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Special Conics in a Hyperbolic Plane

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ABSTRACT

In Euclidean geometry we find three types of special conics, which are distinguished with respect to the Euclidean similarity group: circles, parabolas, and equilateral hyperbolas. They have on one hand special elementary geometric properties (c.f. [7]) and on the other they are strongly connected to the "absolute elliptic involution" in the ideal line of the projectively enclosed Euclidean plane. Therefore, in a hyperbolic plane (h-plane) – and similarly in any Cayley-Klein plane – the analogue question has to consider projective geometric properties as well as hyperbolic-elementary geometric properties. It turns out that the classical concepts "circle", "parabola", and "(equilateral) hyperbola" do not suit very well to the many cases of conics in a hyperbolic plane (c.f. e.g. [10]). Nevertheless, one can consider conics in a h-plane systematicly having one ore more properties of the three Euclidean special conics. Place of action will be the "universal hyperbolic plane" π, i.e., the full projective plane endowed with a hyperbolic polarity ruling distance and angle measure.

Key words: conic section, hyperbolic plane, Thales conic, equilateral hyperbola

MSC2010: 51M09

Specijalne konike u hiperboličnoj ravnini

SAŽETAK

U euklidskoj ravnini s obzirom na euklidsku grupu simetrija razlikujemo tri tipa specijalnih konika: kružnice, parabole i specijalne hiperbole. S jedne strane, one imaju specijalno euklidsko svojstvo (vidi [7]), a s druge su strane čvrsto vezane uz apsolutnu eliptičnu involuciju na idealnom pravcu projektivno proširene euklidske ravnine. Zbog toga, u hiperboličnoj ravnini (h-ravnini) – i slično u svakoj Cayley-Kleinovoj ravnini – treba promatrati i projektivna geometrijska svojstva i elementarno-hiperbolična geometrijska svojstva. Pokazuje se da u brojnim slučajevima konika u hiperboličnoj ravnini klasični koncepti "kružnica", "parabola" i "(jednakostranična) hiperbola" nisu primjenjivi (vidi npr. [10]). Unatoč tome, moguće je sustavno promatranje konika u h-ravnini koje imaju jedno ili više svojstava triju euklidskih specijalnih konika. Proučavanje će se vršiti na "univerzalnoj hiperboličnoj ravnini" π, tj. projektivnoj ravnini u kojoj su udaljenost i mjera kuta definirani apsolutnim polaritetom.

Ključne riječi: konika, hiperbolična ravnina, Talesova konika, jednakostranična hiperbola

1 Introduction

We consider conics in a hyperbolic plane (h-plane) having one ore more properties of the three Euclidean special conics "circle", "parabola" and "equilateral hyperbola". In the projectively enclosed and complexified Euclidean plane circles are conics passing through the (complex conjugate) absolute points I,J on the ideal line u of that plane, parabolas touch this absolute line u, and equilateral hyperbolas intersect the absolute line u in points harmonic to I,J. Besides these projective geometric properties, the three special conics have many Euclidean properties and generations.

Circles are e.g. generated as distance curves of a point, the midpoint, but they are also generated by directly congruent pencils of lines, what results in the remarkable inscribed angle theorem and the theorem of Thales as its special case. Parabolas are e.g. generated as envelope of a leg of a right angle hook sliding along a line, while the other leg passes through a point. The fixed line and point turns out to be vertex tangent and focus of the generated parabola. As a Euclidean conic, a parabola is of course also defined via the Apollonius definition of a conic.

An equilateral hyperbola has orthogonal asymptotes. It is (directly and indirectly) congruent to its conjugate hyperbola, it is generated by indirectly congruent pencils of lines. But the most strange property is that each triangle of points on the hyperbola has its orthocentre on this equilateral hyperbola. The pencil of conics with the vertices of a triangle and its orthocentre as base points consists only of equilateral hyperbolas; (the singular conics are the pairs consisting of the altitude and the corresponding side of the given triangle).

In Figure 1, these three Euclidean cases and their well-known main properties are visualised symbolicly:

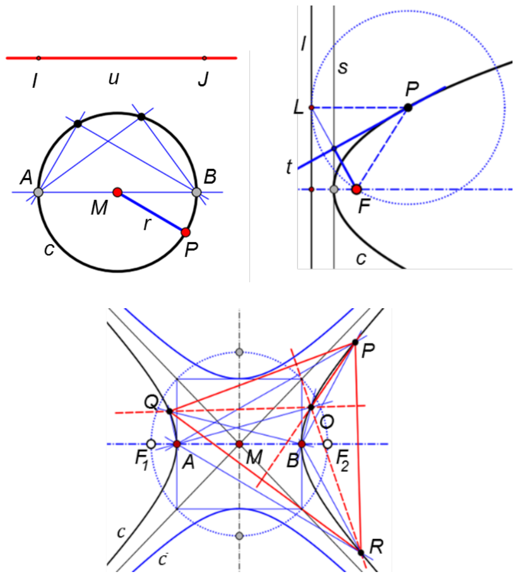

Figure 1: *(Symbolic) visualisation of projective and metric properties of a Euclidean circle, a parabola, and an equilateral hyperbola*

In the following, we shall study conics in a hyperbolic plane being defined by one of the mentioned projective and Euclidean properties. Place of action will be the "universal hyperbolic plane" π, i.e. the full real projective plane endowed with a hyperbolic polarity, see e.g. [15]. This "absolute (regular) hyperbolic polarity" is usually given by the real conic ω and rules orthogonality, h-distance, and h-angle measure. F. Klein's point of view considers h-geometry as a sub-geometry of projective geometry while a puristical point of view allows only the inner domain of ω for being a proper h-plane. In the following, we try to have both points of view in mind, but we will use F. Klein's projective geometric model of a h-plane.

The standard graphic representation of ω is that of a Euclidean circle and this allows us to use e.g. the graphics software "Cinderella" (see [11]), which has the feature "(planar) hyperbolic geometry".

As a first and well-known example we consider circles: In a hyperbolic plane with the (real) "absolute conic" ω a circle c is a conic touching ω twice in algebraic sense (what means disregarding reality and coincidence of the touching points). This projective geometric approach results already in three types of hyperbolic circles: (1) proper circles touching ω in a pair of complex conjugate points, (2) limit circles which hyperosculate ω, and (3) distance circles touching ω in a pair of real distinct points. While the elementary Euclidean definition of a circle as the planar set of points having equal distance from a centre point, the analogue in hyperbolic geometry is true only for h-circles of type (1) and can be modified for h-circles of type (3). For h-circles of type (3) the radius length is not finite, a property, which connects this type rather with Euclidean parabolas than with circles. Euclidean circles can be generated via directly congruent pencils of lines, which expresses the property of a constant angle at circumference and especially the property of Thales. In an h-plane two pencils of orthogonal lines with proper base points generate the so-called "Thales conic" (resp. "Thaloid", as it is called by N. J. Wildberger, see [15]), which is never an h-circle of type (1) and (2), while h-circles of type (3) occur if, and only if the vertices of the h-orthogonal pencils both are ideal points on ω.

Conics with the properties of a Euclidean parabola are treated in [1], where the place of action again is the "universal hyperbolic plane" π. But, for the sake of completeness, we also repeat some of the details here.

A great part of this article will deal with h-conics derived from properties of the Euclidean equilateral hyperbola following the above presented systematic treatment for circles. This results in two special sets of h-conics, the set of "h-equilateral conics" having a harmonic quadrangle of ideal points and the set of h-conics defined by the property that each triangle of conic points has its h-orthocentre also on this conic.

2 Projective geometric classification of h-conics

A given conic c, together with ω, defines as well a pencil of conics $p \cdot c + q \cdot \omega$ as well as a dual pencil of (dual) conics and we distinguish 5 different types of pencils according to the sets of singular conics resp. the sets of common base points and base tangents (see Figures 2 and 3).

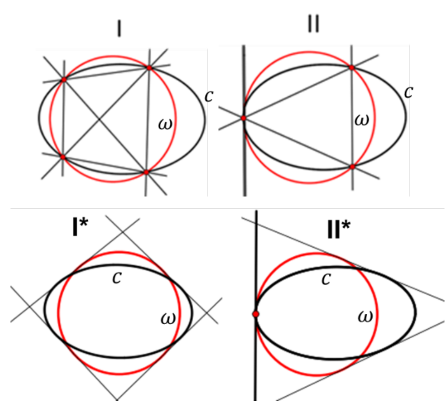

Figure 2: *Conic pencils I and II and its dual pencils I* and II**

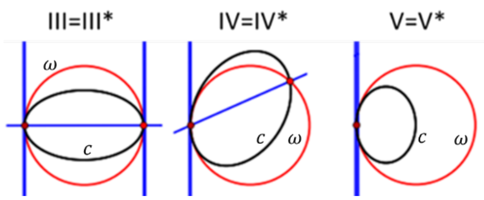

Figure 3: *The self-dual pencils of conics III, IV, and V*

If we take the reality of base points or base lines into account and consider the pencils I and I*, then we have three subcases each, the pencils of type II, II* and III have two such subcases each and there is only one case of pencils of type IV and V. A further distinction can be made concerning the reality of the common polar triangle of c and ω, which acts as the h-midpoint triangle of c: An h-conic can have either one or three real midpoints.

An overview of all possible cases can be found in [5], [6], [9], and [10]. This classification shows that the Euclidean names "ellipse", "parabola" and "hyperbola" need strong modifications (as e.g. "semi-hyperbola", "convex resp. concave hyperbolic parabola" and so on) to express the type of an h-conic, which becomes obvious by its visualisation in some model of the h-plane π.

Pencils $p \cdot c + q \cdot \omega$ of the two subtypes III define h-circles c. They have the well-known property of being distance loci of either points or lines, see Figure 4. h-Conics to type V have no finite radius lenght and are called "limit h-circles" or "horocycles". But as they have similar properties as Euclidean parabolas they can also be considered as special cases of h-parabolas, Figure 5.

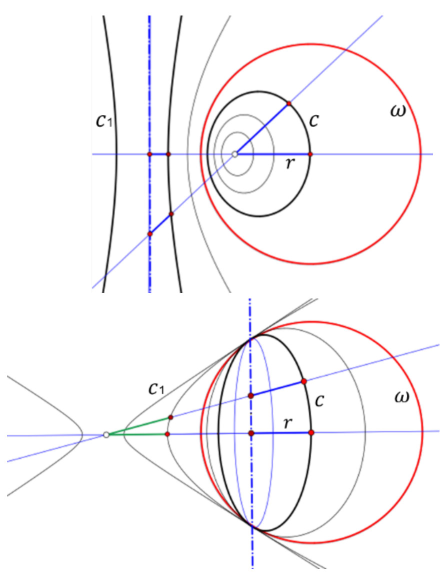

Figure 4: *The different types of h-circles within concentric pencils of h-circles*

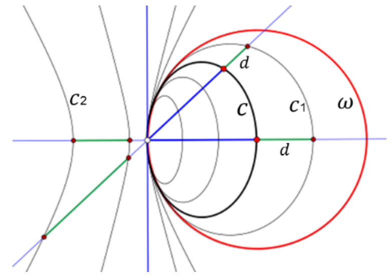

Figure 5: *Pencil of "horocycles" showing the property of Euclidean parabolas, which are translated along their common axis*

Conics c defining pencils of type II and IV can be considered as analogs to Euclidean parabolas and they will be studied in Chapter 5 with respect to their h-metric properties.

Conics c defining pencils of type I are "h-hyperbolas", "semi-hyperbolas" or "h-ellipses" according to the reality of the pencil's base points, which furtheron will be called the "ideal points" of c.

3 Classification with respect to h-orthogonality

We start with an h-conic c, which, together with ω defines a pencil $p \cdot c + q \cdot \omega$ of type I. The quadrangle of its ideal points can be special with respect to the h-orthogonality structure defined by the absolute polarity to ω. There might be h-orthogonal pairs of opposite sides of this base quadrangle (Figure 6). For dual pencils $p \cdot c^* + q \cdot \omega^*$ one can find similar special cases, see Figure 7. Quadrangles (resp. quadrilaterals) with this special property are called "harmonic quadrangles" (resp. "harmonic quadrilaterals") as their (non-trivial) symmetry group is generated by harmonic homologies.

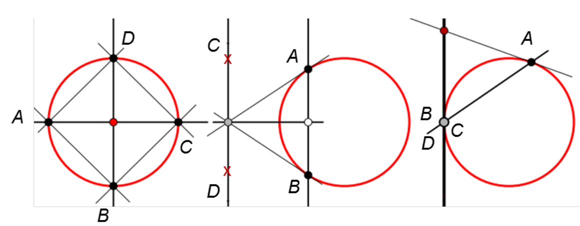

Figure 6: *Pencils $p \cdot c + q \cdot \omega$ with h-orthogonality properties of the base quadrangle (and its degenerate case pencil IV)*

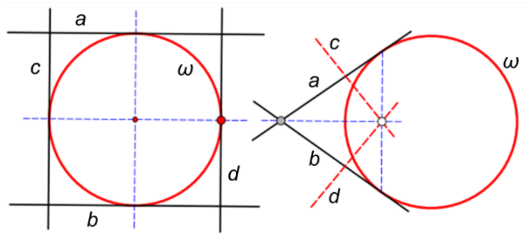

Figure 7: *Dual pencils $p \cdot c^* + q \cdot \omega^*$ of type I^* with h-orthogonality properties of the base quadrilateral*

Figure 6 shows that there occur only h-hyperbolas and semi-hyperbolas and, as a degenerate case, there are h-parabolas, but no h-ellipses. The dual case (Figure 7) contains h-ellipses c, when seen as point conics. It makes sense to define an h-conic possessing a real harmonic ideal quadrangle as "h-equilateral hyperbola" and we will have a closer look to these h-conics in Chapter 6.

4 h-Conics with metric properties of Euclidean circles

We start with the concept of a Euclidean circle and its different Euclidean generations. We have already mentioned that projective h-conics of type III are also a h-distance circles (Figure 4). But they have neither the Thales-property nor the property of the constant angle at circumference. The generation of a conic by h-orthogonal pencils delivers the "Thales conic" x over a segment $[A,B]$, which turns out to be one of the axes of the conic (Figure 8). There is one exeption: Thales-conics over a segment $[A,B]$, $A,B \in \omega$, are h-circles and their radius turns out to be $1/\sqrt{2}$, (Figure 9).

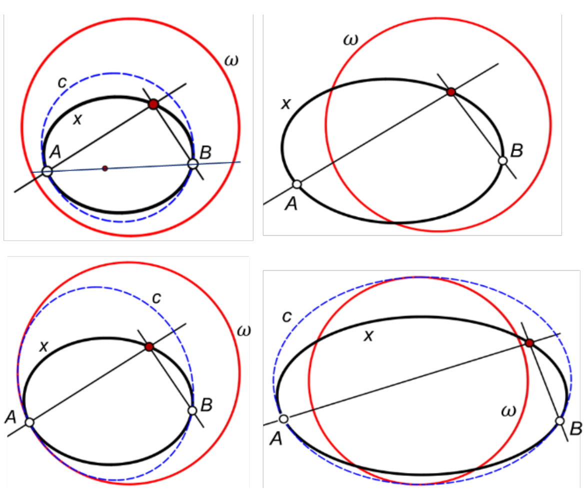

Figure 8: *Different cases of Thales-conics x generated by two h-orthogonal pencils of lines*

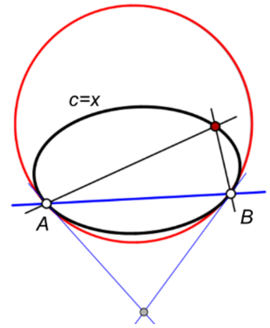

Figure 9: *The exceptional case of a Thales-conic over a segment with endpoints on the absolute conic ω in a h-circle of radius $1/\sqrt{2}$*

Connecting the construction of a Thales-conic x with a kinematic mechanism allows us to construct of points and tangents of x, see [14] and Figure 10.

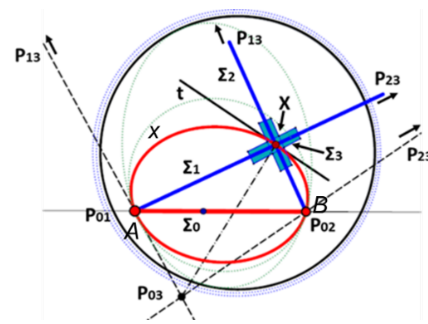

Figure 10: *Kinematic generation of a Thales-conic x applied to construct its tangents*

Arbitrarily chosen direct congruent pencils of lines generate a conic, too. It is simply the Steiner generation of a conic by projective pencils, but this delivers no h-circles, see Figure 11, and it has not the property of inscibed angle theorem either!

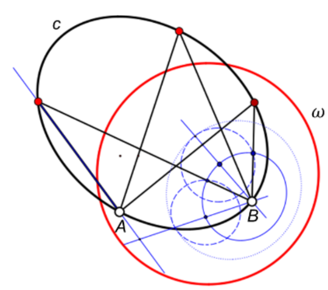

Figure 11: *Conic generated by two directly congruent pencils of lines*

Curves defined by a constant angle at circumference different from a h-right angle, socalled "isoptics of a segment", turn out to be algebraic of degree four! To visualize this one can consider the inverse motion, namely to keep the angle and its legs fixed and move the segment. In the Euclidean plane this motion is the well-known *ellipse motion*, as all points which are fixed connected with the moving system and which are different from the points on the legs of the fixed angle have ellipses as orbits. In the h-plane these orbits turn out to be of order four, too, see Figure 12.

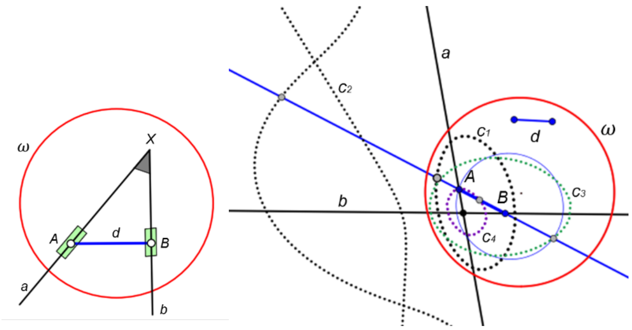

Figure 12: *The h-analog of the Euclidean ellipse motion delivers orbits c_i of degree 4*

Another way to visualise this motion is to start with the Thales-motion and consider the envelop of a line fixed to Thales's right angle hook and passing through its vertex, Figure 13. In the Euclidean case, the envelope is a point of the Thales-circle, in the h-case it is a curve of degree 6.

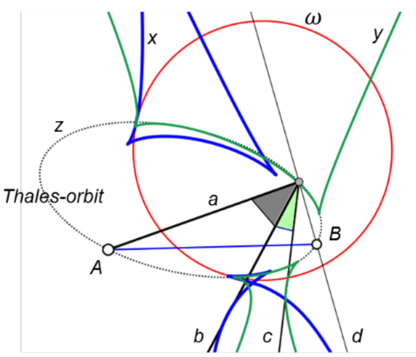

Figure 13: *Moving a fixed angle hook along a Thales-conic such that one leg passes through a bases point A of Thales's construction, the 2^{nd} leg envelops a curve of degree 6.*

5 h-Conics with metric properties of Euclidean parabolas

In this chapter, we strongly refer to [1]. From the projective geometric point of view, we have to distinguish h-parabolas of type II, IV, and V, the latter having also properties of a circle. If one considers the analog of the Euclidean slider crank, there occur h-conics, but they are (in general) not projective h-parabolas, (Figure 14). The proof for the fact that the envelop of the second leg t of the crank slider is a h-conic x is trivial: The line t connects two projectively correlated point series, namely s and the absolute polar line f of F. This line f is also the second vertex tangent of the h-conic x.

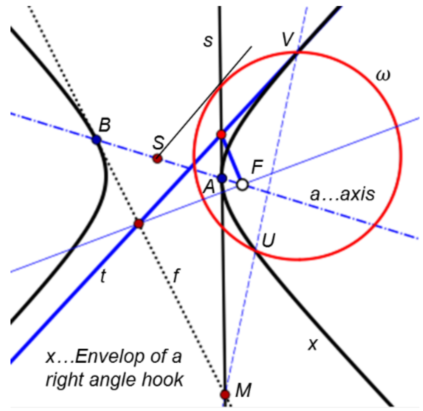

Figure 14: *The h-analog of the Euclidean crank slider motion, defines a conic x with focus F and vertex tangent s.*

Also the h-analog to the construction of a parabola according to Apollonius's definition does in general not deliver h-parabolas in the projective geometric sense, see Figure 15. (Proof: x is Steiner-generated by two projective pencils of lines with centre F, the focus of x, and the absolute Pole L of the directrix line l of x.)

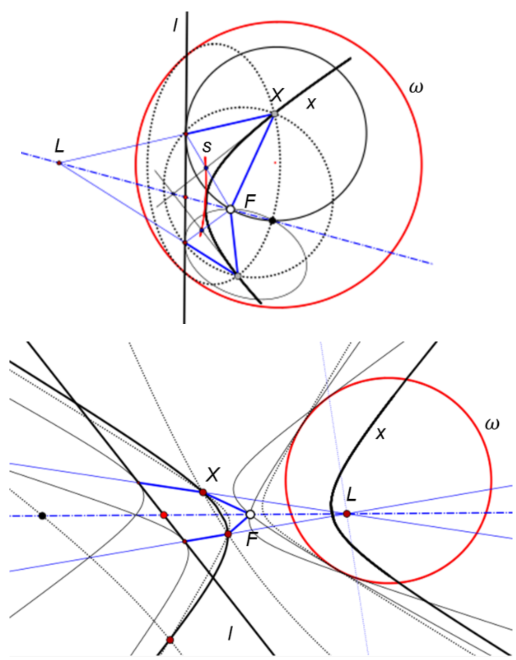

Figure 15: *The h-analog of Apollonius's definition of a Euclidean parabola delivers a conic x.*

Figure 15 also shows that x fulfills a "reflection property" similar to the Euclidean parabola. But while, in the Euclidean case, the diameters of the parabola c are reflected at x, and then pass through the focus of x, in the hyperbolic case the lines through the absolute pole L of the directrix line l are reflected at x and pass through a point, the h-focus F_1. The points L and F_1 act, therefore, as a pair of foci of x.

As all h-conics with two real focal points have the "reflection property", it seems to be obvious that the projective h-parabolas x have the "reflection property", too, i.e., the h-reflections of "diameters" at x pass through a point, the focus of x. It turns out that this is true for all cases, Figure 16. For the limit case of pencil type V it seems to be trivial, as the diameters intersect the "limit circle" x h-orthogonally, a property, which we connect rather to circles then to a parabola and which justifies to call that special h-conic a *limit circle* (horocycle) and not a *limit parabola*.

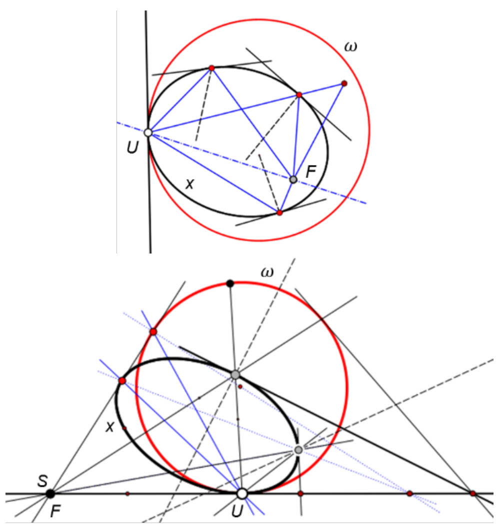

Figure 16: *Reflection property of h-parabolas of the projective types II and IV*

6 h-Conics with properties of Euclidean equilateral hyperbolas

6.1 h-equilateral conics

Here, we continue chapter 3: An "h-equilateral hyperbola" x is an h-conic with a (real) harmonic quadrangle Ω of ideal points, i.e. $x \cap \omega = A, B, C, D$ and $\mathrm{CR}(A, B, C, D) = -1$. Similarly we call an h-equilateral conic x with two real and two conjugate imaginary ideal points an "h-equilateral semi-hyperbola", see Figure 6. For both cases exactly one real pair of sides of the ideal harmonic quadrangle Ω is h-orthogonal and we will call this pair the "asymptotes" a_1, a_2 of x, c.f. also [10]. The other two (real or conjugate imaginary) pairs of sides of the complete quadrangle Ω shall be named as "singular h-equilateral conics" Σ_1, Σ_2. The vertices of the diagonal triangle of Ω are the h-midpoints M_i of x, Figure 17.

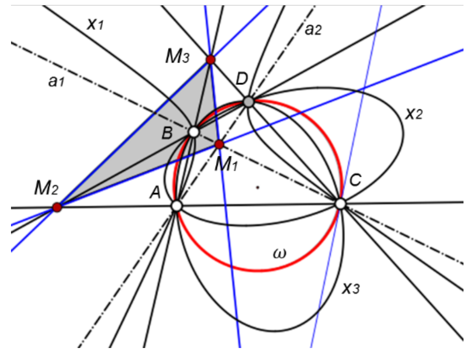

Figure 17: *Concentric h-equilateral hyperbolas x_1, x_2, x_3 with their common asymptotes a_1, a_2, and h-midpoints M_i*

A h-equilateral semi-hyperbola can be visualized in the classical, but projectively closed plane of visual perception as a *Euclidean pencil of circles*, see Figure 18.

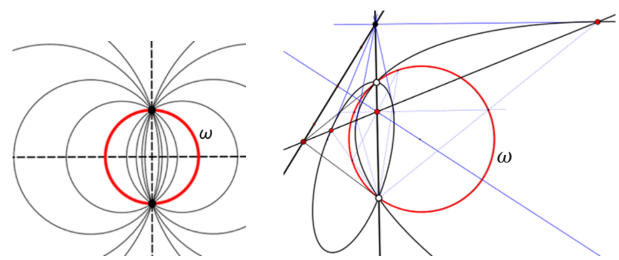

Figure 18: *Left: Euclidean model of h-equilateral semi-hyperbolas, Right: A more projective visualisation of these special h-conics*

Figure 19 shows that even a Thales-conic can be an h-equilateral hyperbola.

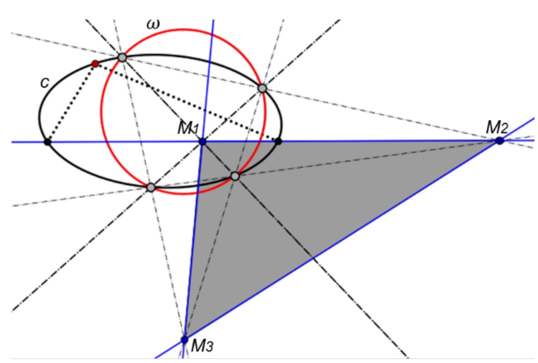

Figure 19: *A Thales-conic can be an h-equilateral hyperbola*

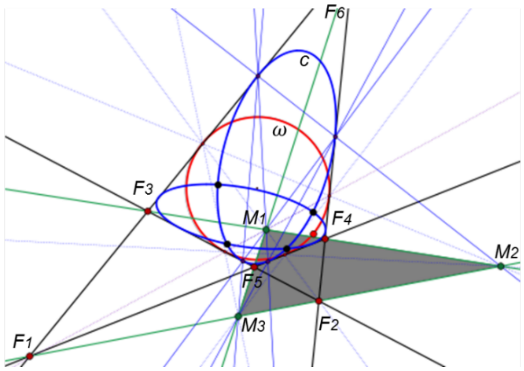

Figure 20: *Dual pencil of h-equilateral dual conics c^**

Figure 20 illustrates the case of a dual pencil of h-equilateral dual conics c^* touching a harmonic (ideal) quadrilateral Ω^*. ("Dual conic" means the set of tangents of a conic c, which is given as a point set.) The three pairs of intersection points of the complete harmonic quadrilateral Ω^* act as the pairs of focal points F_i of the conics c. Again, there exists one real pair of absolute conjugate focal points F_1, F_2, and we call these points the "asymptote points" of c^*. The diagonal points of Ω^* are the midpoints of the conics c to c^*.

If one applies an h-reflection to an h-equilateral hyperbola c in one of its asymptotes, one receives another h-equilateral hyperbola $\bar{c}$, which is h-congruent to c. This repeats a property of Euclidean equilateral hyperbolas. In the Euclidean case c and $\bar{c}$ are "conjugate hyperbolas". This gives a hint to define h-conjugate conics, too, see Figure 21:

Definition 1 *The "h-conjugate conic" y to conic x is concentric with x, has the same ideal points, and thus, the same asymptotes a_1, a_2, and it has the same "axis-quadrangle" Λ. The axis-quadrangle Λ of x has its vertices in the intersection points of vertex tangents w_1, w_2, of x with its asymptotes a_1, a_2.*

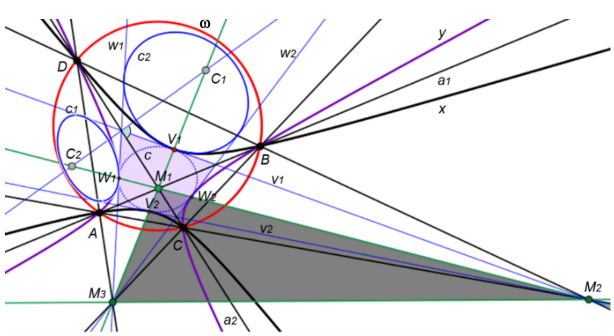

Figure 21: *The h-conjugate conics x and y in the case of x being h-equilateral*

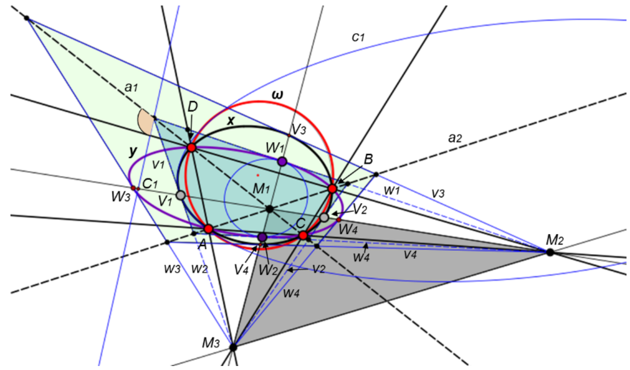

Figure 22: *The h-conjugate conics x and y in the case of x being not h-equilateral*

The special case visualised in Figure 21 also reveals the construction of the hyper-osculating h-circles c_i at the vertices W_1 of x and V_1 of y, which is identical with the Euclidean construction for equilateral hyperbolas. Figure 22 shows the general case of (real) h-conjugate conics. There, too, the classical Euclidean construction of the hyperosculating circles at the vertices is possible. So, we can state the following

Result: If in an arbitrary Cayley-Klein plane (CK-plane) π a conic x has a well-definded real pair of asymptots a_1, a_2, then it has a real CK-conjugate conic y. The CK-normals in vertices of the "axis-quadrangle" of x (defined above) to the asymptotes intersect the axes of x (and y) in the h-centres C_i of the hyperosculating circles c_i of x and y.

6.2 Special h-conics generated by h-congruent pencils of lines

Euclidean equilateral hyperbolas have the property that they can be Steiner-generated by two indirect congruent pencils of lines. Obviously, the result of the analog construction in the h-plane π delivers a conic x, but this conic is, in general, not h-equilateral, see Figure 23. As also a Thales conic can have four real ideal points, see e.g. Figure 19, the sense of the congruence between the two pencils of lines is not essential for receiving a hyperbola as the result of the Steiner-generation.

Remark 1 *In Figure 23 the basis points of both Steiner-generations, the direct and indirect one, are labelled with P, Q. It turns out that the segment $[P,Q]$ is a diameter of the indirectly generated h-conic x as well as of the directly generated h-conic y. It is still an open question, whether any arbitrary h-conic x can be Steiner-generated by h-congruent pencils of lines. In the Euclidean case this is not true.*

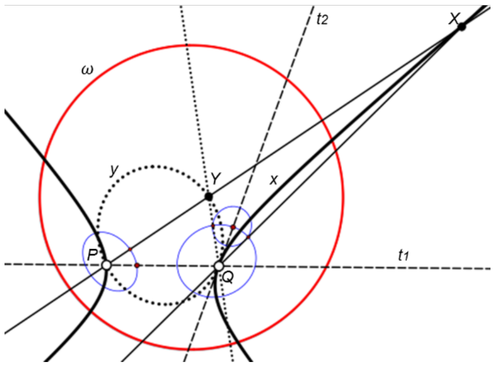

Figure 23: *Generation of a conics x, y via h-congruent pencils of lines. The h-congruence is given by the sense (indirect or direct) and the pair of corresponding lines t_1, t_2. The dotted conic y is the result of directly congruent pencils while the x stems from indirectly congruent pencils.*

6.3 Special h-conics with the "triangle orthocenter property"

It is well-known that Euclidean equilateral hyperbolas x are characterised by the remarkable property that any triangle Δ of hyperbola points has its orthocentre on x. We abbreviate this property as the "triangle orthocentre property" of (Euclidean) equilateral hyperbolas and pose the question whether there exist h-conics with this property, too.

We start with an arbitrary but not right-angled triangle $\Delta = (APQ)$ with O being its h-orthocentre, and we add an arbitrarily chosen fifth point X for defining an h-conic x through these 5 points. We assume x to be regular, see Figure 24.

Theorem 1 *A h-conic x through A,O,P,Q,X, with O the h-orthocentre of $\Delta = (APQ)$ passes also through the h-orthocentre O_X of triangle $\Delta_X = (XPQ)$.*

Proof. Using the labelling of Figure 23 with $a_1 = PA$, $a_2 = QA$, $b_1 = PX$, $b_2 = QX$ and $c_1 = PO$, $c_2 = QO$, $d_1 = PO_X$, $d_2 = QO_X$, we have the following pairs of h-orthogonal lines:

$$a_1 \perp c_2, \quad b_1 \perp d_2, \quad c_1 \perp a_2 \quad \text{and} \quad d_1 \mapsto b_2. \tag{1}$$

The two ordered quadruples (a_1,b_1,c_1,d_1), (c_2,d_2,a_2,b_2) belong to h-orthogonal pencils, which Steiner-generate the Thales conic t over the segment $[P,Q]$. Therefore, we can state that

$$\mathrm{CR}(a_1,b_1,c_1,d_1) = \mathrm{CR}(c_2,d_2,a_2,b_2). \tag{2}$$

By applying permutation rules for cross ratios (see e.g. [2, p. 34]), we infer

$$\mathrm{CR}(a_1,b_1,c_1,d_1) = \mathrm{CR}(c_2,d_2,a_2,b_2) = \mathrm{CR}(a_2,b_2,c_2,d_2), \tag{3}$$

such that also the ordering $a_1 \mapsto a_2$, $b_1 \mapsto b_2$, $c_1 \mapsto c_2$, $d_1 \mapsto d_2$ defines projective pencils, which Steiner-generate the conic x through P,Q,A,X,O. Since (2) holds, we also have $d_1 \cap d_2 = O_X \in x$. □

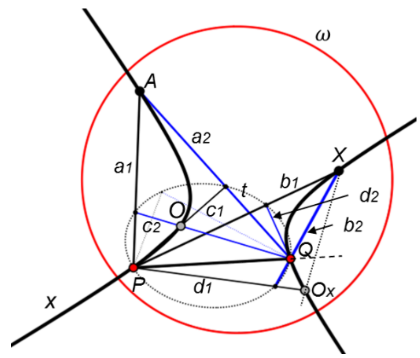

Figure 24: *An h-conic x through A,O,P,Q,X, with O the h-orthocentre of $\Delta = (APQ)$ passes also through the h-orthocentre O_X of the triangle $\Delta_X = (XPQ)$.*

Applying Theorem 1 to different points $X_i \in x$ allows us to go from the basic triangle (A,P,Q) to any other triangle (X_1,X_2,X_3) of conic points, and also this new triangle must have its h-orthocentre on x. So we can state

Theorem 2 *A (regular) h-conic x passing through A,O,P,Q, with O being the h-orthocentre of triangle (A,P,Q) has the triangle orthocentre property, i.e. any triangle of points of x has its h-orthocentre on x.*

One can extend Theorem 1 by the following statement, (see Figure 25):

Theorem 3 *Any conic x through points A,B,C,D and passing through the h-orthocentre O_1 of triangle (ABC) passes also through the h-orthocentres O_i of (BCD), (ABD), and (ACD). Especially, if $A,B,C,D \in \omega$ and (A,B,C,D) is not harmonic, then the diagonal triangle of the quadrangle $(O_1,\ldots,O_4)$ coincides with that of (A,B,C,D), which is the midpoint triangle of x.*

Proof. The first part of theorem 3 is simply a consequence of theorem 2. Now we consider a quadrangle $\Omega = (A,B,C,D)$ of ideal points. Let O_1,O_2 be the h-orthocentres of the triangles (ABC) and (ABD), see Figure 26. The quadrangle Ω admits the h-reflections in the sides of its diagonal triangle (M_1,M_2,M_3). Thereby, the h-symmetry σ with centre $Z := M_3$ and axis $z := M_1M_2$

maps triangle (ABC) to the triangle (ABD) and, as σ is an h-congruence, it maps also O_1 to O_2 which implies that O_1, O_2, Z are collinear. Applying the other possible h-symmetries with centres M_i and axes M_jM_k to the h-orthocenters of the remaining partial triangles Ω completes the proof. □

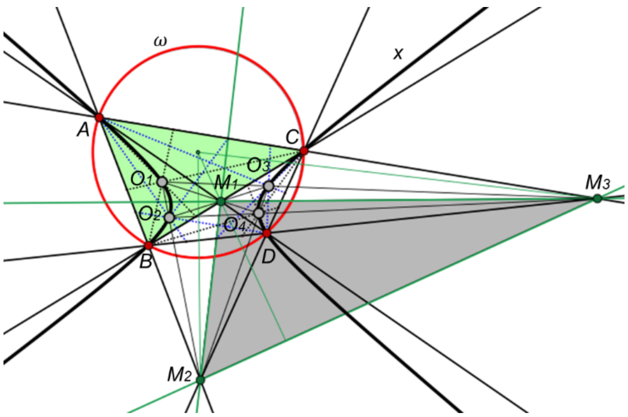

Figure 25: *An h-conic x through A,B,C,D,O_1 with O_1 the h-orthocentre of (ABC) also passes through the h-orthocentres O_i of the remaining partial triangles of (A,B,C,D).*

Remark 2 *The h-symmetry argument used in the proof above suggests an extension of Theorem 3: Each quadrangle (A,B,C,D) admitting three h-symmetries would suit as start figure such that the quadrangle of h-orthocentres of the partial triangles has the diagonal triangle in common with (A,B,C,D).*

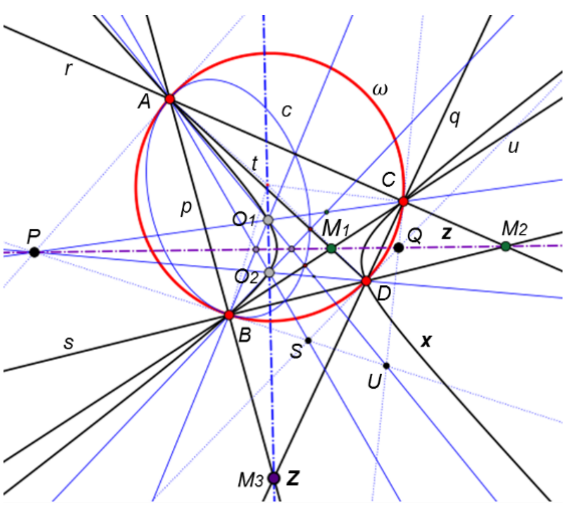

Figure 26: *Applying the h-symmetry $\sigma\colon (A,B,C,D) \mapsto (A,B,D,C)$ proves that O_1,O_2,Z are collinear.*

Remark 3 *Both, the set of h-equilateral hyperbolas possessing one pair of h-orthogonal asymptotes and the set of conics with the triangle orthocentre property ("top-conics") are four-parametric with a three-parametric family of h-conics having both properties, while in Euclidean geometry the two four-parametric families coincide.*

7 Final remarks and conclusion

Conics in Euclidean and non-Euclidean geometries are already widely studied since decades, see e.g. [3], [5], [6] and also the reference list in the monograph on conics [7]. Many references mainly deal with the classification and normal form problem and less with explicite constructions or properties of conics, see e.g. [10], [12]. Explicit constructions can be found e.g. in [1] and [14].

This article aims at a systematic treatment of what can be called "special conics" in a hyperbolic plane. This means that we have to base the investigation on usual classifications of conics from the (projective) universal hyperbolic point of view as well as on the basis of special properties which are non-Euclidean adaptions of properties one can find at Euclidean conics. As one can interpret many of these adaptions simply as Steiner-generations of conics (or its dual), one can widely omit calculations and use synthetic reasoning instead.

Because of the used projective geometric point of view, it is an easy task to transfer the presented results resp. the constructions also to *elliptic geometry*. In an elliptic plane (or its Euclidean spherical model), there are no parabolas even so the constructions for Euclidean parabolas can be performed. Each general conic in the elliptic plane is an *ellipse*, but a spherical conic allows both, the Apollonius-definition of an ellipse and (seen from the complementary side in the spherical model) also that of a hyperbola. As special projective types of (real) conics, one finds one type of "e-circles". All the other metric definitions (as e.g. by the triangle-orthocentre-property) deliver "e-ellipses" with special properties or curves of higher degree.

We conclude with open questions:

It remains open, whether there exist additional special conics in hyperbolic geometry, which have properties one did not consider in Euclidean geometry. One such property which makes no sense in Euclidean geometry but is meaningful in hyperbolic and elliptic planes, is the dual to the Apollonius definition of a conic:

"The tangents of a conic intersect two given lines in angles of constant sum."

For elliptic resp. spherical geometry, this results in a nice application: Given two lines a,b intersecting in C, find points $B \in a$, $A \in b$, such that the spherical triangle ABC has a given area.

As a second open problem occurs, whether each h-conic can be Steiner-generated via two *congruent* pencils of lines. For h-special hyperbolas the generation via congruent pencils leads to pencil vertices on a diameter of the hyperbola. As there is a 5-parametric set of h-conics in the h-plane, and there is also a 5-parametric set of congruent pencils, (namely 2 times two for the vertices and one for the rotation given by start line t_2 to a fixed start line t_1), this question might be answered with "yes", even so it is wrong in Euclidean geometry. But if "yes" is true, how can one find these vertices and the angle of rotation to a given h-conic?

A third question concerns the "h-isoptic curves of a segment", which generalise the incribed angle theorem in Euclidean geometry. Is it possible that the h-isoptic curve, which is irreducible of degree 4 in geneneral, can be reducible in some special cases? This would be similar to the Euclidean case, where the resulting curve of degree 4 always splits into two circular arcs?

A fourth problem might concern the hyperbolic versions of some generalisations of Euclidean Thales-constructions as presented in [13].

Final remark. *Even so the topic of dealing with special conics a specific CK-plane only seems to be what can be called "advanced elementary geometry", it could stimulate research of conics - namely as curves of degree 2 - in arbitrary metric planes, so-called Minkowski planes. Until now "Minkowski conics" are defined only via the Apollonius definition, see e.g.* [4], [8].

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Gunter Weiss
e-mail: weissgunter@hotmail.com

University of Technology Vienna,
Karlsplatz 13, 1040 Vienna, Austria

University of Technology Dresden,
Helmholtzstr. 10, 01069 Dresden, Germany