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Introduction to Planimetry of Quasi-Elliptic Plane

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ABSTRACT

The quasi-elliptic plane is one of nine projective-metric planes where the metric is induced by the absolute figure $\mathcal{F}_{QE} = \{j_1, j_2, F\}$ consisting of a pair of conjugate imaginary lines j_1 and j_2, intersecting at the real point F. Some basic geometric notions, definitions, selected constructions and a theorem in the quasi-elliptic plane will be presented.

Key words: quasi-elliptic plane, perpendicular points, central line, qe-conic classification, hyperosculating qe-circle, envelope of the cental lines

MSC2010: 51A05, 51M10, 51M15

Uvod u planimetriju kvazieliptičke ravnine

SAŽETAK

Kvazieliptička ravnina jedna je od devet projektivno metričkih ravnina. Apsolutnu figuru $\mathcal{F}_{QE} = \{j_1, j_2, F\}$ određuju dva imaginarna pravca j_1 i j_2 i njihovo realno sjecište F.
U ovom radu definirat ćemo osnove pojmove, prikazati odabrane konstrukcije i dokazati jedan teorem.

Ključne riječi: kvazieliptička ravnina, okomite točke, centrala, klasifikacija qe-konika, hiperoskulacijska qe-kružnica, omotaljka centrala

1 Introduction

This paper begins the study of the quasi-elliptic plane from the constructive and synthetic point of view. We will see although the geometry denoted as quasi-elliptic is dual to Euclidean geometry it is a very rich topic indeed and there are many new and unexpected aspects.
In this paper some basic notations concerning the quasi-elliptic conic and some selected constructions and a theorem will be presented. It is known that there exist nine geometries in plane with projective metric on a line and on a pencil of lines which are denoted as Cayley-Klein projective metrics and they have been studied by several authors, such as [2], [3], [4], [8], [9], [10], [13], [14], [15], [16].
The quasi-elliptic geometry, further in text qe-geometry, has elliptic measure on a line and parabolic measure on a pencil of lines. In the quasi-elliptic plane, further in text qe-plane, the metric is induced by the absolute figure $\mathcal{F}_{QE} = \{j_1, j_2, F\}$, i.e. a pair of conjugate imaginary lines j_1 and j_2, incident with the real point F. The lines j_1 and j_2 are called the **absolute lines**, while the point F is called the **absolute point**. In the Cayley-Klein model of the qe-plane only the points, lines and segments inside of one projective angle between the absolute lines are observed. In this paper all points and lines of the qe-plane embedded in the real projective plane $\mathbb{P}_2(\mathbb{R})$ are observed. It is suitable to obtain a line as a basic element, and a point as a pencil of lines (for example a curve is an envelope of lines; quadratic transformation in the qe-plane maps pencil of lines into the second class curve). Using an elliptic involution on the pencil (F) the absolute triple $\mathcal{F}_{QE} = \{j_1, j_2, F\}$ can be given as follows:

- An elliptic involution on the pencil (F) is determined by two arbitrary chosen pairs of corresponding lines a_1, a_2; b_1, b_2. An elliptic involution (F) has the absolute lines j_1 and j_2 for double lines ([1], p.244-245, [6], p.46).
  Notice that the absolute point F can be finite (Figure 1a) or at infinity (Figure 1c).
  In this paper the model were involutory pair of corresponding lines are perpendicular to each other in Euclidean sense (Figure 1b) is used in a way that only the absolute point F is presented.

- The absolute point F is inside the conic k. Pairs of conjugate lines with respect to a conic k determine aforementioned elliptic involution (F). The absolute lines j_1 and j_2 are double lines for the involution (F) and in this case they are a pair of imaginary tangent lines to k from the absolute point F (Figure 1d).

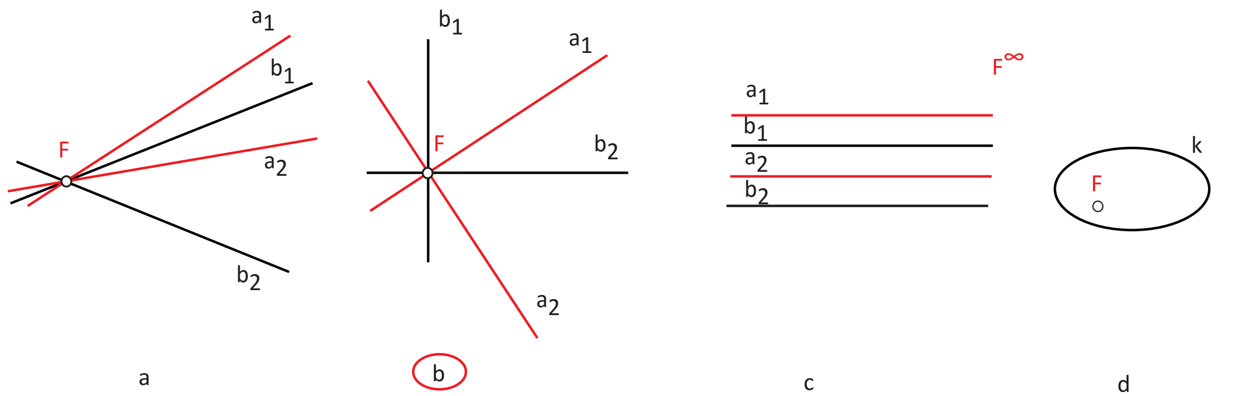

Figure 1

2 Basic notation and selected constructions in the quasi-elliptic plane

For the points and the lines in the qe-plane the following are defined:

- *isotropic lines* - the lines incident with the absolute point F,
- *isotropic points* - the imaginary points incident with one of the absolute line j_1 or j_2,
- *parallel points* - two points incident with the same isotropic line,
- *perpendicular lines* - if at least one of two lines is an isotropic line,
- *perpendicular points* - two points A, A_1 that lie on a pair of corresponding lines a, a_1 of an elliptic (absolute) involution (F).

Remark. The perpendicularity of points in qe-plane is determine by the absolute involution, therefore an elliptic involution (F) is a circular involution in the qe-plane. ([7], p.75)
Notice that the absolute point F is parallel and perpendicular to each point in the qe-plane. Furthermore, in the qe-plane there are no parallel lines.

A brief review of some basic construction

Example 1 *Let the absolute figure $\mathcal{F}_{QE}$ of the qe-plane be given with the involutory pencil (F) (Figure 1b). Let A be the point and p the line which is not incident with the point A in the qe-plane (Figure 2). Construct the point A_1 which is perpendicular to the point A and incident with the given line p.*

Points A, A_1 are perpendicular if they lie on a pair of corresponding lines a, a_1 of an absolute elliptic involution (F), i.e. if they lie on a pair of perpendicular lines in a Euclidean sense ([7], p.71-75).

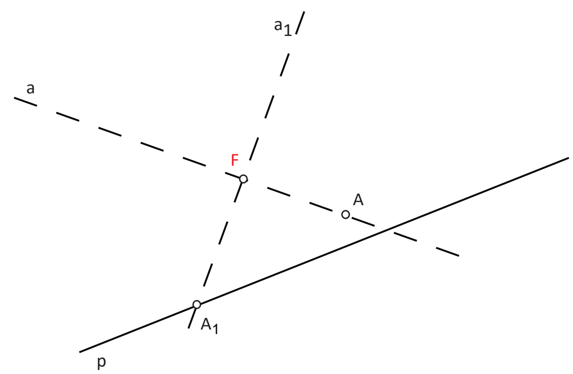

Figure 2

Example 2 *Let the absolute figure $\mathcal{F}_{QE}$ of the qe-plane be given with the involutory pencil (F). Construct the midpoints P_i and the bisectors s_i of a given line segment $\overline{AB}$ $(i = 1, 2)$ (Figure 3).*

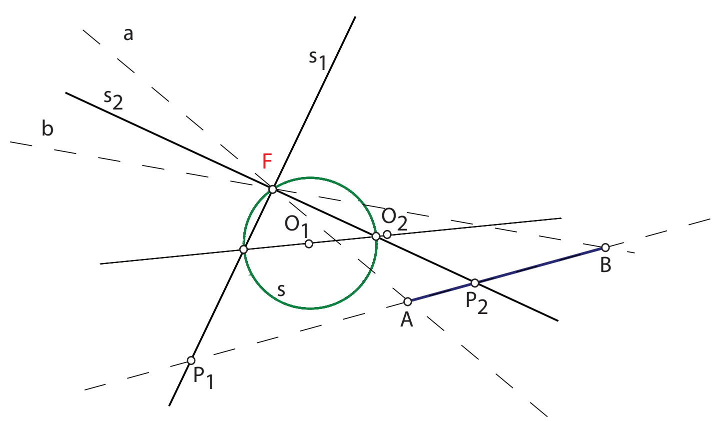

Figure 3

The midpoint of a segment in the qe-plane is dual to an angle bisector in the Euclidean plane, consequently a segment in a qe-plane has two perpendicular midpoints P_1 and P_2 that are in harmonic relation with the points A and B.
A line segment $\overline{AB}$ in the qe-plane has two isotropic bisectors s_1 and s_2 that are a common pair of corresponding lines of two involutions (F) with the center F, denoted as I_1, I_2 . In order to construct the midpoints and bisectors we observe aforementioned involutions (F), a circular involution I_1 is determined by perpendicular corresponding lines in a Euclidean sense and the second hyperbolic involution I_2 is determined by isotropic lines $a = AF$, $b = BF$ as its double lines. The construction is based on the Steiner's construction ([6], p.26, [7], p.74-75). These two pencils will be supplemented by the same Steiner's conic s, which is an arbitrary chosen conic through F. The involutions I_1 and I_2 determine two involutions on the conic s. Let the points O_1 and O_2 be denoted as the centers of these involutions, respectively. The line O_1O_2 intersects the conic s at two points. Isotropic lines s_1 and s_2 through these points are a common pair of these two involutions (F). The intersection points P_1 and P_2 of bisectors s_1 and s_2 with the line AB are midpoints of the line segment $\overline{AB}$.

Example 3 *Let the absolute figure $\mathcal{F}_{QE}$ of the qe-plane be given with the involutory pencil (F). Let two non-isotropic lines a, b be given. Construct an angle bisector between given rays a, b (Figure 4).*

The angle bisector in the qe-plane is dual to a midpoint of a segment in the Euclidean plane. Let V be the vertex of an angle $\angle(a,b)$. Let the isotropic line VF be denoted as f. The angle bisector s is a line in a pencil (V) that is in harmonic relation with triple (a,b,f). The isotropic line f is an isotropic bisector.

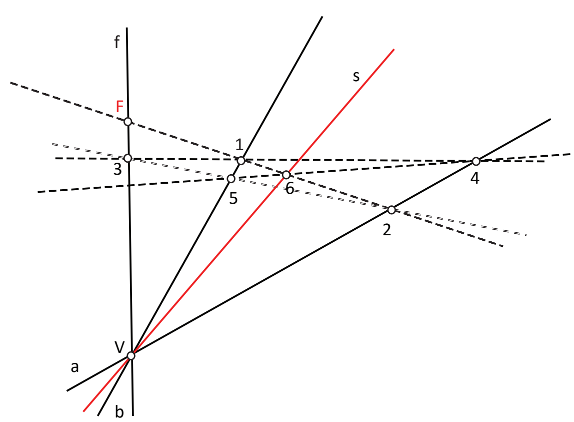

Figure 4

Example 4 *Let the absolute figure $\mathcal{F}_{QE}$ of the qe-plane be given with the involutory pencil (F). Let the lines a, b, c determine a trilateral $\mathcal{ABC}$ with the vertices A, B, C. Construct the ortocentar line of the given trilateral (Figure 5).*

The orthocentar line o of the trilateral in the qe-plane is dual to the orthocenter of a triangle in the Euclidean plane. The points A_1, B_1, C_1 are incident with lines a, b, c and perpendicular to the opposite vertices A, B, C, respectively. The points A_1, B_1, C_1 are collinear and determine a unique ortocentar line.

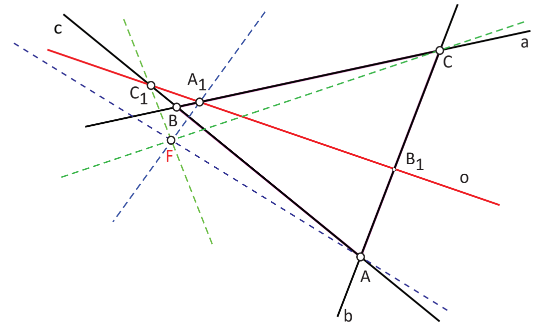

Figure 5

Example 5 *Let the absolute figure $\mathcal{F}_{QE}$ of the qe-plane be given with the involutory pencil (F). Let the lines a, b, c determine a trilateral $\mathcal{ABC}$ with the vertices A, B, C. Construct the centroid line of a trilateral (Figure 6).*

The centroid line o of a trilateral in the qe-plane is dual to the centroid of a triangle in the Euclidean plane. The angel bisectors s_a, s_b, s_c of trilateral intersect opposite sides a, b, c of the trilateral at the points S_A, S_B, S_C, respectively. The points S_A, S_B, S_C are collinear and determine a unique centroid line.

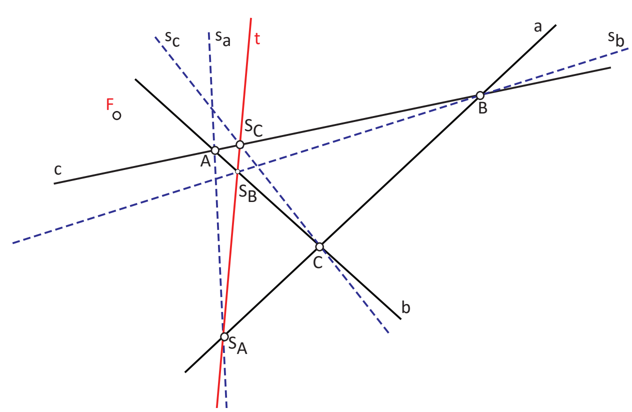

Figure 6

3 Qe-conic classification

There are four types of the second class curves classified according to their position with respect to the absolute figure (Figure 7):

- *qe-hyperbola* (h) - a curve of the second class that has a pair of real and distinct isotropic lines.
  Equilateral qe-hyperbola (h_{EQ}) - a curve of the second class that has isotropic lines as a corresponding lines for the absolute involution (F).
- *qe-ellipse* (e) - a curve of the second class that has a pair of imaginary isotropic lines.
- *qe-parabola* (p) - a curve of the second class where both imaginary isotropic lines coincide.
- *qe-circle* (k) - is a special type of qe-ellipse for which the isotropic lines coincide with the absolute lines j_1 and j_2. In a model of an absolute figure that is used in this paper each qe-conic that has an absolute point F as its Euclidean foci is a qe-circle.

In the projective model of the qe-plane every type of a qe-conic can be represented with every type of Euclidean conics without loss of generality.

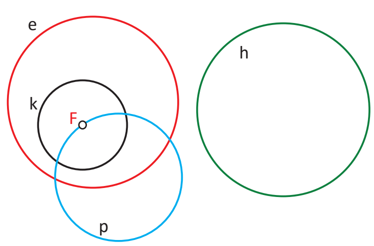

Figure 7

The polar line of the absolute point F with respect to a qe-conic is called the **central line** or the **major diameter** of the second class conic in the qe-plane. The central line of a conic in the qe-plane is dual to a center of a conic in the Euclidean plane. All conics in the qe-plane, except qe-parabolas, have a real non-isotropic central line. The central line of a qe-parabola is isotropic tangent line at the point F.
Dual to the Euclidean diameter of a conic is the point on the central line that is the pole of the isotropic line with respect to a qe-conic. A pair of points incident with the central line that are perpendicular and conjugate with respect to a qe-conic are called the **qe-centers** of the qe-conic. Qe-centers are dual to an axis of Euclidean conic. A qe-ellipse and a qe-hyperbola have two real and distinct qe-centers, while both qe-center of a qe-parabola coincide with the absolute point F.

Each pair of conjugate points incident with the central line with respect to a qe-circle are perpendicular, consequently a qe-circle has infinitely many pairs of qe-centers.
The **isotropic (the minor) diameters** are the lines joining a qe-center to the absolute point F. A qe-ellipse and a qe-hyperbola have two isotropic diameters.
The lines incident with qe-centers of a qe-conic are called the **vertices lines** of a qe-conic in the qe-plane. A qe-hyperbola has two real vertices lines, while a qe-ellipse has four real vertices lines.
A hyperosculating qe-circle of a qe-conic can be constructed only at the vertices lines of a qe-conic.
The intersection points of a qe-conic and vertices lines are called **co-vertices points**.

4 Some construction assignments

Exercise 1 *Construct a qe-circle k determined with the given central line c and the line p (Figure 8).*

In order to construct the qe-circle as a line envelope, a perspective collineation that maps arbitrary chosen qe-circle k_1 into qe-circle k is used. The construction is carried out in the following steps:
The absolute point F is selected for the center of the collineation. Let k_1 be an arbitrary chosen qe-circle with the center F. A polar line c_1 of F, is the central line for chosen qe-circle k_1. Notice that c_1 is the line at infinity. The lines c and c_1 are corresponding lines for the perspective collineation with the center F. Let the point S be the intersection point of the lines p and c. To determine an axis o of the perspective collineation, the point R that is perpendicular to the point S and incident with the line p is constructed. A ray FR of the collineation intersect the qe-circle k_1 at the points R_1 and R_2. Let the line p_1 touch the qe-circle k_1 at a point R_1. The lines p and p_1 are corresponding lines for the perspective collineation with a center F. The axis o passes through the intersection point S_1 of the lines p_1 and p, and it is parallel to c.

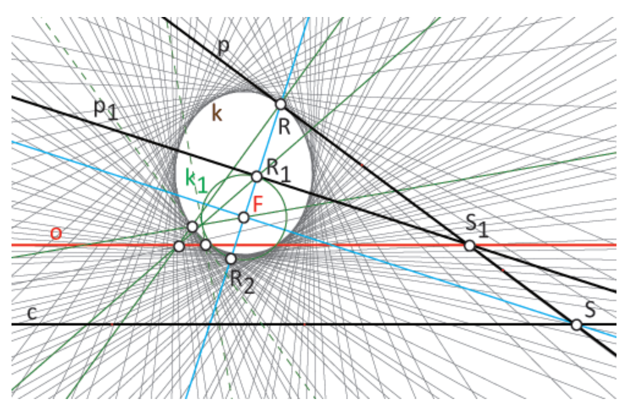

Figure 8

Exercise 2 *Construct the hyperosculating qe-circle of a qe-hyperbola h_1 (Figure 9).*

Let the qe-hyperbola h_1 be given and its central line be denoted as c. A hyperosculating qe-circle of the qe-hyperbola h_1 can be constructed only at the vertices lines. A qe-hyperbola h_1 has two real vertices lines t_1 and t_2. Let the points T_1 and T_2 be co-vertices points. Let the line t_2 and the point T_2 be observed. In order to construct a hyperosculating qe-circle, the point S_2 that is perpendicular to T_2, and incident with the line t_2 is constructed. The central line c_h of a hyperosculating qe-circle is incident with S_2. In order to construct c_h, let the line y_1 of the qe-hyperbola h_1 be arbitrary chosen. The intersection point of h_1 and the line y_1 is denoted as Y_1. The intersection point of lines t_2 and y_1 is denoted as K. The point K_1 is perpendicular to K and incident with joining line T_2Y_1. The line S_2K_1, denoted as c_h, is a central line of a hyperosculating qe-circle. The central line c_h and the line t_2 determine a hyperosculating qe-circle and to construct it the same principle as in Exercise 1 is used.

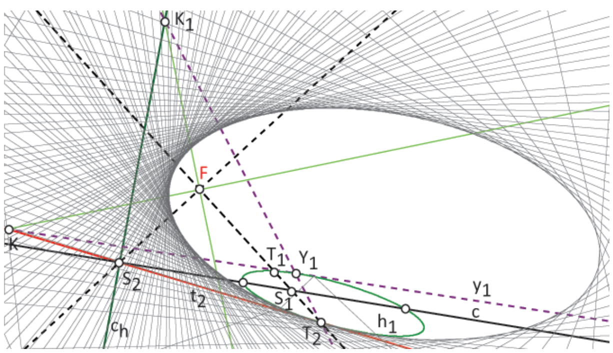

Figure 9

Theorem 1 *Let the lines $\{a,b,c,d\}$ be the base of a pencil of qe-conics in a qe-plane (Figure 10). Then, the envelope of the central lines of all qe-conics in the pencil is a curve of the second class.*

Proof: It is known that the envelope of polar lines of conics in a pencil of conics with respect to a common pole P is a curve of the second class ([6]). Consequently, in the qe-plane, if a common pole P coincides with the absolute point F, than the envelope of its polar lines coincides with the envelope of the central lines in the given pencil. □

In order to construct the envelope of the central lines of all qe-conics in the pencil of qe-conics, denoted as δ_1, we observe involutory pencil (F) of a pairs of isotropic lines of all qe-conics in a pencil. Each qe-conic in a pencil of qe-conics has two real or imaginary isotropic lines. In the given pencil of qe-conics there are three qe-conics degenerated into three pairs of points, denoted as $(1,1')$, $(2,2')$, $(3,3')$. Let the involution (F) be determined with an isotropic lines of any two degenerated qe-conics in a pencil i.e. $(1,1')$, $(3,3')$. The pencil of qe-conics contains two, one or none real qe-parabola.

From the viewpoint of qe-geometry, the envelope δ_1 is a qe-hyperbola if the pencil contains two qe-parabolas. The central lines of these qe-parabolas denoted as, p_1 and p_2 are double lines for the involution (F) and they coincide with the isotropic lines of the envelope δ_1. The envelope is determinate with five lines; the lines p_1, p_2, and central lines of three degenerated qe-conics c_1, c_2, c_3.

The envelope δ_1 is a qe-parabola if the pencil contain one qe-parabola.

The envelope δ_1 is an qe-ellipse if the pencil does not contain qe-parabolas (Figure 10). Double lines for the elliptic involution (F) are imaginary lines.

Pencil will be supplemented by the Steiner's conic s, which is an arbitrary chosen conic through F. Let the point O be denoted as a center of the involution (F).

If the point O is outside the conic s, involutory pencil (F) contains real double lines, and the envelope δ_1 is a qe-hyperbola. If the point O is on the conic s, double lines of involution (F) coincide, and the envelope δ_1 is a qe-parabola.

If the point O is inside the conic s, involutory pencil (F) contains imaginary double lines, and the envelope δ_1 is an qe-ellipse (Figure 10).

If the point O coincides with the center of conic s, double lines of involution (F) coincides with the absolute lines j_1 and j_2, and the envelope δ_1 is a qe-circle (circular involution).

If one of the base lines in a pencil is isotropic line, the pencil of qe-conics contains qe-hyperbolas and one qe-parabola, the envelope δ_1 is a qe-parabola.

If two of the base lines in a pencil are isotropic lines, the envelope δ_1 degenerates into a point.

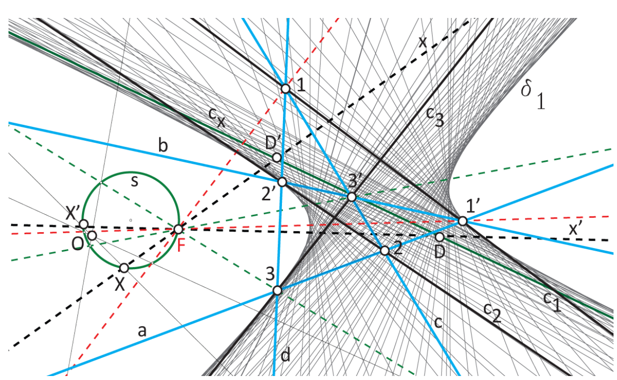

Figure10: *Qe-ellipse - an envelope of the central lines*

Corollary 1 *Let any two degenerated qe-conics in a pencil of qe-conics be given as a pair of perpendicular points i.e. the pencil of equilateral qe-hyperbolas. Than the envelope of the central line is a qe-circle.*

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