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Incenter Symmetry, Euler lines, and Schiffler Points

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ABSTRACT

We look at the four-fold symmetry given by the Incenter quadrangle of a triangle, and the relation with the cirumcircle, which in this case is the nine-point conic of the quadrangle. By investigating Euler lines of Incenter triangles, we show that the classical Schiffler point extends to a set of four Schiffler points, all of which lie on the Euler line. We discover also an additional quadrangle of Incenter Euler points on the circumcircle and investigate its interesting diagonal triangle. The results are framed in purely algebraic terms, so hold over a general bilinear form. We present also a mysterious case of apparent symmetry breaking in the Incenter quadrangle.

Key words: triangle geometry, Euclidean geometry, rational trigonometry, bilinear form, Schiffler points, Euler lines, Incenter hierarchy, circumcircles

MSC2010: 51M05, 51M10, 51N10

"Upisana simetrija", Eulerovi pravci i Schifflerove točke

SAŽETAK

Proučavamo četverostruku simetriju određenu četverovrhom, čiji su vrhovi središta upisanih (pripisanih) kružnica danog trokuta, te vezu s opisanom kružnicom trokuta koja je u ovom slučaju konika devet točaka spomenutog četverokuta. Proučavajući Eulerove pravce takozvanih upisanih trokuta, pokazujemo da je poopćenje klasične Schifflerove točke skup od četiriju točaka koje leže na Eulerovom pravcu. Promatra se četverokut u čijim se vrhovima sijeku Eulerovi pravci upisanih trokuta, te njegov dijagonalni trokut. Kako se koristi algebarski pristup, dobiveni rezultati vrijede za opću bilinearnu formu. Dajemo i primjer svojevrsnog nestanka četverostruke simetrije.

Ključne riječi: geometrija trokuta, euklidska geometrija, racionalna trigonometrija, bilinearna forma, Schifflerove točke, Eulerovi pravci, hijerarhija središta upisanih kružnica, opisane kružnice

1 Introduction

The following is a classical theorem which was first observed by M. Bôcher in 1892. Special cases include the nine-point circle of a triangle, and the nine point hyperbola.

Theorem 1 (Nine point conic) *The six midpoints of a quadrangle (four points) together with the diagonal points lie on a conic.*

This is called the *Nine point conic* of the quadrangle. Bôcher observed that if one of the four points lies on the circumcircle defined by the other three, then the conic is an equilateral hyperbola. If one of the points is the orthocenter of the other three, then the conic is a circle. In Figure 1 we see a general quadrangle $\overline{P_0P_1P_2P_3}$, as well as the six midpoints in dark blue, and the three diagonal points in orange, with these last nine points on the red conic.

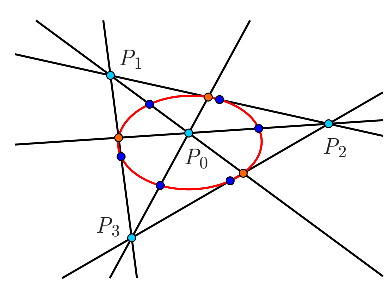

Figure 1: *The Nine point conic of the quadrangle* $\overline{P_1P_2P_3P_4}$

If we consider the above theorem in relation to the Incenter quadrangle $\overline{I_0I_1I_2I_3}$ of a Triangle $\overline{A_1A_2A_3}$, some additional interesting things happen, since this is an orthocentric quadrangle.

Theorem 2 *The Nine point conic of the Incenter quadrangle $\overline{I_0I_1I_2I_3}$ of a Triangle $\overline{A_1A_2A_3}$ is the Circumcircle c of that triangle, and so also the nine-point circle of any three Incenters. Each midpoint of the quadrangle $\overline{I_0I_1I_2I_3}$ is the center of a circle which passes through two Incenters as well as two Points of the Triangle.*

This last lovely fact finds its way routinely into International problem competitions, as has been compiled by E. Chen, who calls it the *Incenter/Excenter lemma* (see [1]). He gives a proof using angle chasing, we will give a more powerful and general argument in the course of this paper. In Figure 2 we see that the **Incenter midpoint** $M = M_{02}$, which is the midpoint of the segment $\overline{I_0I_2}$, is the center of a circle which passes through two points of the triangle, in this case A_1 and A_3, as well as the two Incenters I_0 and I_2.

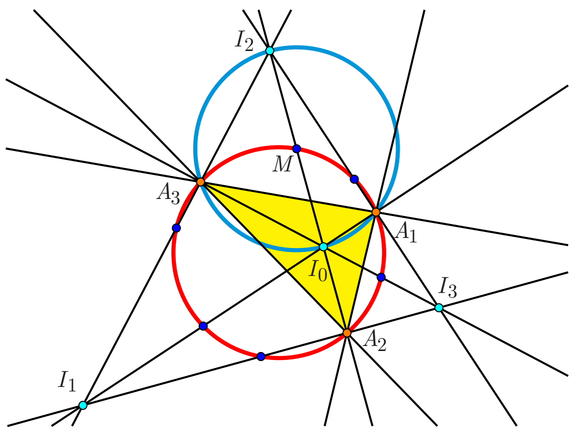

Figure 2: *The Incenter quadrangle and its midpoints*

It is worth noting that obviously each Incenter midpoint lies on an angle bisector, or **Biline**, of the Triangle $\overline{A_1A_2A_3}$, as these are the six lines of the complete quadrangle $\overline{I_0I_1I_2I_3}$.

In C. Kimberling's celebrated list of triangle centers, see [3] and [4], the Incenter I_0 gets pride of place, as the first point X_1 in the entire list. Because his list contains only uniquely defined centers, the other Incenters I_1, I_2 and I_3, which are more usually called excenters, do not get explicit numbered names. In this paper we investigate the fourfold symmetry surrounding Incenter midpoints within the set-up of Rational Trigonometry ([11], [12]), valid for any symmetric bilinear form, as described in [7]. So the theorems in this paper hold also with other bilinear forms, as in Lorentzian planar geometry.

Next to the Incenter, the most famous triangle centers are the Centroid $G = X_2$, the Circumcenter $C = X_3$, and the Orthocenter $H = X_4$, which famously all lie on the Euler line e. In Figure 3 we see both the Euler line and the Incenter quadrangle $\overline{I_0I_1I_2I_3}$ for the Euclidean example that we will exhibit frequently.

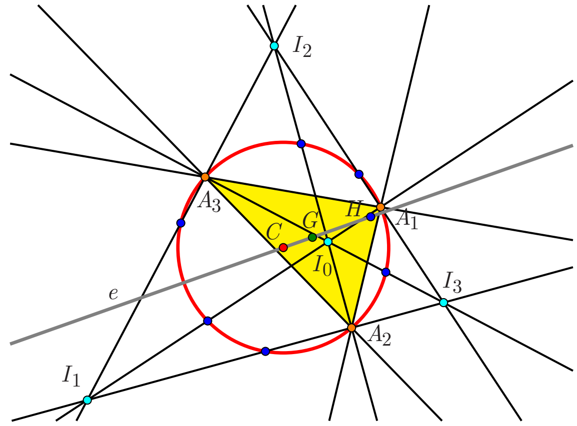

Figure 3: *The Euler line and Incenter quadrangle of* $\overline{A_1A_2A_3}$

The **Schiffler point** $S = X_{21}$ of the triangle $\overline{A_1A_2A_3}$ is another remarkable triangle centre which was discovered more recently by Kurt Schiffler (1896-1986) [9]. This point is the intersection of the Euler lines of the three **Incenter triangles** $\overline{A_1A_2I_0}$, $\overline{A_1A_3I_0}$, $\overline{A_2A_3I_0}$. Pleasantly S lies on the Euler line e of the original triangle $\overline{A_1A_2A_3}$.

This situation is illustrated in Figure 4 which shows the Schiffler point S (in white) of $\overline{A_1A_2A_3}$, the meet of the four **Incenter Euler** lines (in gray), passing through circumcenters (blue) and centroids (green) of the Incenter triangles. Clearly these circumcenters are exactly the midpoints that we observed in the previous diagram. There are several interesting and remarkable properties of the Schiffler point which have been found over the years: see for example ([2], [8], [10]).

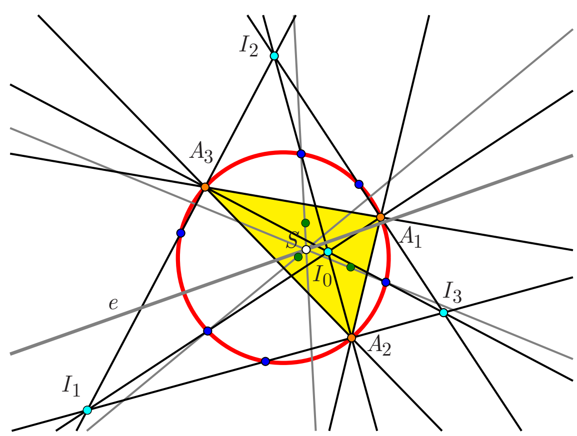

Figure 4: *The Schiffler point S of* $\overline{A_1A_2A_3}$

Since our philosophy, expounded in [7] and [6], is that we ought to consider all four Incenters symmetrically, it is natural for us to expand this story to include Incenter Euler lines from the *other* Incenter triangles obtained by combining two vertices of the original triangle $\overline{A_1A_2A_3}$ and any one of the Incenters. If we agree that $\{i,j,k\} = \{1,2,3\}$, then such an Incenter triangle $\overline{A_jA_kI_l}$ is determined by the pair of indices (i,l), where i runs through $1,2,3$ and l runs through $0,1,2,3$. So let us denote by e_{il} the Incenter Euler line of the triangle $\overline{A_jA_kI_l}$. *Notice that the Point label comes first, followed by the Incenter label.*

This way we get twelve Incenter Euler lines, not just three. When we look at all of these, we meet some remarkable new phenomenon. The first observation is that the standard Schiffler point $S = S_0$ is now but one of four Schiffler points.

Theorem 3 (Four Schiffler points) *The triples $S_0 \equiv e_{10}e_{20}e_{30}$, $S_1 \equiv e_{11}e_{21}e_{31}$, $S_2 \equiv e_{12}e_{22}e_{32}$ and $S_3 \equiv e_{13}e_{23}e_{33}$ of Incenter Euler lines are concurrent. These Schiffler points all lie on the Euler line e of the original triangle $\overline{A_1A_2A_3}$.*

The next result shows that there are other interesting concurrences of the Incenter Euler lines. These are also visible in Figure 5.

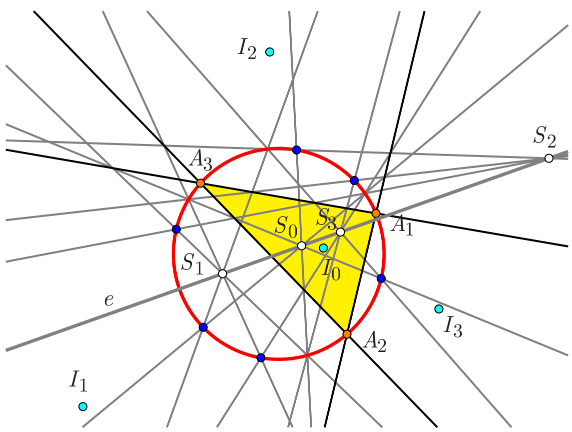

Figure 5: *The four Schiffler points S_0, S_1, S_2 and S_3 on the Euler line e*

Theorem 4 (Four Incenter Euler points) *The triples $P_0 \equiv e_{11}e_{22}e_{33}$, $P_1 \equiv e_{10}e_{23}e_{32}$, $P_2 \equiv e_{20}e_{13}e_{31}$ and $P_3 \equiv e_{30}e_{12}e_{21}$ of Euler lines are concurrent. These points all lie on the Circumcircle of the original Triangle.*

These theorems will form the starting points of the investigations of this paper. We will see that the diagonal triangle $\overline{D_1D_2D_3}$ of the quadrangle $\overline{P_0P_1P_2P_3}$ has some remarkable connections with the original triangle $\overline{A_1A_2A_3}$. We call $\overline{D_1D_2D_3}$ the **Diagonal Incenter Euler triangle**. At the end of the paper, we note a remarkable appearance of symmetry breaking in the original Incenter quadrangle $\overline{I_0I_1I_2I_3}$ which is well worth further investigation.

Throughout the paper our emphasis is on explicit formulas that allow us to give general algebraic proofs. We will give diagrams that illustrate the Euclidean case, but it is an essential strength of this approach that the results hold for a general bilinear form, and we will also include a few pictures from the green geometry coming from Chromogeometry (see [13] and [14]).

1.1 Quadrance and spread

In this section we briefly summarize the main facts needed from rational trigonometry in the general affine setting (see [11], [12]). We work in the standard two-dimensional affine or vector space over a field, consisting of affine **points**, or row vectors $v = [x,y]$. Sometimes it will be convenient to represent such a vector projectively, as the projective row vector $[x:y:1]$; this makes dealing with fractional entries easier. A **line** l is the proportion $l \equiv \langle p:q:r \rangle$, or equivalently a projective column vector $[p:q:r]^T$, provided that p and q are not both zero. Incidence between the point v and the line l above is given by the relation

$$px + qy + r = 0.$$

Our notation is that the line determined by two points A and B is denoted AB, while the point where two non-parallel lines l and m meet is denoted lm. If three lines k, l and m are concurrent at a point A, we will sometimes write $A = klm$.

A metrical structure is determined by a non-degenerate symmetric 2×2 matrix D: this gives a symmetric bilinear form on vectors

$$v \cdot u \equiv vDu^T.$$

Non-degenerate means $\det D \neq 0$, and implies that if $v \cdot u = 0$ for all vectors u, then $v = 0$.

Two vectors v and u are then **perpendicular** precisely when $v \cdot u = 0$. Since the matrix D is non-degenerate, for any vector v there is, up to a scalar, exactly one vector u which is perpendicular to v. Two lines l and m are **perpendicular** precisely when they have perpendicular direction vectors.

The bilinear form determines the **quadrance** of a vector v as

$$Q(v) \equiv v \cdot v$$

and similarly the **quadrance** between points A and B is

$$Q(A,B) \equiv Q\left(\overrightarrow{AB}\right).$$

A vector v is **null** precisely when $Q(v) = v \cdot v = 0$, in other words precisely when v is perpendicular to itself. A line is **null** precisely when it has a null direction vector.

The **spread** between non-null vectors v and u is the number

$$s(v,u) \equiv 1 - \frac{(v\cdot u)^2}{Q(v)\,Q(u)} = 1 - \frac{(v\cdot u)^2}{(v\cdot v)\,(u\cdot u)}$$

and the **spread** between any non-null lines l and m with direction vectors v and u is defined to be $s(l,m) \equiv s(v,u)$.

1.2 Standard coordinates

This paper employs the novel approach to planar affine triangle geometry initiated in [7] and continued in [6], which allows us to frame the subject in a much wider and more general algebraic fashion, valid over an arbitrary field, not of characteristic two.

The basic idea with standard coordinates is to take any particular triangle, and apply a combination of a translation and an invertible linear transformation to send it to the **standard Triangle** $\overline{A_1A_2A_3}$ with

$$A_1 \equiv [0,0], \quad A_2 \equiv [1,0] \quad \text{and} \quad A_3 \equiv [0,1]. \tag{1}$$

Our convention is to use capital letters to refer to objects associated to this standard Triangle. The Lines of the Triangle are

$$l_1 \equiv A_2A_3 = \langle 1:1:-1\rangle,$$
$$l_2 \equiv A_1A_3 = \langle 1:0:0\rangle,$$
$$l_3 \equiv A_2A_1 = \langle 0:1:0\rangle.$$

The **Midpoints** of the Triangle are clearly

$$M_1 = \left[\frac{1}{2},\frac{1}{2}\right], \quad M_2 = \left[0,\frac{1}{2}\right], \quad M_3 = \left[\frac{1}{2},0\right]$$

while the corresponding **Median lines** are

$$d_1 \equiv A_1M_1 = \langle 1:-1:0\rangle,$$
$$d_2 \equiv A_2M_2 = \langle 1:2:-1\rangle,$$
$$d_3 \equiv A_3M_3 = \langle 2:1:-1\rangle.$$

The **Centroid** is the common meet of the Medians, namely

$$G = X_2 = \left[\frac{1}{3},\frac{1}{3}\right].$$

These objects are defined independent of any metrical structure: they are purely affine notions.

A metrical structure may be imposed by a general invertible 2×2 matrix

$$D \equiv \begin{pmatrix} a & b \\ b & c \end{pmatrix}. \tag{2}$$

We note that the determinant of D is $ac-b^2$. The quantity

$$d \equiv a+c-2b$$

will also prove to be useful.

Because the effect of a linear transformation on a bilinear form is the familiar congruence, it suffices to understand the particular standard Triangle with respect to such a general quadratic form. This is the basic, but powerful, idea behind standard coordinates. The idea now is to find all relevant information about the original triangle in terms of the corresponding information about the standard Triangle expressed in terms of the numbers a,b and c.

So we have moved from considering *a general triangle with respect to a specific bilinear form* to the more general situation of a *specific triangle with respect to a general quadratic form*. This system of standard coordinates allows a systematic augmentation of Kimberling's *Encyclopedia of Triangle Centers (*[3], [4], [5]) to *arbitrary quadratic forms* and *general fields*.

The **Midlines** m_1, m_2 and m_3 of the Triangle are the lines through the midpoints M_1, M_2 and M_3 perpendicular to the respective sides— these are usually called **perpendicular bisectors**. They are also the altitudes of $\overline{M_1M_2M_3}$ and are given by:

$$m_1 = \langle 2(b-a) : 2(c-b) : a-c\rangle,$$
$$m_2 = \langle 2b : 2c : -c\rangle,$$
$$m_3 = \langle 2a : 2b : -a\rangle.$$

The Midlines m_1, m_2, m_3 meet at the **Circumcenter**

$$C = X_3 = \frac{1}{2(ac-b^2)}[c(a-b), a(c-b)]. \tag{3}$$

The **Circumcircle** c of $\overline{A_1A_2A_3}$ is the unique circle with equation $Q(X,C) = R$ that passes through A_1, A_2 and A_3, and this turns out to be the equation in $X = [x,y]$ given by

$$ax^2 + 2bxy + cy^2 - ax - cy = 0. \tag{4}$$

The **Orthocenter** of the Triangle is

$$H = X_4 = \frac{b}{ac-b^2}[c-b, a-b].$$

The **Euler line** CG is

$$e = \left\langle 2b^2 - 3ab + ac : -2b^2 + 3cb - ac : b(a-c)\right\rangle. \tag{5}$$

The fact that this line passes through each of C, G and H can be checked by making the following computations via projective coordinates:

$$\left[c(a-b) : a(c-b) : 2\left(ac-b^2\right)\right]$$
$$\left[2b^2 - 3ab + ac : -2b^2 + 3cb - ac : b(a-c)\right]^T = 0,$$
$$[1:1:3]\left[2b^2 - 3ab + ac : -2b^2 + 3cb - ac : b(a-c)\right]^T = 0,$$
$$\left[b(c-b) : b(a-b) : ac-b^2\right]$$
$$\left[2b^2 - 3ab + ac : -2b^2 + 3cb - ac : b(a-c)\right]^T = 0.$$

The existence of Incenters of our standard Triangle however is more subtle: this leads to number theoretic conditions that depend on certain quantities being squares in our field.

2 The four Incenters

A **biline** of the non-null vertex $\overline{l_1 l_2}$ is a line b which passes through $l_1 l_2$ and satisfies $s(l_1, b) = s(b, l_2)$. The existence of Bilines (and hence Incenters) of the standard Triangle depends on number theoretical considerations of a particularly simple kind which we recall from [7].

Theorem 5 (Existence of Triangle bilines) *The Triangle $\overline{A_1A_2A_3}$ has Bilines at each vertex precisely when we can find numbers u, v, w in the field satisfying*

$$ac = u^2, \quad ad = v^2, \quad cd = w^2. \tag{6}$$

In this case we can choose u, v, w so that $acd = uvw$ and

$$du = vw \quad cv = uw \quad \text{and} \quad aw = uv. \tag{7}$$

We are interested in formulas for triangle centers of the standard Triangle $\overline{A_1A_2A_3}$, assuming the existence of Bilines. These formulas will then involve the entries a, b and c of D from (2), as well as the secondary quantities u, v and w. The **quadratic relations** (6 and 7) play a major role in simplifying formulas.
The four Incenters are, from [7],

$$I_0 = \frac{1}{d+v-w}[-w, v], \quad I_1 = \frac{1}{d-v+w}[w, -v],$$
$$I_2 = \frac{1}{d+v+w}[w, v], \quad I_3 = \frac{1}{d-v-w}[-w, -v].$$

It is important to note that I_1, I_2 and I_3 may be obtained from I_0 by changing signs of: both v and w, just w, and just v respectively. This four-fold symmetry will hold more generally and it means that we can generally just record the formulas for objects which are associated to I_0. We refer to this as the **basic u, v, w symmetry**.

2.1 Incenter midpoints

We now look at meets of Midlines and the Circumcircle. Somewhat surprisingly, it turns out that the existence of these meets is entirely aligned with the existence of Incenters.

Theorem 6 (Incenter midpoints) *The three Midlines m_1, m_2 and m_3 meet the Circumcircle c precisely when Incenters exist, that is when we can find u, v and w satisfying the quadratic relations. In this case, the Midline m_1 meets the Circumcircle in points*

$$M_{01} \equiv \frac{1}{2(b-u)}[c-u, a-u], \quad M_{23} \equiv \frac{1}{2(b+u)}[c+u, a+u]$$

which are the midpoints of $\overline{I_0I_1}$ and $\overline{I_2I_3}$ respectively; the Midline m_2 meets the Circumcircle in points

$$M_{13} \equiv \frac{1}{2(b-a+v)}[c, v-a], \quad M_{02} \equiv \frac{1}{2(a-b+v)}[-c, v+a]$$

which are the midpoints of $\overline{I_1I_3}$ and $\overline{I_0I_2}$ respectively; and the Midline m_3 meets the Circumcircle in points

$$M_{03} \equiv \frac{1}{2(b-c+w)}[w-c, a], \quad M_{12} \equiv \frac{1}{2(c-b+w)}[w+c, -a]$$

which are the midpoints of $\overline{I_0I_3}$ and $\overline{I_1I_2}$ respectively.

Proof. The proofs of these are straightforward, as we have the equations of the Midlines and the Circumcircle c, and finding midpoints of a segment just involves taking the averages of the coordinates. However we must be prepared to use the quadratic relations to make simplifications. □

This theorem motivates us to call the points M_{ij} the **Incenter midpoints** of the Triangle.

2.2 Incenter Euler lines

For each Incenter triangle $\overline{A_jA_kI_l}$ we may now compute its Euler line, which we call an **Incenter Euler** line of the original triangle $\overline{A_1A_2A_3}$. This may be done by joining the circumcenter of the Incenter triangle, which is an Incenter midpoint, to the centroid of that Incenter triangle, whose coordinates are just formed by taking affine averages of the points of the given triangle.
For example the Euler line e_{30} of $\overline{A_1A_2I_0}$ is the join of the Incenter midpoint

$$M_{03} = \frac{1}{2(b-c+w)}[w-c, a] = [w-c : a : 2(b-c+w)]$$

and the centroid

$$\frac{1}{3}\left[\frac{d+v-2w}{d+v-w}, \frac{v}{d+v-w}\right] = [d+v-2w : v : 3(d+v-w)].$$

Using a Euclidean cross product and simplifying using the quadratic relations, we find that

$$e_{30} = \left\langle \begin{array}{c} 6ab-3ac+2au-3av-4bu+3aw+2bv+2cu-2cv-3a^2 : \\ au-2bc-2ab-2bu+aw-2bv+cu+2bw-cv+4b^2 : \\ ac-2ab-au+av+2bu-2aw-cu+cv+a^2 \end{array} \right\rangle.$$

Note that we can obtain e_{i1}, e_{i2}, e_{i3} from e_{i0} by changing the signs of (v, w), (u, w) and (u, v) respectively. So for example by applying the basic u, v, w symmetry we find that

$$e_{31} = \left\langle \begin{array}{c} 6ab-3ac+2au+3av-4bu-3aw-2bv+2cu+2cv-3a^2 : \\ au-2bc-2ab-2bu-aw+2bv+cu-2bw+cv+4b^2 : \\ ac-2ab-au-av+2bu+2aw-cu-cv+a^2 \end{array} \right\rangle.$$

So it suffices if we exhibit also

$$e_{20} = \left\langle \begin{array}{c} au-2bc-2ab-2bu+aw-2bv+cu+2bw-cv+4b^2 : \\ 6bc-3ac+2au-4bu+2aw+2cu-2bw-3cv+3cw-3c^2 : \\ ac-2bc-au+2bu-aw-cu+2cv-cw+c^2 \end{array} \right\rangle$$

and

$$e_{10} = \left\langle \begin{array}{c} 8ab-3ac+2bc+au-3av-2bu+2aw \\ +4bv+cu-2bw-cv-3a^2-4b^2 : \\ 3ac-2ab-8bc-au+2bu-aw-2bv \\ -cu+4bw+2cv-3cw+4b^2+3c^2 : \\ (a-c)(a-2b+c+v-w) \end{array} \right\rangle .$$

3 Schiffler points

The Incenter Euler lines also figure prominently in the classical Schiffler point. We will now see that there is in fact a four-fold symmetry inherent here.

Theorem 7 (Four Schiffler points) *The triples $S_0 \equiv e_{10}e_{20}e_{30}$, $S_1 \equiv e_{11}e_{21}e_{31}$, $S_2 \equiv e_{12}e_{22}e_{32}$ and $S_3 \equiv e_{13}e_{23}e_{33}$ of Incenter Euler lines are concurrent. These points all lie on the Euler line.*

Proof. The concurrences of the lines e_{10}, e_{20}, e_{30} is

$$S_0 = \left[\begin{array}{c} \left(2a^2-5ab+6ac+2b^2-7bc+2c^2\right)u \\ -c\,(5a-5b+2c)+\left(5ac-3ab-2bc+2a^2\right)w \\ -c\left(-10ab+5ac-2bc+5a^2+2b^2\right) : \\ \left(2a^2-7ab+6ac+2b^2-5bc+2c^2\right)u \\ +\left(2ab-5ac+3bc-2c^2\right)v+a\,(5c-5b+2a)\,w \\ +a\left(2ab-5ac-2b^2+10bc-5c^2\right) : \\ \left(6a^2-15ab+16ac+2b^2-15bc+6c^2\right)u \\ +\left(4b^2+9bc-6c^2-13ac\right)v \\ +\left(6a^2-9ab+13ca-4b^2\right)w \\ +\left(4ab^2-13a^2c+22abc-13ac^2-4b^3+4b^2c\right) \end{array} \right].$$

The other three Schiffler points S_1, S_2 and S_3 may be computed to be exactly the corresponding points when we perform the three basic u, v, w symmetries, namely negating v and w to get S_1, negating u and w to get S_2, and negating u and v to get S_3. □

The Euler line e we know is (5), so we can check directly that $eS_0 = 0$ identically, without use of the quadratic relations. The statement also holds for the other Schiffler points.

In Figure 6 we see an example from green geometry with the bilinear form $x_1y_2 + x_2y_1$, showing the four Schiffler points of the triangle $\overline{A_1A_2A_3}$ on the green Euler line e (for more about chromogeometry and geometry in Lorentz spaces see for example [13], [14]).

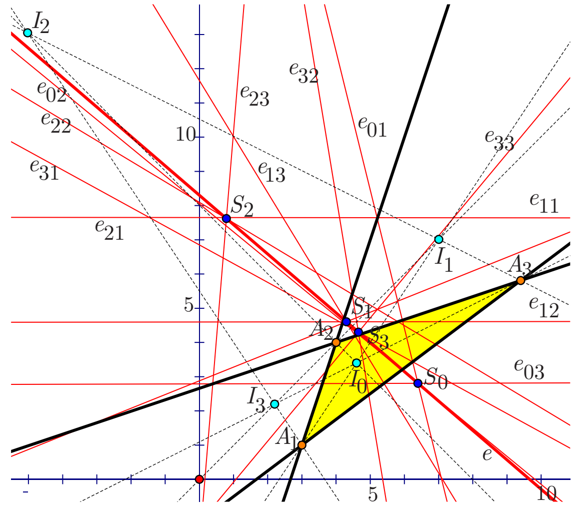

Figure 6: *Green Schiffler points lying on the green Euler line of $\overline{A_1A_2A_3}$*

4 Incenter Euler points

Theorem 8 (Four Incenter Euler points) *The triples $P_0 \equiv e_{11}e_{22}e_{33}$, $P_1 \equiv e_{10}e_{23}e_{32}$, $P_2 \equiv e_{20}e_{13}e_{31}$, and $P_3 \equiv e_{30}e_{12}e_{21}$ of Euler lines are concurrent. These points all lie on the Circumcircle c of the original triangle.*

Proof. The proof requires using the quadratic relations involving u, v and w. For example to show the concurrency $P_0 \equiv e_{11}e_{22}e_{33}$ we create the determinant of the 3×3 matrix with rows given by the Euler lines. This expression is a polynomial of degree six in a, b, c and u, v and w. By successive applications of the quadratic relations involving u, v and w we can step by step reduce this polynomial until it eventually equals 0. Alternatively we can use the cross product to determine the common meets of these lines: here is the formula for P_0:

$$P_0 = \left[\begin{array}{c} \left(2b^2-5bc-ab+2c^2+2ac\right)u+\left(ac-3bc+2c^2\right)v \\ +(3ac-ab-2bc)\,w+c\left(a^2-4ab+3ac+2b^2-2bc\right) : \\ b\,(2b-c-a)\,u+c\,(a-b)\,v+a\,(c-b)\,w \\ +a\left(2b^2-ac-2bc+c^2\right) : \\ b\,(2b-c-a)\,u+c\,(a-b)\,v+\left(5ac-ab-4b^2\right)w \\ +\left(a^2c-6abc+5ac^2+4b^3-4b^2c\right) \end{array} \right].$$

The formulas for P_1, P_2 and P_3 follow by the basic u, v, w symmetry. The Circumcircle c of the standard Triangle we know has equation $ax^2 + 2bxy + cy^2 - ax - cy = 0$. By substitution, we find, after using the quadratic relations, that P_0 satisfies this equation, and the other points are similar. □

We will call the points P_0, P_1, P_2 and P_3 the **Incenter Euler points** of the triangle.

4.1 Lines of the Incenter Euler quadrangle $\overline{P_0P_1P_2P_3}$

The lines of the Incenter Euler quadrangle have the following equations:

$$P_0P_1 = \left\langle \begin{array}{c} c\left(ab-a^2-2ac+2b^2\right)v+a\left(ab-3ca+2b^2\right)w: \\ c\left(2b^2+bc-3ac\right)v+a\left(2b^2-2ac+bc-c^2\right)w: \\ c\left(a^2+3ac-3ba-bc\right)v+a\left(3ac-ba+c^2-3bc\right)w \end{array} \right\rangle,$$

$$P_2P_3 = \left\langle \begin{array}{c} 3a^2cw-2ac^2v-a^2bw-a^2cv-2ab^2w+2b^2cv+abcv: \\ ac^2w-3ac^2v-2ab^2w+bc^2v+2a^2cw+2b^2cv-abcw: \\ 3ac^2v+a^2bw+a^2cv-ac^2w-bc^2v-3a^2cw-3abcv+3abcw \end{array} \right\rangle,$$

$$P_0P_2 = \left\langle \begin{array}{c} 4b^3u-4ab^2u+a^2bu+2ac^2u+2a^2cu+2ab^2w \\ +a^2bw-2b^2cu-3a^2cw-3abcu: \\ -(c-2b)\left(abu-2b^2u+abw+bcu-acw\right): \\ -(a-c)\left(abu-2b^2u+abw+bcu-acw\right) \end{array} \right\rangle,$$

$$P_1P_3 = \left\langle \begin{array}{c} 4b^3u-4ab^2u+a^2bu+2ac^2u+2a^2cu-2ab^2w \\ -a^2bw-2b^2cu+3a^2cw-3abcu: \\ -(c-2b)\left(abu-2b^2u-abw+bcu+acw\right): \\ -(a-c)\left(abu-2b^2u-abw+bcu+acw\right) \end{array} \right\rangle,$$

$$P_0P_3 = \left\langle \begin{array}{c} (a-2b+2c+2w)\left(abu-2b^2u-acv+bcu+bcv\right): \\ (a-2b+c)\left(2ac-13bc+2b^2+10c^2\right)u+ \\ c\left(-4ab+7ac-23bc+10b^2+10c^2\right)v: \\ (a-c)\left(2b^2u+2c^2u+2c^2v-abu+\right. \\ \left. 2acu+acv-5bcu-3bcv\right) \end{array} \right\rangle,$$

$$P_1P_2 = \left\langle \begin{array}{c} (a-2b+2c-2w)\left(abu-2b^2u+acv+bcu-bcv\right): \\ (a-2b+c)\left(2ac-13bc+2b^2+10c^2\right)u+ \\ c\left(4ab-7ac+23bc-10b^2-10c^2\right)v: \\ (a-c)\left(2b^2u+2c^2u-2c^2v-abu+\right. \\ \left. 2acu-acv-5bcu+3bcv\right) \end{array} \right\rangle.$$

4.2 Diagonal points of the Incenter Euler quadrangle $\overline{P_0P_1P_2P_3}$

Remarkably, the diagonal points of the Incenter Euler quadrangle $\overline{P_0P_1P_2P_3}$ have a particularly simple form, and in fact generally lie on the lines of the original triangle!

Theorem 9 *If $a^2c-ab^2-2abc+ac^2+2b^3-b^2c \neq 0$ and $c-2b \neq 0$ and $a-2b \neq 0$ and $a \neq c$, then the diagonal points of the quadrangle $\overline{P_0P_1P_2P_3}$ are*

$$D_1 \equiv (P_0P_1)(P_2P_3) = \left[\frac{a-2b}{a-c}, \frac{2b-c}{a-c}\right],$$

$$D_2 \equiv (P_0P_2)(P_1P_3) = \left[0, \frac{a-c}{2b-c}\right],$$

$$D_3 \equiv (P_0P_3)(P_1P_2) = \left[\frac{a-c}{a-2b}, 0\right],$$

which lie on the lines L_1, L_2 and L_3 respectively.

Proof. These are calculations that rely on the previous formulas for the Incenter Euler quadrangle lines, and involve simplifications using the quadratic relations, as well as cancellation of the terms that appear in the conditions of the theorem. □

We call D_1, D_2 and D_3 the **Diagonal Incenter Euler points** of the Triangle, and $\overline{D_1D_2D_3}$ the **Diagonal Incenter Euler triangle** of the Triangle $\overline{A_1A_2A_3}$. These two triangles, shown in Figure 7, have a remarkable relationship!

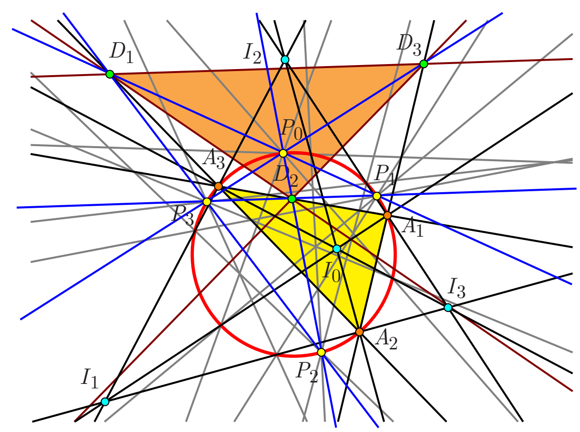

Figure 7: *The Diagonal Incenter Euler triangle $\overline{D_1D_2D_3}$ of the Triangle $\overline{A_1A_2A_3}$*

Theorem 10 *The signed area of the oriented Diagonal Incenter Euler triangle $\overrightarrow{D_1D_2D_3}$ is negative two times the signed area of the oriented original Triangle $\overrightarrow{A_1A_2A_3}$.*

Proof. This is a consequence of the formulas above for D_1, D_2 and D_3, together with the identities

$$\det\begin{pmatrix} \frac{a-2b}{a-c} & \frac{2b-c}{a-c} & 1 \\ 0 & \frac{a-c}{2b-c} & 1 \\ \frac{a-c}{a-2b} & 0 & 1 \end{pmatrix} = -2$$

and

$$\det\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} = 1.$$ □

Theorem 11 *The orthocenter of the Diagonal Incenter Euler triangle $\overline{D_1D_2D_3}$ is the Circumcenter C of the original Triangle $\overline{A_1A_2A_3}$.*

Proof. It is a straightforward calculation to show that the Circumcenter of $\overline{A_1A_2A_3}$ given by (3) is indeed also the orthocenter of $\overline{D_1D_2D_3}$. □

5 The Incenter Euler transformation

We may now define a transformation Γ at the level of triangles, where $\Gamma\left(\overline{A_1A_2A_3}\right)$ is the Diagonal Incenter Euler triangle $\overline{D_1D_2D_3}$. This gives a canonical second triangle associated to a given triangle, with one vertex of the new triangle on each of the lines of the original, where the signed area is multiplied by -2, and where the Circumcenter of the original Triangle becomes the orthocenter of the new triangle.

But now this transformation Γ allows one to transfer whole-scale triangle centers from $\overline{A_1A_2A_3}$ to $\overline{D_1D_2D_3}$. Generally every triangle center of $\overline{A_1A_2A_3}$ will then play a distinguished triangle center role for $\overline{D_1D_2D_3}$. Conceivably there are some particular exceptions, such as when one of the factors $a^2c-ab^2-2abc+ac^2+2b^3-b^2c$ or $c-2b\neq 0$ or $a-2b\neq 0$ or $a\neq c$ is zero.

This implies that Kimberling's list may well have a hall-of mirrors aspect, where once we identify a triangle center say X_i we consider the corresponding point for $\overline{A_1A_2A_3}$ to be a possibly new X_j of $\overline{D_1D_2D_3}$. This gives a natural mapping of Kimberling's list to itself. It seems an interesting question to identify what points go to what points. Could a computer be programmed to answer this question?

6 Incenter Euler line meets on the Lines

We have seen that the Incenter Euler lines meet at Incenter Midpoints (six) , at Incenter Euler points (four) and at Schiffler points (four). But there is more.

Theorem 12 *The Incenter Euler lines also meet at twelve points on the original Lines of the triangle, with four such meets on each Line.*

Proof. The calculation of these points are straightforward, the meets are, using projective coordinates:

$$e_{10}e_{13}=\begin{bmatrix} 0:(a-c)\,(du+(b-c)\,v): \\ 3\,(b-c)\,(a-2b+c)\,u+ \\ \left(2b^2-6bc+3c^2+ac\right)v \end{bmatrix},$$

$$e_{10}e_{12}=\begin{bmatrix} (a-c)\,(du+(a-b)\,w):0: \\ 3a^2u+6b^2u+3a^2w+2b^2w-9abu+3acu \\ -6abw-3bcu+acw \end{bmatrix},$$

$$e_{11}e_{12}=\begin{bmatrix} 0:-(a-c)\,(du-(b-c)\,v): \\ 6b^2u+2b^2v+3c^2u+3c^2v-3abu+3acu \\ +acv-9bcu-6bcv \end{bmatrix},$$

$$e_{11}e_{13}=\begin{bmatrix} (a-c)\,(du-(a-b)\,w):0: \\ 3a^2u+6b^2u-3a^2w-2b^2w-9abu+3acu \\ +6abw-3bcu-acw \end{bmatrix},$$

$$e_{20}e_{21}=\left[c\,(aw-bv+bw-cv):0:2b^2w+acw-3bcv\right],$$

$$e_{22}e_{23}=\left[c\,(aw+bv+bw+cv):0:2b^2w+acw+3bcv\right],$$

$$e_{20}e_{23}=\begin{bmatrix} -c\,(au-3bu+bv+2cu-2cv): \\ (c-2b)\,(bu-cu+cv): \\ -\left(2b^2u+3c^2u-3c^2v+acu-6bcu+3bcv\right) \end{bmatrix},$$

$$e_{21}e_{22}=\begin{bmatrix} c\,(au-3bu-bv+2cu+2cv): \\ (c-2b)\,(cu-bu+cv): \\ 2b^2u+3c^2u+3c^2v+acu-6bcu-3bcv \end{bmatrix},$$

$$e_{30}e_{31}=\left[0:a\,(aw-bv+bw-cv):-2b^2v+3abw-acv\right],$$

$$e_{32}e_{33}=\left[0:(aw+bv+bw+cv):2b^2v+3abw+acv\right],$$

$$e_{31}e_{33}=\begin{bmatrix} (a-2b)\,(bu-au+aw): \\ -a\,(2au-3bu-2aw+cu+bw): \\ -\left(3a^2u+2b^2u-3a^2w-6abu+acu+3abw\right) \end{bmatrix},$$

$$e_{30}e_{32}=\begin{bmatrix} (a-2b)\,(au-bu+aw): \\ a\,(2au-3bu+2aw+cu-bw): \\ 3a^2u+2b^2u+3a^2w-6abu+acu-3abw \end{bmatrix}.$$

□

7 The mystery of apparent symmetry breaking

There is another very intriguing aspect of this entire story that invites further exploration. The lines of the Diagonal Incenter Euler triangle $\overline{D_1D_2D_3}$ of the Triangle $\overline{A_1A_2A_3}$ with Incenters I_0, I_1, I_2 and I_3 can be easily computed to be

$$D_1D_2=[a+2b-2c:2b-c:c-a]$$
$$D_2D_3=[a-2b:2b-c:c-a]$$
$$D_1D_3=[2b-a:2b-2a+c:a-c].$$

It is first of all remarkable that the formulas for these lines are simple linear expressions in the numbers a,b and c of the matrix for the bilinear form. In Figure 7 we notice that the line D_2D_3 appears to pass through I_3, but the other two lines D_1D_2 and D_1D_3 do not pass through any of the other Incenters. If this were true, it would imply a completely remarkable, even seemingly impossible, symmetry breaking.

Why should the Incenter I_3 be singled out in this fashion? This very curious situation may at first confound the experienced geometer, as it did us when we first observed it. The reader might enjoy creating such a diagram and determining to what extent this phenomenon holds, and trying to find an explanation of it. We will address this challenge in a future paper.

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