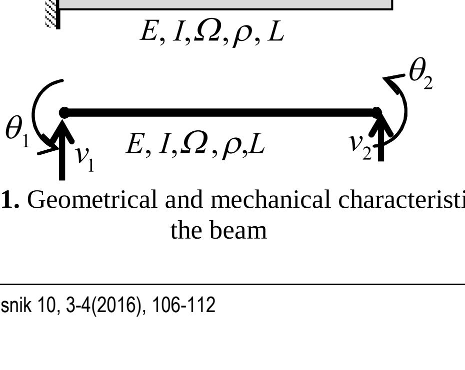

Figure 1. Geometrical and mechanical characteristics of the beam

Each node is defined by two degrees of freedom; the vertical displacement v and the slope $\theta = \dfrac{dv(x)}{dx}$. Therefore, the element shown in Figure (1) using the free-body diagram then has four degrees of freedom. The displacement expression can be expressed by a polynomial function by using four constants.

$$v(x) = a_1 + a_2 x + a_3 x^2 + a_4 x^3 = \langle \phi(x) \rangle \{a\} \quad (1)$$

The slope expression is

$$\theta(x) = \frac{dv(x)}{dx} = \langle \phi'(x) \rangle \{a\} \quad (2)$$

At the level of the element nodes, we can write

$$\{q_e\} = \begin{Bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & L & L^2 & L^3 \\ 0 & 1 & 2L & 3L^2 \end{bmatrix} \{a\} = [A]\{a\} \quad (3)$$

In the compacted form, the relation (3) can be written as

$$\{a\} = [A]^{-1}\{q_e\} \quad (4)$$

The substitution of equation (4) into (1), the displacement field becomes

$$v(x) = \langle \phi(x) \rangle [A]^{-1} \{q_e\} \quad (5)$$

or,

$$v(x) = \langle N(x) \rangle \{q_e\} \quad (6)$$

Where,

$$\langle N(x) \rangle = \langle \phi(x) \rangle [A]^{-1} \quad (7)$$

The shape functions (7) can be derived as

$$N_1(x) = (1 - \frac{3}{L^2}x^2 + \frac{2}{L^3}x^3) \quad (8.1)$$

$$N_2(x) = (x - \frac{2}{L}x^2 + \frac{1}{L^2}x^3) \quad (8.2)$$

$$N_3(x) = (\frac{3}{L^2}x^2 - \frac{2}{L^3}x^3) \quad (8.3)$$

$$N_4(x) = (-\frac{1}{L}x^2 + \frac{1}{L^2}x^3) \quad (8.4)$$

The strain energy of the element can be developed as

$$U_e = \frac{1}{2}\int_L \frac{M^2(x)}{EI}dx = \frac{1}{2}\int_L EI(\frac{d^2v(x)}{dx^2})dx \quad (9)$$

The substitution of the displacement expression (6) in the relationship (9) gives

$$U_e = \frac{1}{2}\langle q_e \rangle EI \int_0^L \left\{\frac{d^2N_i}{dx^2}\right\}\left\langle \frac{d^2N_i}{dx^2}\right\rangle dx \{q_e\} \quad (10)$$

The stiffness matrix of the finite element is therefore

$$\left[K_e\right] = EI\int_0^L \left\{\frac{\partial^2 N_i}{\partial x^2}\right\}\left\langle\frac{\partial^2 N_i}{\partial x^2}\right\rangle dx \tag{11}$$

The parameters of the stiffness matrix (11) are

$$\left[K_e\right] = \begin{bmatrix} \frac{12EI}{L^3} & \frac{6EI}{L^2} & -\frac{12EI}{L^3} & \frac{6EI}{L^2} \\ \frac{6EI}{L^2} & \frac{4EI}{L} & -\frac{6EI}{L^2} & \frac{2EI}{L} \\ -\frac{12EI}{L^3} & -\frac{6EI}{L^2} & \frac{12EI}{L^3} & -\frac{6EI}{L^2} \\ \frac{6EI}{L^2} & \frac{2EI}{L} & -\frac{6EI}{L^2} & \frac{4EI}{L} \end{bmatrix} \tag{12}$$

In the same manner, the mass matrix can be deduced by

$$\left[M_e\right] = \rho\Omega\int_0^L \left\{N_i\,(x)\right\}\left\langle N_i\,(x)\right\rangle dx \tag{13}$$

The introduction of the relation (8) into (13), the consistent mass matrix can be derived

$$\left[M_e\right] = \frac{\rho\Omega L}{420}\begin{bmatrix} 156 & 22L & 54 & -13L \\ 22L & 4L^2 & 13L & -3L^2 \\ 54 & 13L & 156 & -22L \\ -13L & -3L^2 & -22L & 4L^2 \end{bmatrix} \tag{14}$$

If a lumped mass hypothesis is considered, the mass matrix can be written as

$$\left[M_e\right] = \frac{\rho\Omega L}{2}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{15}$$

Applying boundary conditions $v(0) = 0.$ and $\frac{\partial v(0)}{\partial x} = 0.$, the Lagrange's equations becomes

$$\left[K_e\right]\left\{q_e\right\} = \omega^2\left[M_e\right]\left\{q_e\right\} \tag{16}$$

Case 1: Lumped mass hypothesis

$$\left|\begin{bmatrix} \frac{12}{L^3} & -\frac{6}{L^2} \\ -\frac{6}{L^2} & \frac{4}{L} \end{bmatrix} - \frac{\omega^2\rho\Omega L}{2EI}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right| = 0. \tag{17}$$

The free vibration pulsation is then

$$\omega = \frac{\sqrt{6}}{L^2}\sqrt{\frac{EI}{\rho\Omega}} = \frac{2.4495}{L^2}\sqrt{\frac{EI}{\rho\Omega}} \tag{18}$$

Case 2: Consistent mass hypothesis

$$\left|\begin{bmatrix} 12 & -6L \\ -6L & 4L^2 \end{bmatrix} - \frac{\omega^2\rho\Omega L^4}{210EI}\begin{bmatrix} 78 & -11L \\ -11L & 4L^2 \end{bmatrix}\right| = 0. \tag{19}$$

Set $\lambda = \frac{\omega^2\rho\Omega L^4}{210EI}$, the solutions of the previous equation are

$$\lambda_1 = 0.0594295 \tag{20.1}$$

$$\lambda_2 = 5.76914 \tag{20.2}$$

The corresponding circular frequencies can be computed as

$$\omega_{c1} = \frac{3.533}{L^2}\sqrt{\frac{EI}{\rho\Omega}} \tag{21.1}$$

$$\omega_{c2} = \frac{34.807}{L^2}\sqrt{\frac{EI}{\rho\Omega}} \tag{21.2}$$

When comparing the obtained pulsations (18) and (21), it is necessary to adopt the consistent mass hypothesis in the study.

2.2. Analytical method

In the Bernoulli-Euler's beam theory, shear strains are neglected. These mean that the plane section before loading remains plane after loading.

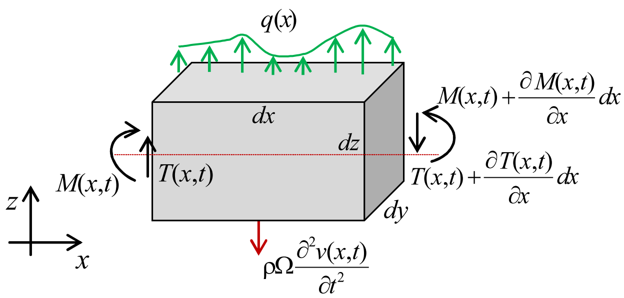

Figure 2. Free body diagram of the element dx

2.2.1. Equation of motion

Consider a free body diagram of a beam element (Figure 2) where $M(x,t)$ and $T(x,t)$ are the bending moment and the shear force at x and time t, respectively. $q(x)$ is the external loading and $\rho\Omega\frac{\partial^2 v(x,t)}{\partial t^2}$ is the inertia force acting on the beam element.

The equilibrium equation according to the vertical axis is

$$T(x,t) - \left(T(x,t) + \frac{\partial T(x,t)}{\partial x}dx\right) - q(x)dx - \rho\Omega\frac{\partial^2 v(x,t)}{\partial t^2}dx = 0 \tag{22}$$

or,

$$\frac{\partial T(x,t)}{\partial x} = q(x)dx - \rho\Omega\frac{\partial^2 v(x,t)}{\partial t^2} \tag{23}$$

Since $T(x,t) = \frac{\partial M(x,t)}{\partial x}$ and $M(x,t) = EI\frac{\partial^2 v(x)}{\partial x^2}$, the equation (23) can be written in the general partial differential equation as

$$\frac{\partial^4 v(x,t)}{\partial x^4} + \frac{\rho\Omega}{EI}\frac{\partial^2 v(x,t)}{\partial t^2} = \frac{q(x)}{EI}. \tag{24}$$

For a free vibration, the external loads are not considered in the analysis and the equation (24) becomes

$$\frac{\partial^4 v(x,t)}{\partial x^4}+\frac{\rho\Omega}{EI}\frac{\partial^2 v(x,t)}{\partial t^2}=0. \quad (25)$$

The general solution of the equation (25) by using a separated variable method can be introduced.

$$v(x,t)=X(x).\mathrm{Y}(t) \quad (26)$$

The relationships (25) and (26) deal to

$$\frac{EI}{\rho\Omega}\frac{\frac{d^4 \mathrm{X}(x)}{dx^4}}{X(x)}=-\frac{\frac{d^2Y(t)}{dt^2}}{Y(t)}=\omega^2=cste \quad (27)$$

The equation (27) can be written as

$$\frac{d^4v(x)}{dx^4}=\frac{\omega^2\rho\Omega}{EI}X(x) \quad (28.1)$$

$$\frac{d^2Y(t)}{dt^2}=\omega^2Y(t) \quad (28.2)$$

Set $\alpha^4=\frac{\omega^2\rho\Omega}{EI}$, the first differential equation (28.1) can be written

$$\frac{d^4X(x)}{dx^4}-\alpha^4X(x)=0. \quad (29)$$

The spatial solution of the equation (29) is

$$X(x)=Ce^{\beta x} \quad (30)$$

Substituting the equation (30) in the equation (29)

$$(\beta^4-\alpha^4)Ce^{\beta x}=0. \quad (31)$$

The spatial solution of the equation (31) can be deduced as

$$X(x)=c_1e^{i\alpha x}+c_2e^{-i\alpha x}+c_3e^{\alpha x}+c_4e^{-\alpha x} \quad (32)$$

Thus, the general displacement field is

$$v(x,t)=\left[c_1e^{i\alpha x}+c_2e^{-i\alpha x}+c_3e^{\alpha x}+c_4e^{-\alpha x}\right].e^{i\omega t} \quad (33)$$

This spatial solution (32) can be written as

$$X(x,\omega)=A_1\cos\alpha x+A_2\,\mathrm{e}^{-\alpha x}+A_3\sin\alpha x+A_4\,\mathrm{e}^{\alpha x} \quad (34)$$

The application of the boundary conditions of a clamped-free beam are

$$X(x=0,\omega)=0 \quad (35.1)$$

$$\frac{dX}{dx}(x=0,\omega)=0 \quad (35.2)$$

$$\frac{dX}{dx}(x=0,\omega)=0 \quad (35.2)$$

$$\frac{d^2X}{dx^2}(x=L,\omega)=0 \quad (35.3)$$

$$\frac{d^3X}{dx^3}(x=L,\omega)=0 \quad (35.4)$$

By applying the boundary conditions, we can build the following matrix.

$$\left[A(\lambda)\right]=\begin{bmatrix}1 & 1 & 0 & 1\\ 0 & -1 & 1 & 1\\ -\cos\lambda & e^{-\lambda} & -\sin\lambda & e^{\lambda}\\ \sin\lambda & -e^{-\lambda} & -\cos\lambda & e^{\lambda}\end{bmatrix} \quad (36)$$

With $\lambda=\alpha.L$

The determinant of the matrix $A(\lambda)$ must be null.

$$-4(1+\cos\lambda.\cosh(\lambda)=0 \quad (37)$$

The first five solutions of the equation (37) are regrouped in the table (1).

Table 1. First five eigen-values

Solution	1	2	3	4	5
$\lambda=\alpha L$	1.8751	4.69409	7.85476	10.9955	14.1372

2.3. Spectral element method

The partial differential equation (25) can be solved by using the Fourier decomposition of the displacement field into the sum of the harmonic vibration as

$$v(x,t)=\frac{1}{N}\sum_{n=0}^{N-1}W_n(x).\mathrm{e}^{i\omega_n t} \quad (38)$$

By substituting the equation (38) into the relationship (25), we get the eigen-values of the problem for $\omega=\omega_n$.

$$\frac{d^4\,\mathrm{W}(x)}{dx^4}-\alpha^4\,\mathrm{W}(x)=0 \quad (39)$$

Then the solution of the equation (39) is

$$\mathrm{W}(x,\omega)=c_1\cos\alpha x+c_2\sin\alpha x+c_3\cosh\alpha x+c_4\sinh\alpha x \quad (40)$$

The nodal displacement and the slope at both ends can be expressed as

$$\left\{q_e\right\}=\begin{Bmatrix}v_1\\ \theta_1\\ v_2\\ \theta_2\end{Bmatrix}\begin{bmatrix}1 & 0 & 1 & 0\\ 0 & \alpha & 0 & \alpha\\ \cos\alpha L & \sin\alpha L & \cosh\alpha L & \sinh\alpha L\\ -\alpha\cos\alpha L & \alpha\cos\alpha L & \alpha\sinh\alpha L & \alpha\cosh\alpha L\end{bmatrix} \quad (41)$$

or,

$$\left\{q_e\right\}=\left[B(\omega)\right]\left\{c\right\} \quad (42)$$

When using the equation (40), transverse shear forces and bending moments at element node level are

$$\left\{F_e\right\}=\begin{Bmatrix}T_1\\ M_1\\ -T_2\\ -M_2\end{Bmatrix}=EI\begin{bmatrix}0 & -\alpha^3 & 0 & \alpha^3\\ \alpha^2 & 0 & -\alpha^2 & 0\\ -\alpha^3.\sin\alpha L & \alpha^3.\cos\alpha L & -\alpha^3.\sinh\alpha L & -\alpha^3.\cos\alpha L\\ -\alpha^2.\cos\alpha L & -\alpha^2.\sin\alpha L & \alpha^2.\cosh\alpha L & \alpha^2.\sinh\alpha L\end{bmatrix}\begin{Bmatrix}c_1\\ c_2\\ c_3\\ c_4\end{Bmatrix} \quad (43)$$

or,

$$\left\{F_e\right\}=EI\left[F(\omega)\right]\left\{c\right\}=EI\left[F(\omega)\right].\left[B(\omega)\right]^{-1}\left\{q_e\right\} \quad (44)$$

The quantity $(\mathrm{EI}.\left[F(\omega)\right].\left[B(\omega)\right]^{-1})$ is called the spectral stiffness matrix.

Finally, the parameters of the spectral element matrix of the Euler-Bernoulli beam are given by

$$K_{11}=K_{33}=\eta(\alpha L)^3\left[\cos\alpha L.\sinh\alpha L+\sin\alpha L.\cosh\alpha L\right] \quad (45.1)$$

$$K_{22}=K_{44}=\eta\alpha L^3\left[-\cos\alpha L.\sinh\alpha L+\sin\alpha L.\cosh\alpha L\right] \quad (45.2)$$

$$K_{12}=-K_{34}=\eta\alpha^2 L^3.\sin\alpha L.\sinh\alpha L \quad (45.3)$$

$$K_{13}=-\eta(\alpha L)^3\left[\sin\alpha L+\sinh\alpha L\right] \quad (45.4)$$

$$K_{14}=-K_{23}=\eta\alpha^2 L^3\left[-\cos\alpha L+\cosh\alpha L\right] \quad (45.5)$$

$$K_{24}=\eta\alpha L^3\left[-\sin\alpha L+\sinh\alpha L\right] \quad (45.6)$$

With, $h=\dfrac{1}{1-\cos a\,L.\cosh a\,L}$

After obtaining the spectral element matrix and applying the boundary conditions, the determinant of the spectral element matrix is zero.

$$\left|K\right|=K_{33}.K_{44}-K_{34}^2=0 \quad (46)$$

The equation (46) leads to compute pulsations of free vibration.

$$tgh^2(\alpha L)-\sin^2(\alpha L)=0 \quad (47)$$

The solutions of the previous equation are, sequentially

$\alpha_1=1.8751$, $\alpha_2=4.69409$, $\alpha_3=7.85475$,

$\alpha_4=10.9955$, $\alpha_5=14.1372$, ….

They are the exact solutions of the partial differential equation (39).

3. OBTAINED RESULTS

In this section, a clamped-free beam (Figure 3) is considered. Mechanical and geometrical data are regrouped in Table 3.

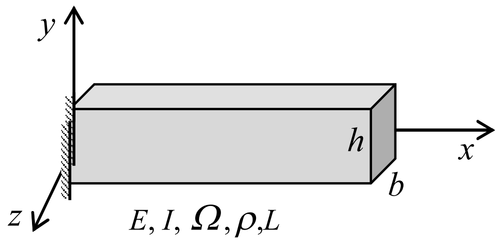

Figure 3. Studied beam

Table 3. Properties of the beam used

b (cm)	h (cm)	L (m)	E (N/m²)	ρ (kg/m³)
20	20	1	10^{11}	1000

Table 4. First five modes of vibration (rad/s)

Mode	**1**	**2**	**3**	**4**	**5**	**Ratio**
2 elements (SFEM)	203	1272.156	3565.087	6980.219	-	0.0%
20 elements (FEM)	203	1272.16	3562.03	6980.22	11540.73	0.155%
Exact Solution	203	1272.156	3565.087	6980.219	11538.940	-

3.1. Finite element method

The obtained results by using the finite element method are regrouped in Table (4). The first five modes of the clamped-free vibration beam are regrouped. The beam is meshed of different meshes as: 2, 5, 10, 15 and 20 elements.

Table (4) shows that the convergence towards the exact results is validated when 20 finite beam elements are used. Moreover, higher-order modes of vibration have a notable influence on the cantilever beam vibration. To accurate the dynamic results, it's necessary to use an important number of elements and a higher order of modes of vibration simultaneously.

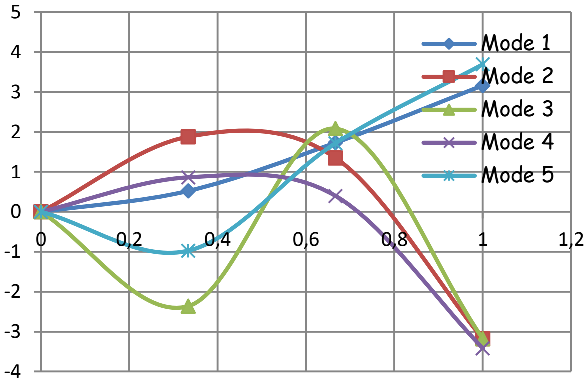

Figure 4. Transversal vibration of the first five modes

Figures (4-5) illustrate the shape modes of the vibration for the first five modes of the vibration. Figures (4-5) show the transversal and rotational shape modes of the vibration.

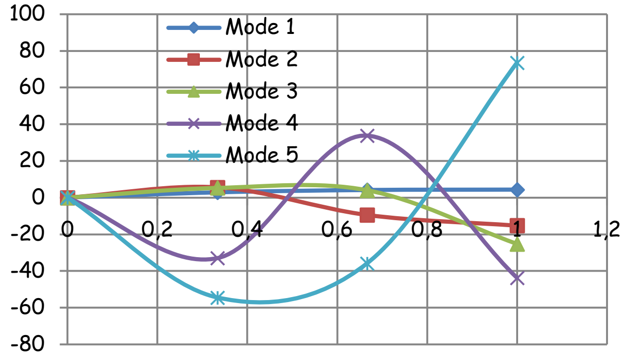

Figure 5. Rotational vibration of the first five modes

3.2. Spectral finite element method

Table (5) shows a comparison between the results obtained by the spectral element method (2 elements), the finite element method (20 elements) and the analytical solution. The analysis shows that the numerical computing can be achieved in the spectral element results and analytical method when the number of elements must be superior then 20.

Figure 6 shows the comparison between the diagonal stiffness parameters evaluated by using the FEM and SFEM. The finite element diagonal parameters are under-estimated compared to the SFEM ones.

Table 5. Comparison between the results of the methods used

Mode	**1**	**2**	**3**	**4**	**5**	**Ratio**
1 element	204	2009.60				57.96%
2 elements	203	1283	4339	12594		80.42%
5 elements	203	1273	3575	7062	11721	1.577%
10 elements	203	1272.20	3563	6986.90	11568	0.251%
15 elements	203	1272.18	3562.50	6983.20	11560.35	0.185%
20 elements	203	1272.16	3562.03	6980.22	11540.73	0.155%
Exact Solution (rad/s)	**203**	**1272.156**	**3565.087**	**6980.219**	**11538.940**	**-**

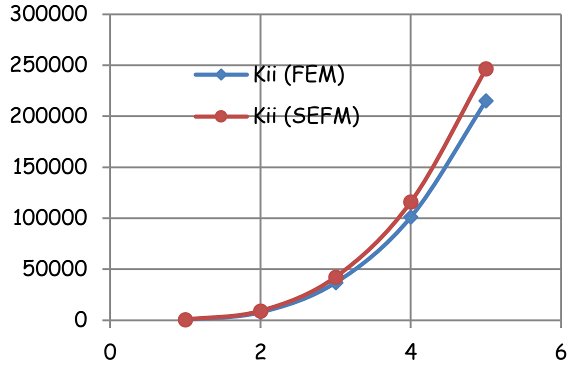

Figure 6. Diagonal stiffness parameters

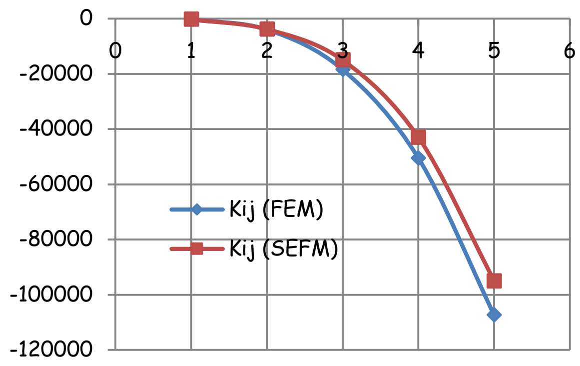

Figure 7. Non-diagonal stiffness parameters

In the same context, Figure 7 shows the comparison between the non-diagonal stiffness parameters computed with the FEM and SFEM. The figure illustrates that the finite element non-diagonal parameters are over-estimated compared to the SFEM ones.Rods and Euler-Bernoulli beams behave in the same manner corresponding to the diagonal and non-diagonal stiffness parameter variations evaluated with FEM and SFEM [13]. Finally, what can bring the effect of shear forces on the free vibration of beam responses [14]?

4. CONCLUSIONS

In this work, the spectral finite element method (SFEM) is described as a numerical approach to predict the free-vibration of the Euler-Bernoulli beam responses. The exactness of the SFEM is validated by using only one to two elements. On the contrary, the finite element method requires important finite elements to reach an accurate solution. In this case, to predict the solution by using the FEM it is necessary to incorporate higher modes of vibration in the analysis.

The comparison is performed by the calculation of circular frequencies. The study shows that the proposed method is practical, efficient and can be used as a reference for the convergence criteria of structural dynamics via the finite element method.

The convergence of SFEM is principally due to the fact that the diagonal finite element method parameters are under-estimated, but the non-diagonal ones are over-estimated compared to the SFEM stiffness parameters.

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