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Quadrangle Centroids in Universal Hyperbolic Geometry

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ABSTRACT

We study relations between the eight projective quadrangle centroids of a quadrangle in universal hyperbolic geometry which are analogs of the barycentric centre of a Euclidean quadrangle. We investigate the number theoretical conditions for such centres to exist, and show that the eight centroids naturally form two quadrangles which together with the original one have three-fold perspective symmetries. The diagonal triangles of these three quadrangles are also triply perspective.

Key words: Universal hyperbolic geometry, projective geometry, centroids, quadrangles, diagonal triangles, perspectivities

MSC2010: 51M05, 51M10, 51N10

Težišta četverokuta u univerzalnoj hiperboličnoj geometriji

SAŽETAK

Promatramo veze između osam projektivnih težišta četverokuta u univerzalnoj hiperboličnoj geometriji koji su analogoni baricentričnom središtu euklidskog četverokuta. Određujemo teoretske uvjete postojanja tih središta i pokazujemo da osam težišta tvore dva četverokuta koji zajedno s danim četverokutom imaju trostruku perspektivnu simetriju. I dijagonalni trokuti ovih četverokuta su trostruko perspektivni.

Ključne riječi: Univerzalna hiperbolična geometrija, projektivna geometrija, težišta, četverovrsi, dijagonalni trokuti, perspektiviteti

1 Introduction: classical theories of the centroid

In this paper we investigate quadrangle centroids in the general setting of universal hyperbolic geometry (UHG) using the novel algebraic orientation of *standard quadrangle coordinates*. We will associate to a hyperbolic quadrangle with midpoints two other perspectively related *quadrangles of centroids* as in Figure 1, and will identify a number of perspective relations between these quadrangles, and also of their associated diagonal triangles.

Recall that the three median lines of a Euclidean or affine triangle $\overline{ABC}$ are concurrent at the **centroid** G, which can be viewed as either the centre of mass of the three points of the triangle, or as the centre of mass of a uniform lamina, or mass distribution, on the triangle – the two notions coincide.

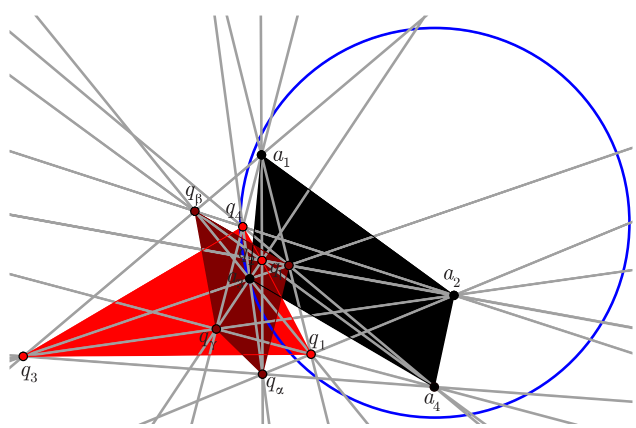

Figure 1: *Three perspectively related quadrangles in UHG*

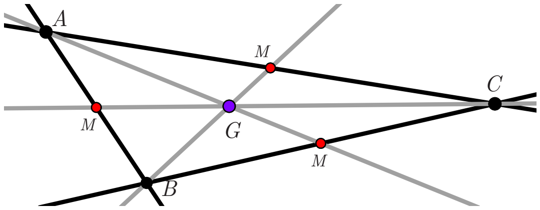

Figure 2: *The centroid of a triangle*

For a Euclidean or affine quadrangle the story is a bit more subtle; there are two notable centroid points. For simplicity let's assume temporarily that the Euclidean quadrangle $\square \equiv \overline{ABCD}$ is convex, and consider the four subtriangles

$$\triangle_A \equiv BCD, \quad \triangle_B \equiv ACD, \quad \triangle_C \equiv ABD \quad \text{and} \quad \triangle_D \equiv ABC$$

with respective centroids G_A, G_B, G_C, and G_D. If we consider the quadrangle $\square$ to have a uniformly distributed mass, then the **type I centroid** G_1 is the point at which this mass balances. The lines G_AG_C and G_BG_D are both lines of balance of the uniform mass distribution, so their meet

$$G_1 = (G_AG_C)(G_BG_D)$$

is also a point of balance, as in Figure 3.

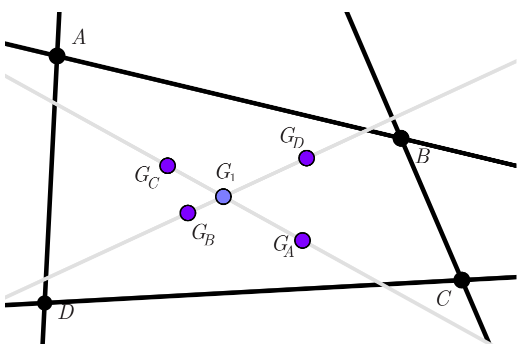

Figure 3: *The type 1 centroid of a quadrangle*

On the other hand the **type II centroid**, or **barycenter**, G_2 is the center of mass of the four vertices. This is the common meet of the lines of balance AG_A, BG_B, CG_C and DG_D as in Figure 4.

The barycenter G_2 can also be found as the meet of the three **bimedian lines**, which are the joins of the midpoints of opposite sides, which we write as the triple concurrence

$$G_2 = (M_{AB}M_{CD})(M_{AD}M_{BC})(M_{AC}M_{BD})$$

as in Figure 5.

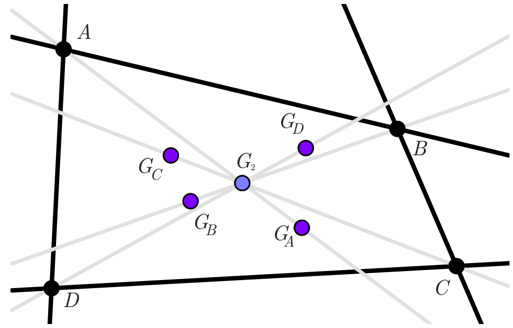

Figure 4: *The type II centroid or barycenter of a quadrangle*

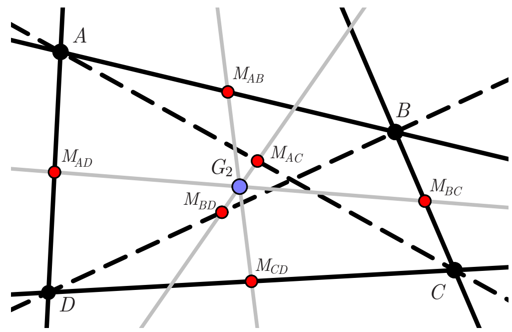

Figure 5: *Bimedian line construction of type II centroid*

Both these ideas extend to non-convex shapes, but on account of the dependence on the order of the points, there are two other possible type I centroids, namely the points $(G_AG_B)(G_CG_D)$ and $(G_AG_D)(G_BG_C)$, depending on the quadrangle's orientation. In contrast, the barycenter G_2 has a purely affine nature and is uniquely defined even in the non-convex situation.

How do these notions extend to hyperbolic, elliptic and other non-Euclidean geometries? Both the books of Sommerville [13] and Coolidge [5], after constructing a projective metric in a Cayley Klein geometry (as in [10], [11]), discuss how a side has two midpoints, a three-point system has four centroids, and finally that a four-point system has eight centroids. The absence of proofs suggests that this was a reasonably well-known 19th century or early 20th century configuration.

UHG is a broad algebraic generalization of hyperbolic geometry which stems from rational trigonometry ([15]) and is related to, but distinctly different from, the more familiar Cayley Klein geometry; and since we want this paper to be largely self contained we will review this; the reader can refer to papers [16], [17], [18], [19] and [20] for further details. We then study quadrangle centroids using the purely algebraic approach of UHG, which works over general fields as well as arbitrary non-degenerate symmetric bilinear forms, and allows us to give concrete computational proofs of results. The generality of this theory is actually a powerful aspect to the arguments that we employ, which require specific transformations to place particular points in standard positions to simplify the algebra, and then the structure constants of the general bilinear form become ingredients in our formulas.

In this general situation there are either zero, one or two midpoints of a side [4], and which case it is depends on number theoretic considerations. So we need to understand the explicit algebraic consequences of the assumptions that ensure existence of quadrilateral centroids. This is highly dependent on the underlying field, so things becomes clearer if we do not assume an algebraically closed field, where such subtleties are largely lost, and indeed where there are serious logical hurdles. So our results are meaningful over, for example, the rational numbers, and also over finite fields, where additional combinatorial aspects arise that are largely invisible to classical synthetic geometry.

Our goal is to prove theorems using explicit general formulas that may help future researchers to make further algebraic explorations in as wide a context as possible. While hyperbolic triangles have seen some recent study, (see [14], [19] and [20]), opportunities for investigation with hyperbolic quadrangles are also rich.

2 Universal hyperbolic geometry and quadrangles

We will now review briefly the projective metrical framework of universal hyperbolic geometry (see [17], [20]) and introduce basic notation for triangles and quadrangles. The general projective linear algebraic setting covers both elliptic, hyperbolic and more general metrical geometries, and works over a general field, not of characteristic two. UHG shares the underlying framework of Cayley Klein geometry—a projective plane with a distinguished conic or its algebraic equivalent, namely a symmetric bilinear form. But it utilises purely algebraic metrical notions, namely *projective quadrance* and *spread,* instead of *hyperbolic distance* and *angle*.

Projective quadrance and spread can be introduced also in a projective framework using cross ratios as described in [17], but we will frame them in the context of a three-dimensional (affine) vector space over a field not of characteristic two. To differentiate between affine and projective linear algebra we will use the convention of writing familiar affine vectors and matrices with round brackets, and projective vectors and matrices, which are determined only up to a (non-zero) multiple, with square brackets.

A **(projective) point** is a *non-zero* projective row vector a shown in either of two ways:

$$a = \begin{bmatrix} x & y & z \end{bmatrix} \equiv [x:y:z]$$

while a **(projective) line** is a *non-zero* projective column vector L shown in either of two ways:

$$L = \begin{bmatrix} l \\ m \\ n \end{bmatrix} \equiv \langle l:m:n \rangle.$$

The point $a=[x:y:z]$ and line $L=\langle l:m:n\rangle$ are **incident** precisely when

$$0 = aL \equiv \begin{bmatrix} x & y & z \end{bmatrix} \begin{bmatrix} l \\ m \\ n \end{bmatrix} = [x:y:z]\,\langle l:m:n\rangle = lx+my+nz.$$

Note that the product of two projective matrices is only determined up to scalars, but in particular the condition of having a zero product is well-defined.

The **join** a_1a_2 of distinct points $a_1 \equiv [x_1:y_1:z_1]$ and $a_2 \equiv [x_2:y_2:z_2]$ is then the line

$$\begin{aligned} a_1a_2 &\equiv [x_1:y_1:z_1] \times [x_2:y_2:z_2] \\ &= \langle y_1z_2 - z_1y_2 : z_1x_2 - x_1z_2 : x_1y_2 - y_1x_2 \rangle \end{aligned}$$

and the **meet** L_1L_2 of two distinct lines $L_1 = \langle l_1:m_1:n_1\rangle$ and $L_2 = \langle l_2:m_2:n_2\rangle$ is the point

$$\begin{aligned} L_1L_2 &\equiv \langle l_1:m_1:n_1\rangle \times \langle l_2:m_2:n_2\rangle \\ &= [m_1n_2 - n_1m_2 : n_1l_2 - l_1n_2 : l_1m_2 - m_1l_2]. \end{aligned}$$

The cross product here is the usual Euclidean cross product, which is well-defined on projective points and lines.

We'll say a **side** $\overline{a_1a_2} = \{a_1,a_2\}$ is a set of two points; a **vertex** $\overline{L_1L_2} = \{L_1,L_2\}$ is a set of two lines; a **triangle** $\overline{a_1a_2a_3} \equiv \{a_1,a_2,a_3\}$ is a set of three non-collinear points; and a **quadrangle** $\overline{a_1a_2a_3a_4} \equiv \{a_1,a_2,a_3,a_4\}$ is a set of four points, where no three are collinear. The quadrangle $\overline{a_1a_2a_3a_4} \equiv \{a_1,a_2,a_3,a_4\}$ has six distinct sides $\overline{a_1a_2}$, $\overline{a_3a_4}$, $\overline{a_1a_3}$, $\overline{a_2a_4}$, $\overline{a_1a_4}$, and $\overline{a_2a_3}$, with corresponding lines $L_{12} \equiv a_1a_2$, $L_{34} \equiv a_3a_4$, $L_{13} \equiv a_1a_3$, $L_{24} \equiv a_2a_4$, $L_{14} \equiv a_1a_4$, and $L_{23} \equiv a_2a_3$.

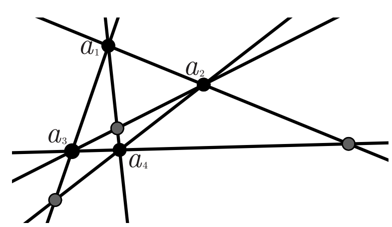

Figure 6: *A projective quadrangle and its diagonal points*

The four **subtriangles** of the quadrangle $\overline{a_1a_2a_3a_4}$ are

$$\triangle_1 \equiv \overline{a_2a_3a_4}, \quad \triangle_2 \equiv \overline{a_1a_3a_4},$$
$$\triangle_3 \equiv \overline{a_1a_2a_4}, \quad \triangle_4 \equiv \overline{a_1a_2a_3}.$$

Two sides of a quadrangle are **opposite** if they have no point in common. Each of the six sides of the quadrangle are contained in exactly two distinct subtriangles, while opposite sides are precisely the pairs of sides which don't belong to any one of the subtriangles.

The **diagonal points** of the quadrangle $\overline{a_1a_2a_3a_4}$ are the meets of the associated lines of opposite sides, that is

$$d_\alpha \equiv L_{12}L_{34}, \quad d_\beta \equiv L_{13}L_{24} \quad \text{and} \quad d_\gamma \equiv L_{14}L_{23}. \tag{1}$$

Then $\overline{d_\alpha d_\beta d_\gamma}$ is the **diagonal triangle** of the standard quadrangle. The indices α, β and γ correspond to the three ways of dividing four points into two pairs, these play a distinguished role in the subject.

3 Projective bilinear forms and metrical structure

We now impose metrical structure through a general symmetric bilinear form, possibly non-Euclidean, on an underlying three-dimensional vector space, as studied for example in [12]. This generalises the notion of a dot product, and is a standard notion in projective geometry, where a bilinear form can be used to define objects such as conics sections and quadrics, as described for example in [21] and [11]. The bilinear form induces metrical structure to the associated projective space, which can be viewed as one-dimensional subspaces of the vector space. This gives a far-reaching methodology for imbedding measurement into purely algebraic geometry. For a given affine vector v we will denote the associated projective vector by $a = [v] = [\lambda v]$ where $\lambda \neq 0$, and similarly for an affine matrix A we denote the associated projective matrix as $\mathbf{A} = [A] = [\mu A]$ where $\mu \neq 0$.

Let us fix a general invertible symmetric 3×3 matrix A and its adjugate B in the general form

$$A = \begin{pmatrix} a & d & f \\ d & b & g \\ f & g & c \end{pmatrix} \qquad \text{and}$$
$$B = \begin{pmatrix} g^2 - bc & dc - fg & fb - dg \\ dc - fg & f^2 - ac & ga - df \\ fb - dg & ga - df & d^2 - ab \end{pmatrix} \tag{2}$$

and denote by $\mathbf{A}$ and $\mathbf{B}$ the associated symmetric projective matrices. By assumption the determinant

$$D \equiv \det A = abc + 2fdg - ag^2 - bf^2 - cd^2$$

is non-zero. The **dual** of a projective point a is then defined to be the line $a^\perp \equiv \mathbf{A}a^T$, and the **dual** of the projective line L is the point $L^\perp \equiv L^T\mathbf{B}$. Since the projective matrices $\mathbf{A}$ and $\mathbf{B}$ are inverses, and both are symmetric,

$$(aL)^T = (a\mathbf{AB}L)^T = L^T\mathbf{BA}a^T = L^\perp a^\perp,$$

and so the point a is incident with the line L precisely when their duals are incident, and in addition $\left(a^\perp\right)^\perp = a$ and $\left(L^\perp\right)^\perp = L$. Hence duality preserves projective theorems.

Two points a_1 and a_2 are **perpendicular,** written as $a_1 \perp a_2$, precisely when one is incident with the dual of the other, that is when $0 = a_1a_2^\perp = a_1\mathbf{A}a_2^T$. Symmetrically two lines L_1 and L_2 are **perpendicular,** written as $L_1 \perp L_2$, precisely when one is incident with the dual of the other, that is when $0 = L_1^\perp L_2 = L_1^T\mathbf{B}L_2$.

A point a is **null** precisely when it is self perpendicular, and a line L is **null** precisely when it is self perpendicular, that is when respectively

$$a\mathbf{A}a^T = 0 \quad \text{and} \quad L^T\mathbf{B}L = 0.$$

The **null conic** (or **absolute**) then consists of null points, and the null lines are the tangents to this conic. The above notions of perpendicularity and duality are symmetric, and algebraically capture the synthetic notion of polarity with respect to the null conic in the projective plane.

The standard cases in UHG are those of *hyperbolic* and *elliptic geometries* [19], which arise respectively from the choices

$$\mathbf{A} = \mathbf{J} \equiv \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \quad \text{and} \quad \mathbf{A} = \mathbf{I} \equiv \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

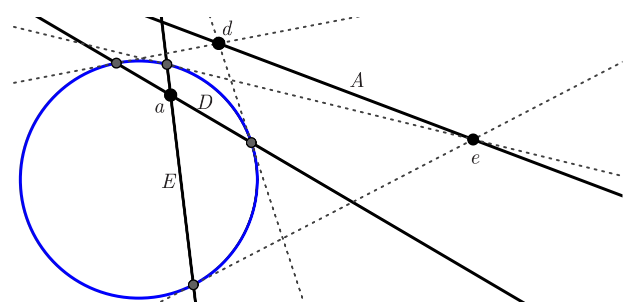

Figure 7: *Polars of points and lines, interior and exterior*

Figures in this paper come from the hyperbolic situation where the null conic has equation $x^2 + y^2 - z^2 = 0$, which meets the viewing plane $z = 1$ in the usual unit circle $x^2 + y^2 = 1$ shown in blue in Figure 7. In this Figure we see that the duals of the points a, d and e are the lines A, D and E respectively, so this is just the classical pole/polar duality with respect to the circle. The points a and d are perpendicular, as are a and e, and so are the dual lines.

The inverse pair of symmetric projective matrices $\mathbf{A}$ and $\mathbf{B}$ coming from (2) that determine perpendicularity and duality in UHG also allow us to define the **(projective) quadrance** $q(a_1,a_2)$ between the two points a_1 and a_2, and the **(projective) spread** $S(L_1,L_2)$ between the two lines L_1 and L_2 as the respective quantities

$$q(a_1,a_2) \equiv 1-\frac{\left(a_1\mathbf{A}a_2^T\right)^2}{\left(a_1\mathbf{A}a_1^T\right)\left(a_2\mathbf{A}a_2^T\right)} \qquad \text{and}$$

$$S(L_1,L_2) \equiv 1-\frac{\left(L_1^T\mathbf{B}L_2\right)^2}{\left(L_1^T\mathbf{B}L_1\right)\left(L_2^T\mathbf{B}L_2\right)}.$$

By homogeneity these are well-defined numbers, even if the various ingredients are only defined up to non-zero multiplicative scalars. It should be noted that in rational trigonometry, the notions of quadrance and spread are affine versions of the above definitions: but they are different.

In the special case of hyperbolic geometry, these metrical measurements are closely related to the hyperbolic distance $d(a_1,a_2)$ and the angle $\theta(L_1,L_2)$ as described in [17] provided we restrict our attention to the interior of the null conic, and the relations are

$$q(a_1,a_2) = -\sinh^2\left(d(a_1,a_2)\right),$$
$$S(L_1,L_2) = \sin^2\left(\theta(L_1,L_2)\right).$$

With the algebraic orientation of UHG, the avoidance of all transcendental functions means that calculations can be performed with full precision, that we can work over a general field, that quadrance and spread really mirror the fundamental projective duality between points and lines, and that metrical theorems extend to arbitrary non-degenerate symmetric bilinear forms. These are very serious advantages! As explained in [17] and [18], projective quadrance and spread may also be defined in terms of cross ratios, independent of the relative positions of the points and lines with respect to the null conic. This is also related to the approach of Brauner in [3].

A **hyperbolic circle** with fixed center a and quadrance k is the locus of points x which satisfy $q(a,x)=k$. This is a conic. In Figure 8 we see circles centered at the external point a and their quadrances; these are conics which are tangential to the null conic and include what in the classical literature are called "equi-distant curves". This notation is not really necessary, as these conics are just circles.

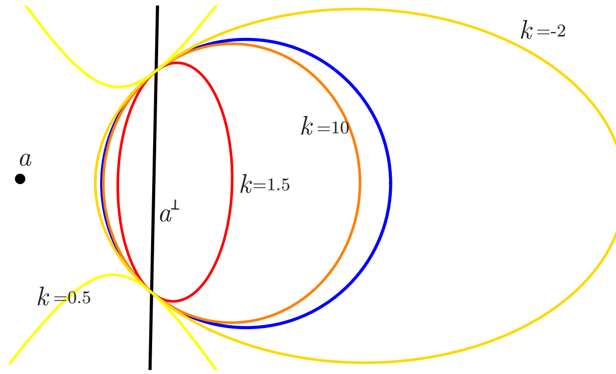

Figure 8: *Hyperbolic circles with center a and different quadratnces*

4 Midpoints

As defined in [17], the point m is a **midpoint** of the side $\overline{a_1a_2}$ precisely when m is a point incident with the line a_1a_2 which satisfies

$$q(a_1,m) = q(m,a_2).$$

A **midline** M of the side $\overline{a_1a_2}$ is a line passing through a midpoint m which is perpendicular to a_1a_2. In Figure 9 we see two midpoints of the side $\overline{a_1a_2}$, and a synthetic construction of them, along with the associated midlines. Note that such a construction does not always work: the relative positions of the side $\overline{a_1a_2}$ and the null conic determine when this happens.

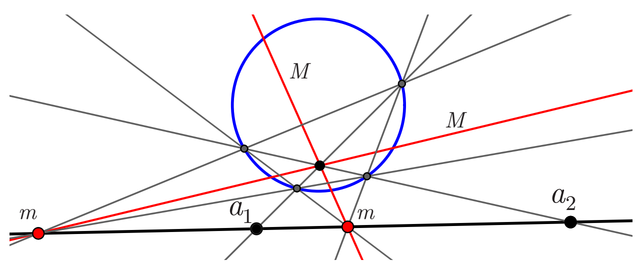

Figure 9: *Midpoints and midlines of a side*

The following theorem was given by Wildberger [17] in the purely hyperbolic case. Since midpoints are at the heart of this paper, we give a more general proof which applies to an arbitrary bilinear form.

Theorem 1 (Side midpoints) *Suppose that a_1 and a_2 are non-null, non-perpendicular points, forming a non-null side $\overline{a_1a_2}$. Then $\overline{a_1a_2}$ has a non-null midpoint m precisely when $1-q(a_1,a_2)$ is a square (within the field), and in this case there are exactly* two *perpendicular midpoints m.*

Proof. We prove the theorem first for specific points and a general bilinear form. Let $a_1 \equiv [u_1] = [1:1:1]$ and $a_2 \equiv [u_2] = [-1:-1:1]$ where $u_1 \equiv (1,1,1)$ and $u_2 \equiv (-1,-1,1)$ are affine row vectors, and the general bilinear form be given by the invertible pair of symmetric projective matrices $\mathbf{A}$ and $\mathbf{B}$ with affine representatives given by (2). By the definition of quadrance

$$1-q(a_1,a_2) = \frac{\left(a_1\mathbf{A}a_2^T\right)^2}{\left(a_1\mathbf{A}a_1^T\right)\left(a_2\mathbf{A}a_2^T\right)} = \frac{(c-b-a-2d)^2}{A_1A_2}$$

where

$$A_1 \equiv u_1Au_1^T = a+b+c+2(d+f+g)$$
$$A_2 \equiv u_2Au_2^T = a+b+c+2(d-f-g).$$

Since a_1 and a_2 are non-null points, each of A_1 and A_2 are nonzero. An arbitrary point m on the line $a_1a_2 \equiv \langle 1:-1:0\rangle$ has the form $m = [x-y:x-y:x+y]$, which is null precisely when

$$m\mathbf{A}m^T = (a+b+2d)(x-y)^2+c(x+y)^2+2(f+g)(x^2-y^2)=0$$

in which case the quadrance between m and any other point is undefined. Thus we assume that m is non-null, which gives the quadrances

$$q(a_1,m) = \frac{4y^2\left(c(a+b+d)-(f+g)^2\right)}{A_1\left((a+b+2d)(x-y)^2+c(x+y)^2+2(f+g)(x^2-y^2)\right)}$$
$$q(a_2,m) = \frac{4x^2\left(c(a+b+d)-(f+g)^2\right)}{A_2\left((a+b+2d)(x-y)^2+c(x+y)^2+2(f+g)(x^2-y^2)\right)}.$$

By assumption $\overline{a_1a_2}$ is non-null, and so the above expressions are equal precisely when $y^2A_2 = x^2A_1$ or

$$\frac{1}{A_1A_2} = \frac{y^2}{A_1^2x^2}.$$

This occurs precisely when $1-q(a_1,a_2)$ is a square. In such a case if we define a number σ_{12} by

$$\frac{1}{A_1A_2} = \sigma_{12}^2$$

then we get solutions $x=1$ and $y=\pm\sigma_{12}A_1$, so the midpoints are

$$m \equiv [u_1 \pm \sigma_{12}A_1 \cdot u_2] = [1\pm\sigma_{12}A_1 : 1\pm\sigma_{12}A_1 : 1\mp\sigma_{12}A_1].$$

These two points are perpendicular since $m_+\mathbf{A}m_-^T = 0$ by a computation.

Despite the proof being for a specific side, the general bilinear form allows us to use the *Fundamental theorem of projective geometry* to transform this side to a general side with any specific bilinear form. This gives us the result. □

What we have shown in the proof is that for points a_1 and a_2 if we can find affine vectors v and u such that $a_1 = [v]$, $a_2 = [u]$ and $vAv^T = uAu^T$, then the midpoints are $[v+u]$ and $[v-u]$. This is a particularly useful observation.

We point out an important variant of midpoint introduced in [20]: the point s is a **sydpoint** of the side $\overline{a_1a_2}$ precisely when s is a point incident with the line a_1a_2 which satisfies

$$q(a_1,s) = -q(s,a_2).$$

Alkhaldi and Wildberger have shown that the side $\overline{a_1a_2}$ has *sydpoints* precisely when the number $q(a_1,a_2)-1$ is a square. Thus over the rationals a side approximately has two midpoints or two sydpoints [20]. Over a finite field things can behave rather differently, depending on the field. For example over $\mathbb{F}_5$ a side has midpoints precisely when it has sydpoints, as the only squares are 1 and -1. It is remarkable that virtually everything in this paper can be extended to the situation where the sides of a quadrangle have either midpoints or sydpoints, but we will leave that to a future discussion.

4.1 Centroids and Circumcenters

We now review familiar facts about medians and centroids of triangles from Cayley Klein geometry and classical hyperbolic geometry, as described for example in [13] and [6], but in the setting of UHG. This way we get a seamless extension of these notions also to points outside the traditional hyperbolic disk; in Cayley Klein geometry it can be awkward to smoothly make this extension on account of the more limited nature of distance and angle, and their dependence on transcendental functions.

We assume that we have a (hyperbolic) triangle $\overline{a_1a_2a_3}$ all of whose sides have midpoints m as in Figure 10.

The **median lines** (or just **medians**) D of $\overline{a_1a_2a_3}$ are then the joins of corresponding midpoints of sides with opposite points of the triangle. There are six medians, two passing through each point a_i. The following results are well-known, although the terminology is somewhat novel, see for example the above references and [19].

Theorem 2 (Centroids) *The median lines D of the triangle $\overline{a_1a_2a_3}$ are concurrent three at a time, meeting at four distinct* ***centroid points*** *g.*

We note that each centroid is associated with a distinct set of three midpoints used to construct it, one from each side.

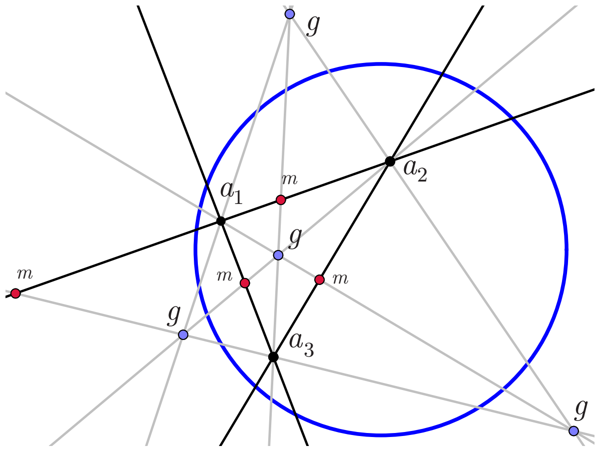

Figure 10: *Centroids of a triangle $\overline{a_1a_2a_3}$*

Theorem 3 (Circumcenters) *The six midpoints m of the triangle $\overline{a_1a_2a_3}$ are collinear three at a time, lying on four distinct* ***circumlines*** *C. The six midlines M of $\overline{a_1a_2a_3}$ are concurrent three at a time, meeting at four distinct* ***circumcenters*** *c which are dual to the circumlines C. The circumcenters are the centers of the four hyperbolic circles which go through the points of the triangle.*

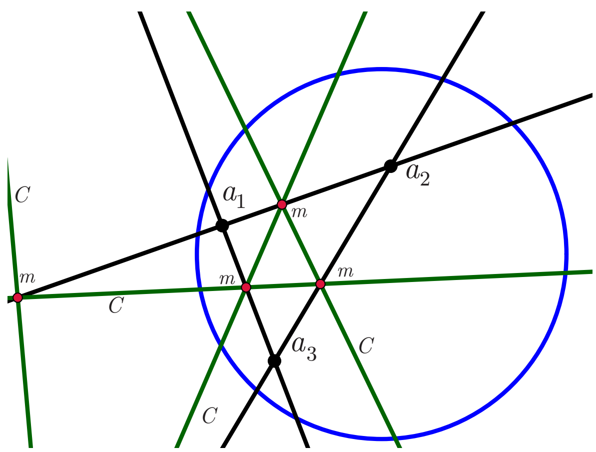

Figure 11: *Circumlines of a triangle $\overline{a_1a_2a_3}$*

The centroid and circumcenter configurations are closely related through perspectivities. As noted in [5], the median/centroid configuration is that of a quadrangle, as there are four points, and six lines concurrent at the points in threes, giving a Desargues configuration.

The quadrangle of centroids has the original triangle as its diagonal triangle. So if we look at a subtriangle of this centroid quadrangle, then it is perspective with the original triangle at the fourth centroid point, as illustrated in Figure 12.

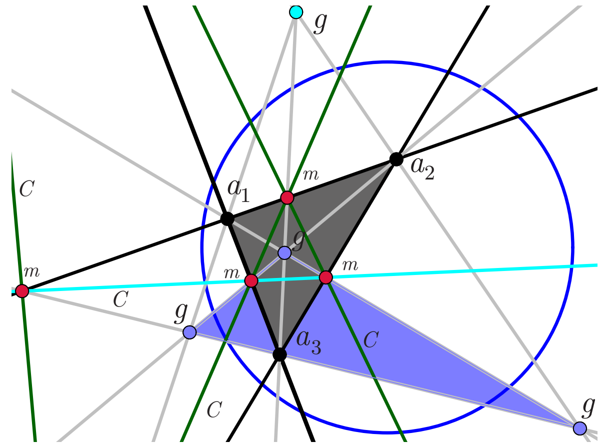

Figure 12: *The triangle $\overline{a_1a_2a_3}$ is perspective with a subtriangle of centroids from a Circumline C.*

From Desargues' theorem we know that these two triangles are perspective from a line which is precisely a circumline. This can be done for each centroid, giving a unique correspondence between centroids and circumcenters (through the circumlines). In this paper we will concentrate on the centroid story, as it is quite rich and interesting in its own right, and leave the circumcenter picture for another occasion.

5 Standard quadrangles

The fundamental theorem of projective geometry states that *any two projective frames in a projective plane can be mapped onto each other by exactly one invertible projective linear transformation*. This allows us to transform the study of a general quadrangle under a fixed bilinear form to that of a fixed standard quadrangle with a general bilinear form. This powerful technique for tackling metrical structure was employed in [19] and [20] in the context of hyperbolic triangle geometry, in [1] and [2] to study hyperbolic parabolas, and used in the papers [9] and [8], as well as the thesis of Nguyen Le [7] in the Euclidean case.

In this paper the **standard quadrangle** $\square \equiv \overline{a_1a_2a_3a_4}$ is given by the four points

$$a_1 \equiv [1:1:1], \quad a_2 \equiv [-1:-1:1],$$
$$a_3 \equiv [1:-1:1], \quad a_4 \equiv [-1:1:1]$$

with corresponding affine vectors

$$u_1 \equiv (1,1,1), \quad u_2 \equiv (-1,-1,1),$$
$$u_3 \equiv (1,-1,1), \quad u_4 \equiv (-1,1,1).$$

After a suitable projective transformation, any given quadrangle can be transformed to the standard quadrangle, and since the metrical structure will change correspondingly by a congruence, we may assume that this is then given by the general pair of matrices A and B of (2). We will also use a consistent notational convention throughout this paper, which will emphasize the quadrangle point(s) used to construct an object, so that for example whenever an object has a 1 in its label, it relates to the point a_1 in some way.

We make some assumptions to avoid degenerate situations. The first is that none of the points a_1, a_2, a_3, a_4 or lines

$$\begin{aligned} L_{12} &\equiv a_1a_2 = \langle 1:-1:0\rangle, & L_{13} &\equiv a_1a_3 = \langle 1:0:-1\rangle, \\ L_{14} &\equiv a_1a_4 = \langle 0:1:-1\rangle, & L_{34} &\equiv a_3a_4 = \langle 1:1:0\rangle, \\ L_{24} &\equiv a_2a_4 = \langle 1:0:1\rangle, & L_{23} &\equiv a_2a_3 = \langle 0:1:1\rangle \end{aligned}$$

of $\square$ are null. The second is that the four points of the quadrangle *do not lie on a single hyperbolic circle*. The reason for this will be explained in the last section.

Let us also record that the diagonal points of the quadrangle are

$$\begin{aligned} &d_\alpha \equiv L_{12}L_{34} = [0:0:1], \quad d_\beta \equiv L_{13}L_{24} = [0:1:0], \\ &d_\gamma \equiv L_{14}L_{23} = [1:0:0]. \end{aligned} \tag{3}$$

It will be useful to define the numbers

$$\begin{aligned} A_1 &\equiv u_1Au_1^T = a+b+c+2(d+f+g), \\ A_2 &\equiv u_2Au_2^T = a+b+c+2(d-f-g), \\ A_3 &\equiv u_3Au_3^T = a+b+c+2(-d+f-g), \\ A_4 &\equiv u_4Au_4^T = a+b+c+2(-d-f+g) \end{aligned}$$

which by assumption are all non-zero. We are going to express things in terms of the entries a, b, c, d, f and g of the matrix A of (2) which determines the metrical structure.

Theorem 4 (Quadrangle quadrances) *The quadrances of the standard quadrangle $\overline{a_1a_2a_3a_4}$ are*

$$\begin{aligned} q(a_1,a_2) &= 4\frac{c(a+b+2d)-(f+g)^2}{A_1A_2}, \\ q(a_3,a_4) &= 4\frac{c(a+b-2d)-(f-g)^2}{A_3A_4}, \\ q(a_1,a_3) &= 4\frac{b(a+c+2f)-(d+g)^2}{A_1A_3}, \\ q(a_2,a_4) &= 4\frac{b(a+c-2f)-(d-g)^2}{A_2A_4}, \\ q(a_1,a_4) &= 4\frac{a(b+c+2g)-(d+f)^2}{A_1A_4}, \\ q(a_2,a_3) &= 4\frac{a(b+c-2g)-(d-f)^2}{A_2A_3}. \end{aligned}$$

These satisfy

$$\begin{aligned} 1-q(a_1,a_2) &= \frac{(c-b-a-2d)^2}{A_1A_2}, \\ 1-q(a_3,a_4) &= \frac{(c-b-a+2d)^2}{A_3A_4}, \\ 1-q(a_1,a_3) &= \frac{(a-b+c+2f)^2}{A_1A_3}, \\ 1-q(a_2,a_4) &= \frac{(a-b+c-2f)^2}{A_2A_4}, \\ 1-q(a_1,a_4) &= \frac{(b-a+c+2g)^2}{A_1A_4}, \\ 1-q(a_2,a_3) &= \frac{(b-a+c-2g)^2}{A_2A_3}. \end{aligned}$$

Proof. Computations yield these results. $\square$

5.1 Conditions for midpoints, or sigma relations

Now we also want to impose the conditions that ensure that all six sides $\overline{a_ia_j}$ for $i \neq j$ have midpoints, so that each of the subtriangles of $\overline{a_1a_2a_3a_4}$ has centroids. To find elegant proofs of our theorems we will require some careful bookkeeping with regards to the labelling of midpoints. There is an interesting combinatorial aspect to this.

From the Side midpoint theorem we know that the non-null side $\overline{a_ia_j}$ has midpoints precisely when $1-q(a_i,a_j)$ is a square. From the second half of the previous theorem, the condition for all six sides having midpoints is equivalent to the existence of six non-zero **sigma values**

$$\sigma_{12}, \sigma_{34}, \sigma_{13}, \sigma_{24}, \sigma_{14}, \sigma_{23}$$

in the chosen field, satisfying the following **quadratic relations**

$$\begin{aligned} &\frac{1}{A_1A_2} = \sigma_{12}^2, \quad \frac{1}{A_1A_3} = \sigma_{13}^2, \quad \frac{1}{A_1A_4} = \sigma_{14}^2, \\ &\frac{1}{A_3A_4} = \sigma_{34}^2, \quad \frac{1}{A_2A_4} = \sigma_{24}^2, \quad \frac{1}{A_2A_3} = \sigma_{23}^2. \end{aligned}$$

We can further take the product of these quadratic relations in threes, say

$$\frac{1}{A_1A_2} = \sigma_{12}^2, \quad \frac{1}{A_1A_3} = \sigma_{13}^2 \quad \text{and} \quad \frac{1}{A_2A_3} = \sigma_{23}^2$$

and by possibly changing the sign of any or all of the *sigma values* to arrange the following four **cubic relations**

$$\begin{aligned} &\frac{1}{A_1A_2A_3} = \sigma_{12}\sigma_{23}\sigma_{13} \equiv \sigma_4, \quad \frac{1}{A_1A_2A_4} = \sigma_{12}\sigma_{24}\sigma_{14} \equiv \sigma_3, \\ &\frac{1}{A_1A_3A_4} = \sigma_{13}\sigma_{34}\sigma_{14} \equiv \sigma_2, \quad \frac{1}{A_2A_3A_4} = \sigma_{23}\sigma_{34}\sigma_{24} \equiv \sigma_1. \end{aligned}$$

There are three ways of expressing each A_i using the quadratic and cubic relations, given by

$$A_1 = \frac{\sigma_{23}}{\sigma_{12}\sigma_{13}} = \frac{\sigma_{24}}{\sigma_{12}\sigma_{14}} = \frac{\sigma_{34}}{\sigma_{13}\sigma_{14}},$$
$$A_2 = \frac{\sigma_{13}}{\sigma_{12}\sigma_{23}} = \frac{\sigma_{14}}{\sigma_{12}\sigma_{24}} = \frac{\sigma_{34}}{\sigma_{23}\sigma_{24}},$$
$$A_3 = \frac{\sigma_{12}}{\sigma_{13}\sigma_{23}} = \frac{\sigma_{14}}{\sigma_{13}\sigma_{34}} = \frac{\sigma_{24}}{\sigma_{23}\sigma_{34}},$$
$$A_4 = \frac{\sigma_{12}}{\sigma_{14}\sigma_{24}} = \frac{\sigma_{13}}{\sigma_{14}\sigma_{34}} = \frac{\sigma_{23}}{\sigma_{24}\sigma_{34}}.$$

These show how *each point is in exactly three subtriangles*. For example A_1 has three representations, each relating to the subtriangles $\triangle_2$, $\triangle_3$ and $\triangle_4$ which contain a_1.

From this point on most (if not all) of the expressions and formulas involve these sigma values and relations defined in this section, so it is valuable to appreciate the symmetries involved, which boil down to the following relations

$$\frac{\sigma_{23}}{\sigma_{13}} = \frac{\sigma_{24}}{\sigma_{14}}, \quad \frac{\sigma_{23}}{\sigma_{12}} = \frac{\sigma_{34}}{\sigma_{14}}, \quad \frac{\sigma_{24}}{\sigma_{12}} = \frac{\sigma_{34}}{\sigma_{13}}$$

or simply

$$\sigma_{12}\sigma_{34} = \sigma_{13}\sigma_{24} = \sigma_{14}\sigma_{23} \equiv \eta.$$

Let's also introduce

$$\sigma_{234} \equiv \sigma_{12}\sigma_{13}\sigma_{14}, \quad \sigma_{134} \equiv \sigma_{12}\sigma_{23}\sigma_{24},$$
$$\sigma_{124} \equiv \sigma_{13}\sigma_{23}\sigma_{34}, \quad \sigma_{123} \equiv \sigma_{14}\sigma_{24}\sigma_{34}.$$

5.2 Midpoints of the quadrangle

By the Side midpoint theorem, for a side $\overline{a_i a_j}$ with midpoints, where $a_i = [v_i]$ and $a_j = [v_j]$, we are able to normalize the affine vectors v_i and v_j such that the midpoints have the simple forms

$$m^{(ij)} \equiv [v_i + v_j] \quad \text{and} \quad m^{(ji)} \equiv [v_i - v_j].$$

Despite the arbitrariness of the ordering we will use the convention that the side corresponding to the points a_i and a_j for $i < j$ will be given as $\overline{a_i a_j}$ and *not as* $\overline{a_j a_i}$, and so the midpoint $m^{(ij)} = [v_i + v_j]$ can be seen as having **positive** (or **ascending**) **orientation** and the midpoint $m^{(ji)} = [v_i - v_j]$ as having **negative** (or **descending**) **orientation**.

In the end of the proof of the Side midpoints theorem we saw that the midpoints for the side $\overline{a_1 a_2}$ could also be written as

$$m \equiv [v_1 \pm \sigma_{12} A_1 v_2] = [1 \pm \sigma_{12} A_1 : 1 \pm \sigma_{12} A_1 : 1 \mp \sigma_{12} A_1].$$

In light of the above discussion, we'll say that $m^{(12)} = [v_1 + \sigma_{12} A_1 v_2]$ has **ascending orientation** and $m^{(21)} = [v_1 - \sigma_{12} A_1 v_2]$ has **descending orientation**.

But these aren't the only forms of the midpoints; the η sigma relations allow us to rewrite the midpoints in two different forms. For example the side $\overline{a_1 a_2}$ midpoints can be rewritten as

$$\begin{aligned} m &= [\sigma_{13} \pm \sigma_{23} : \sigma_{13} \pm \sigma_{23} : \sigma_{13} \mp \sigma_{23}] \\ &= [\sigma_{14} \pm \sigma_{24} : \sigma_{14} \pm \sigma_{24} : \sigma_{14} \mp \sigma_{24}]. \end{aligned}$$

There are exactly two sigma representations for every midpoint, as *every side is in exactly two subtriangles*. For example the side $\overline{a_1 a_2}$ is in the subtriangles $\triangle_3$ and $\triangle_4$ which have corresponding midpoint representations

$$[\sigma_{14} \pm \sigma_{24} : \sigma_{14} \pm \sigma_{24} : \sigma_{14} \mp \sigma_{24}] \quad \text{and}$$

$$[\sigma_{13} \pm \sigma_{23} : \sigma_{13} \pm \sigma_{23} : \sigma_{13} \mp \sigma_{23}]$$

respectively. For this reason we will list the remaining midpoints of the sides in direct reference to the subtriangles in the next section.

6 Subtriangles and their centroids

We now give some precise notational conventions and listings for the subtriangles and their centroids.

6.1 Subtriangle 1

Subtriangle $\triangle_1$ has lines

$$L_{34} \equiv a_3 a_4 = \langle 1 : 1 : 0 \rangle, \quad L_{24} \equiv a_2 a_4 = \langle 1 : 0 : 1 \rangle,$$
$$L_{23} \equiv a_2 a_3 = \langle 0 : 1 : 1 \rangle$$

which correspond to the three sides of the quadrangle associated with the points a_2, a_3 and a_4. Note that the subscripts here really describe sets, although we write them as lists for brevity. As stated before, each midpoint of the sides of the quadrangle has exactly two sigma representations corresponding to the two subtriangles that a side belongs to.

In this light the sigma representations for the midpoints of these sides corresponding to the subtriangle $\triangle_1$, are given as follows;

$$m^{(34)} \equiv [\sigma_{23} - \sigma_{24} : \sigma_{24} - \sigma_{23} : \sigma_{23} + \sigma_{24}],$$
$$m^{(43)} \equiv [\sigma_{23} + \sigma_{24} : -\sigma_{24} - \sigma_{23} : \sigma_{23} - \sigma_{24}],$$
$$m^{(24)} \equiv [-\sigma_{23} - \sigma_{34} : \sigma_{34} - \sigma_{23} : \sigma_{34} + \sigma_{23}],$$
$$m^{(42)} \equiv [\sigma_{23} - \sigma_{34} : \sigma_{34} + \sigma_{23} : \sigma_{34} - \sigma_{23}],$$
$$m^{(23)} \equiv [\sigma_{34} - \sigma_{24} : -\sigma_{24} - \sigma_{34} : \sigma_{34} + \sigma_{24}],$$
$$m^{(32)} \equiv [\sigma_{34} + \sigma_{24} : \sigma_{24} - \sigma_{34} : \sigma_{34} - \sigma_{24}].$$

The corresponding medians are

$$D_1^{(34)} \equiv a_2 m^{(34)} = \langle -\sigma_{24} : \sigma_{23} : \sigma_{23} - \sigma_{24} \rangle,$$
$$D_1^{(43)} \equiv a_2 m^{(43)} = \langle \sigma_{24} : \sigma_{23} : \sigma_{23} + \sigma_{24} \rangle,$$
$$D_1^{(24)} \equiv a_3 m^{(24)} = \langle \sigma_{34} : \sigma_{23} + \sigma_{34} : \sigma_{23} \rangle,$$
$$D_1^{(42)} \equiv a_3 m^{(42)} = \langle \sigma_{34} : \sigma_{34} - \sigma_{23} : -\sigma_{23} \rangle,$$
$$D_1^{(23)} \equiv a_4 m^{(23)} = \langle \sigma_{24} + \sigma_{34} : \sigma_{34} : \sigma_{24} \rangle,$$
$$D_1^{(32)} \equiv a_4 m^{(32)} = \langle \sigma_{24} - \sigma_{34} : -\sigma_{34} : \sigma_{24} \rangle.$$

These medians are concurrent three at a time giving the four subtriangle centroids for $\triangle_1$:

$$g_1^1 \equiv D_1^{(34)} D_1^{(24)} D_1^{(23)} = \begin{bmatrix} \sigma_{23}\sigma_{24} - \sigma_{23}\sigma_{34} + \sigma_{24}\sigma_{34} : \\ \sigma_{23}\sigma_{24} + \sigma_{23}\sigma_{34} - \sigma_{24}\sigma_{34} : \\ \sigma_{23}\sigma_{24} - \sigma_{23}\sigma_{34} - \sigma_{24}\sigma_{34} \end{bmatrix},$$

$$g_1^2 \equiv D_1^{(34)} D_1^{(42)} D_1^{(32)} = \begin{bmatrix} \sigma_{23}\sigma_{24} + \sigma_{23}\sigma_{34} - \sigma_{24}\sigma_{34} : \\ \sigma_{23}\sigma_{24} - \sigma_{23}\sigma_{34} + \sigma_{24}\sigma_{34} : \\ -\sigma_{23}\sigma_{24} + \sigma_{23}\sigma_{34} + \sigma_{24}\sigma_{34} \end{bmatrix},$$

$$g_1^3 \equiv D_1^{(43)} D_1^{(24)} D_1^{(32)} = \begin{bmatrix} \sigma_{23}\sigma_{24} + \sigma_{23}\sigma_{34} + \sigma_{24}\sigma_{34} : \\ \sigma_{23}\sigma_{24} - \sigma_{23}\sigma_{34} - \sigma_{24}\sigma_{34} : \\ -\sigma_{23}\sigma_{24} + \sigma_{23}\sigma_{34} - \sigma_{24}\sigma_{34} \end{bmatrix},$$

$$g_1^4 \equiv D_1^{(43)} D_1^{(42)} D_1^{(23)} = \begin{bmatrix} \sigma_{23}\sigma_{24} - \sigma_{23}\sigma_{34} - \sigma_{24}\sigma_{34} : \\ \sigma_{23}\sigma_{24} + \sigma_{23}\sigma_{34} + \sigma_{24}\sigma_{34} : \\ -\sigma_{23}\sigma_{24} - \sigma_{23}\sigma_{34} + \sigma_{24}\sigma_{34} \end{bmatrix}.$$

In Figure 13 we see how these centroids appear.

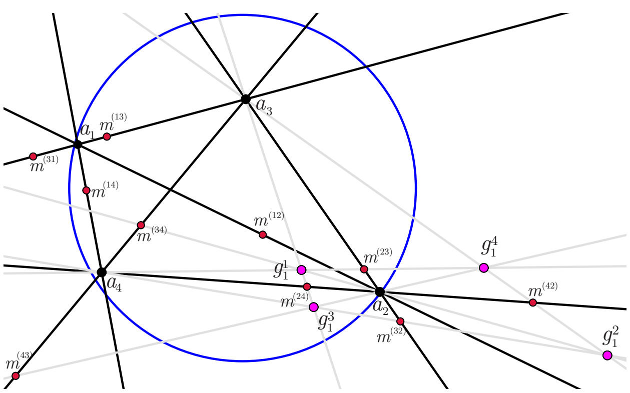

Figure 13: *Labelling for the centroids of* $\triangle_1 = \overline{a_2 a_3 a_4}$

The labelling of a centroid reflects both the subtriangle it is associated with and the three distinct midpoints used in its construction, at least one of which is of positive orientation. The centroid g_1^2 for example is associated with the midpoints $m^{(32)}$, $m^{(42)}$ and $m^{(34)}$ used to construct it, exactly one of which is positively oriented, namely $m^{(34)}$, which lies on the median $D_1^{(34)} = a_2 m^{(34)}$. Thus the superscript of the centroid g_1^2 comes from the point a_2 which is incident with the median through the positively oriented midpoint.

Similarly the centroids g_1^3 and g_1^4 are associated with precisely one positively oriented midpoint $m^{(24)}$ and $m^{(23)}$ respectively, which lie on the medians $D_1^{(24)} = a_3 m^{(24)}$ and $D_1^{(23)} = a_4 m^{(23)}$ respectively, and so are associated with the points a_3 and a_4 respectively. Finally the centroid g_1^1 is distinct from the other centroids in the sense that it is associated with *three* positively oriented midpoints, and is thus associated with the point a_1. This method for labelling the centroids can be extended to the rest of the subtriangles in the natural way. So our centroid labelling depends on and is consistent with the prior labelling of midpoints. We now proceed to the other subtriangles, where the pattern is the same. But it is useful to see all of these laid out explicitly.

6.2 Subtriangle 2

Subtriangle $\triangle_2$ has lines

$$L_{14} \equiv a_1 a_4 = \langle 0 : 1 : -1 \rangle,$$
$$L_{13} \equiv a_1 a_3 = \langle 1 : 0 : -1 \rangle,$$
$$L_{34} \equiv a_3 a_4 = \langle 1 : 1 : 0 \rangle,$$

with midpoints

$$m^{(14)} \equiv [\sigma_{13} - \sigma_{34} : \sigma_{13} + \sigma_{34} : \sigma_{13} + \sigma_{34}],$$
$$m^{(41)} \equiv [\sigma_{13} + \sigma_{34} : \sigma_{13} - \sigma_{34} : \sigma_{13} - \sigma_{34}],$$
$$m^{(13)} \equiv [\sigma_{14} + \sigma_{34} : \sigma_{14} - \sigma_{34} : \sigma_{14} + \sigma_{34}],$$
$$m^{(31)} \equiv [\sigma_{14} - \sigma_{34} : \sigma_{14} + \sigma_{34} : \sigma_{14} - \sigma_{34}],$$
$$m^{(34)} \equiv [\sigma_{13} - \sigma_{14} : \sigma_{14} - \sigma_{13} : \sigma_{13} + \sigma_{14}],$$
$$m^{(43)} \equiv [\sigma_{13} + \sigma_{14} : -\sigma_{13} - \sigma_{14} : \sigma_{13} - \sigma_{14}],$$

medians

$$D_2^{(14)} \equiv a_3 m^{(14)} = \langle \sigma_{13} + \sigma_{34} : \sigma_{34} : -\sigma_{13} \rangle,$$
$$D_2^{(41)} \equiv a_3 m^{(41)} = \langle \sigma_{34} - \sigma_{13} : \sigma_{34} : \sigma_{13} \rangle,$$
$$D_2^{(13)} \equiv a_4 m^{(13)} = \langle \sigma_{34} : \sigma_{14} + \sigma_{34} : -\sigma_{14} \rangle,$$
$$D_2^{(31)} \equiv a_4 m^{(31)} = \langle \sigma_{34} : \sigma_{34} - \sigma_{14} : \sigma_{14} \rangle,$$
$$D_2^{(34)} \equiv a_1 m^{(34)} = \langle \sigma_{13} : -\sigma_{14} : \sigma_{14} - \sigma_{13} \rangle,$$
$$D_2^{(43)} \equiv a_1 m^{(43)} = \langle \sigma_{13} : \sigma_{14} : -\sigma_{14} - \sigma_{13} \rangle,$$

and centroids

$$g_2^2 \equiv D_2^{(14)} D_2^{(13)} D_2^{(34)} = \begin{bmatrix} \sigma_{13}\sigma_{14} + \sigma_{13}\sigma_{34} - \sigma_{14}\sigma_{34} : \\ \sigma_{13}\sigma_{14} - \sigma_{13}\sigma_{34} + \sigma_{14}\sigma_{34} : \\ \sigma_{13}\sigma_{14} + \sigma_{13}\sigma_{34} + \sigma_{14}\sigma_{34} \end{bmatrix},$$

$$g_2^1 \equiv D_2^{(41)} D_2^{(31)} D_2^{(34)} = \begin{bmatrix} \sigma_{13}\sigma_{14} - \sigma_{13}\sigma_{34} + \sigma_{14}\sigma_{34} : \\ \sigma_{13}\sigma_{14} + \sigma_{13}\sigma_{34} - \sigma_{14}\sigma_{34} : \\ \sigma_{13}\sigma_{14} - \sigma_{13}\sigma_{34} - \sigma_{14}\sigma_{34} \end{bmatrix},$$

$$g_2^4 \equiv D_2^{(41)} D_2^{(13)} D_2^{(43)} = \begin{bmatrix} \sigma_{13}\sigma_{14} + \sigma_{13}\sigma_{34} + \sigma_{14}\sigma_{34} : \\ \sigma_{13}\sigma_{14} - \sigma_{13}\sigma_{34} - \sigma_{14}\sigma_{34} : \\ \sigma_{13}\sigma_{14} + \sigma_{13}\sigma_{34} - \sigma_{14}\sigma_{34} \end{bmatrix},$$

$$g_2^3 \equiv D_2^{(14)} D_2^{(31)} D_2^{(43)} = \begin{bmatrix} \sigma_{13}\sigma_{14} - \sigma_{13}\sigma_{34} - \sigma_{14}\sigma_{34} : \\ \sigma_{13}\sigma_{14} + \sigma_{13}\sigma_{34} + \sigma_{14}\sigma_{34} : \\ \sigma_{13}\sigma_{14} - \sigma_{13}\sigma_{34} + \sigma_{14}\sigma_{34} \end{bmatrix}.$$

6.3 Subtriangle 3

Subtriangle $\triangle_3$ has lines

$$L_{12} \equiv a_1 a_2 = \langle 1 : -1 : 0 \rangle, \quad L_{24} \equiv a_2 a_4 = \langle 1 : 0 : 1 \rangle,$$
$$L_{14} \equiv a_1 a_4 = \langle 0 : 1 : -1 \rangle$$

with midpoints

$$m^{(12)} \equiv [\sigma_{14} - \sigma_{24} : \sigma_{14} - \sigma_{24} : \sigma_{14} + \sigma_{24}],$$
$$m^{(21)} \equiv [\sigma_{14} + \sigma_{24} : \sigma_{14} + \sigma_{24} : \sigma_{14} - \sigma_{24}],$$
$$m^{(24)} \equiv [-\sigma_{12} - \sigma_{14} : \sigma_{14} - \sigma_{12} : \sigma_{12} + \sigma_{14}],$$
$$m^{(42)} \equiv [\sigma_{12} - \sigma_{14} : \sigma_{14} + \sigma_{12} : \sigma_{14} - \sigma_{12}],$$
$$m^{(14)} \equiv [\sigma_{12} - \sigma_{24} : \sigma_{12} + \sigma_{24} : \sigma_{12} + \sigma_{24}],$$
$$m^{(41)} \equiv [\sigma_{12} + \sigma_{24} : \sigma_{12} - \sigma_{24} : \sigma_{12} - \sigma_{24}],$$

medians

$$D_3^{(12)} \equiv a_4 m^{(12)} = \langle \sigma_{24} : \sigma_{14} : \sigma_{24} - \sigma_{14} \rangle,$$
$$D_3^{(21)} \equiv a_4 m^{(21)} = \langle \sigma_{24} : -\sigma_{14} : \sigma_{24} + \sigma_{14} \rangle,$$
$$D_3^{(24)} \equiv a_1 m^{(24)} = \langle \sigma_{12} : -\sigma_{12} - \sigma_{14} : \sigma_{14} \rangle,$$
$$D_3^{(42)} \equiv a_1 m^{(42)} = \langle \sigma_{12} : \sigma_{14} - \sigma_{12} : -\sigma_{14} \rangle,$$
$$D_3^{(14)} \equiv a_2 m^{(14)} = \langle \sigma_{12} + \sigma_{24} : -\sigma_{12} : \sigma_{24} \rangle,$$
$$D_3^{(41)} \equiv a_2 m^{(41)} = \langle \sigma_{24} - \sigma_{12} : \sigma_{12} : \sigma_{24} \rangle,$$

and centroids

$$g_3^3 \equiv D_3^{(12)} D_3^{(24)} D_3^{(14)} = \begin{bmatrix} \sigma_{12}\sigma_{14} - \sigma_{12}\sigma_{24} - \sigma_{14}\sigma_{24} : \\ \sigma_{12}\sigma_{14} - \sigma_{12}\sigma_{24} + \sigma_{14}\sigma_{24} : \\ \sigma_{12}\sigma_{14} + \sigma_{12}\sigma_{24} + \sigma_{14}\sigma_{24} \end{bmatrix},$$

$$g_3^4 \equiv D_3^{(12)} D_3^{(42)} D_3^{(41)} = \begin{bmatrix} \sigma_{12}\sigma_{14} - \sigma_{12}\sigma_{24} + \sigma_{14}\sigma_{24} : \\ \sigma_{12}\sigma_{14} - \sigma_{12}\sigma_{24} - \sigma_{14}\sigma_{24} : \\ \sigma_{12}\sigma_{14} + \sigma_{12}\sigma_{24} - \sigma_{14}\sigma_{24} \end{bmatrix},$$

$$g_3^1 \equiv D_3^{(21)} D_3^{(24)} D_3^{(41)} = \begin{bmatrix} \sigma_{12}\sigma_{14} + \sigma_{12}\sigma_{24} + \sigma_{14}\sigma_{24} : \\ \sigma_{12}\sigma_{14} + \sigma_{12}\sigma_{24} - \sigma_{14}\sigma_{24} : \\ \sigma_{12}\sigma_{14} - \sigma_{12}\sigma_{24} - \sigma_{14}\sigma_{24} \end{bmatrix},$$

$$g_3^2 \equiv D_3^{(21)} D_3^{(42)} D_3^{(14)} = \begin{bmatrix} \sigma_{12}\sigma_{14} + \sigma_{12}\sigma_{24} - \sigma_{14}\sigma_{24} : \\ \sigma_{12}\sigma_{14} + \sigma_{12}\sigma_{24} + \sigma_{14}\sigma_{24} : \\ \sigma_{12}\sigma_{14} - \sigma_{12}\sigma_{24} + \sigma_{14}\sigma_{24} \end{bmatrix}.$$

6.4 Subtriangle 4

Finally subtriangle $\triangle_4$ has lines

$$L_{23} \equiv a_2 a_3 = \langle 0 : 1 : 1 \rangle, \quad L_{13} \equiv a_1 a_3 = \langle 1 : 0 : -1 \rangle,$$
$$L_{12} \equiv a_1 a_2 = \langle 1 : -1 : 0 \rangle,$$

with midpoints

$$m^{(23)} \equiv [\sigma_{13} - \sigma_{12} : -\sigma_{12} - \sigma_{13} : \sigma_{12} + \sigma_{13}],$$
$$m^{(32)} \equiv [\sigma_{12} + \sigma_{13} : \sigma_{12} - \sigma_{13} : \sigma_{13} - \sigma_{12}],$$
$$m^{(13)} \equiv [\sigma_{12} + \sigma_{23} : \sigma_{12} - \sigma_{23} : \sigma_{12} + \sigma_{23}],$$
$$m^{(31)} \equiv [\sigma_{12} - \sigma_{23} : \sigma_{12} + \sigma_{23} : \sigma_{12} - \sigma_{23}],$$
$$m^{(12)} \equiv [\sigma_{13} - \sigma_{23} : \sigma_{13} - \sigma_{23} : \sigma_{13} + \sigma_{23}],$$
$$m^{(21)} \equiv [\sigma_{13} + \sigma_{23} : \sigma_{13} + \sigma_{23} : \sigma_{13} - \sigma_{23}],$$

medians

$$D_4^{(23)} \equiv a_1 m^{(23)} = \langle -\sigma_{12} - \sigma_{13} : \sigma_{12} : \sigma_{13} \rangle,$$
$$D_4^{(32)} \equiv a_1 m^{(32)} = \langle \sigma_{12} - \sigma_{13} : -\sigma_{12} : \sigma_{13} \rangle,$$
$$D_4^{(13)} \equiv a_2 m^{(13)} = \langle -\sigma_{12} : \sigma_{12} + \sigma_{23} : \sigma_{23} \rangle,$$
$$D_4^{(31)} \equiv a_2 m^{(31)} = \langle \sigma_{12} : \sigma_{23} - \sigma_{12} : \sigma_{23} \rangle,$$
$$D_4^{(12)} \equiv a_3 m^{(12)} = \langle \sigma_{13} : \sigma_{23} : \sigma_{23} - \sigma_{13} \rangle,$$
$$D_4^{(21)} \equiv a_3 m^{(21)} = \langle -\sigma_{13} : \sigma_{23} : \sigma_{23} + \sigma_{13} \rangle,$$

and centroids

$$g_4^4 \equiv D_4^{(12)} D_4^{(13)} D_4^{(23)} = \begin{bmatrix} \sigma_{12}\sigma_{13} - \sigma_{12}\sigma_{23} + \sigma_{13}\sigma_{23} : \\ \sigma_{12}\sigma_{13} - \sigma_{12}\sigma_{23} - \sigma_{13}\sigma_{23} : \\ \sigma_{12}\sigma_{13} + \sigma_{12}\sigma_{23} + \sigma_{13}\sigma_{23} \end{bmatrix},$$

$$g_4^3 \equiv D_4^{(12)} D_4^{(31)} D_4^{(32)} = \begin{bmatrix} \sigma_{12}\sigma_{13} - \sigma_{12}\sigma_{23} - \sigma_{13}\sigma_{23} : \\ \sigma_{12}\sigma_{13} - \sigma_{12}\sigma_{23} + \sigma_{13}\sigma_{23} : \\ \sigma_{12}\sigma_{13} + \sigma_{12}\sigma_{23} - \sigma_{13}\sigma_{23} \end{bmatrix},$$

$$g_4^2 \equiv D_4^{(21)} D_4^{(13)} D_4^{(32)} = \begin{bmatrix} \sigma_{12}\sigma_{13} + \sigma_{12}\sigma_{23} + \sigma_{13}\sigma_{23} : \\ \sigma_{12}\sigma_{13} + \sigma_{12}\sigma_{23} - \sigma_{13}\sigma_{23} : \\ \sigma_{12}\sigma_{13} - \sigma_{12}\sigma_{23} + \sigma_{13}\sigma_{23} \end{bmatrix},$$

$$g_4^1 \equiv D_4^{(21)} D_4^{(31)} D_4^{(23)} = \begin{bmatrix} \sigma_{12}\sigma_{13} + \sigma_{12}\sigma_{23} - \sigma_{13}\sigma_{23} : \\ \sigma_{12}\sigma_{13} + \sigma_{12}\sigma_{23} + \sigma_{13}\sigma_{23} : \\ \sigma_{12}\sigma_{13} - \sigma_{12}\sigma_{23} - \sigma_{13}\sigma_{23} \end{bmatrix}.$$

7 Finding the quadrangle centroids

What we would like to do is use centroids of subtriangles to construct centroids of the quadrangle $\overline{a_1a_2a_3a_4}$. In Figure 14 we see some of the sixteen subtriangle centroids, using different colours for each subtriangle. It turns out that these sixteen centroids enjoy some interesting relations.

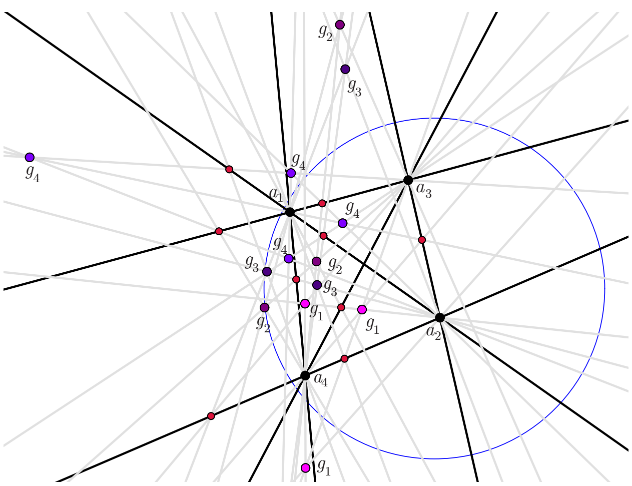

Figure 14: *Subtriangle centroids of the quadrangle* $\overline{a_1a_2a_3a_4}$

7.1 Midpoint consistencies

Define the **set of associated midpoints** (or **associated midpoints**) S_j^i for a centroid point g_j^i of a Triangle $\triangle_i$ to be the (unique) set of three distinct midpoints used to construct it. For example the centroid g_4^4 of the subtriangle $\triangle_4$ is constructed from the median lines $D_4^{(12)}$, $D_4^{(13)}$, and $D_4^{(23)}$, and hence

$$S_4^4 \equiv \left\{ m^{(12)}, m^{(13)}, m^{(23)} \right\}.$$

A set $\left\{ g_1^{i_1}, g_2^{i_2}, g_3^{i_3}, g_4^{i_4} \right\}$ containing one centroid from each subtriangle of the quadrangle $\overline{a_1a_2a_3a_4}$ is said to be **midpoint consistent** if the union of the associated midpoint sets

$$S_m \equiv \cup_k S_k^{i_k}$$

contains exactly one midpoint from each of the six sides of the quadrangle. For example the set

$$S = \left\{ g_1^1, g_2^2, g_3^3, g_4^4 \right\}$$

is midpoint consistent since the union of associated midpoints is

$$S_m = \left\{ m^{(12)}, m^{(34)}, m^{(13)}, m^{(24)}, m^{(14)}, m^{(23)} \right\}$$

which has exactly six elements, one from each side.

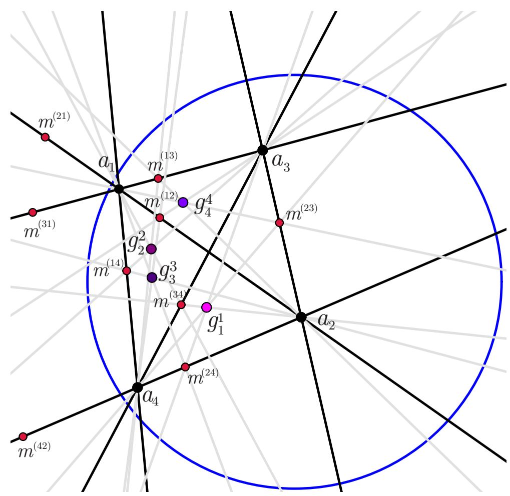

Figure 15: *A midpoint consistent set of subtriangle centroids with six associated midpoints*

On the other hand the set

$$S = \left\{ g_1^1, g_2^2, g_3^4, g_4^3 \right\}$$

is not midpoint consistent, since the union of associated midpoints

$$S_m = \left\{ m^{(12)}, m^{(34)}, m^{(13)}, m^{(31)}, m^{(42)}, m^{(14)}, m^{(41)}, m^{(32)} \right\}$$

contains both midpoints for the sides $\overline{a_1a_3}$ and $\overline{a_1a_4}$, and in addition contains eight points.

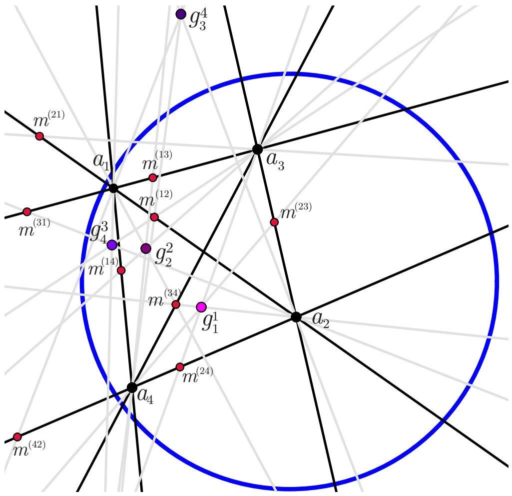

Figure 16: *A set of subtriangle centroids which are not midpoint consistent*

Theorem 5 (Midpoint consistent centroids) *There are exactly eight distinct sets of subtriangle centroids*

$$S = \left\{g_1^{i_1}, g_2^{i_2}, g_3^{i_3}, g_4^{i_4}\right\}$$

which are midpoint consistent.

Proof. We can see this by trying to construct a midpoint consistent set of subtriangle centroids $S = \left\{g_1^{i_1}, g_2^{i_2}, g_3^{i_3}, g_4^{i_4}\right\}$. First let's choose $i_1 = 1$, that is let g_1^1 be in the set. The centroid g_1^1 has associated midpoints $S_1^1 \equiv \left\{m^{(23)}, m^{(24)}, m^{(34)}\right\}$, and so for any other centroid in S, the midpoints $m^{(32)}, m^{(42)}$, and $m^{(43)}$ cannot be in their respective set of associated midpoints.

So looking at the centroids $g_2^{i_2}$ of the subtriangle $\triangle_2$ and their associated midpoints

$$g_2^2 : S_2^2 \equiv \left\{m^{(13)}, m^{(14)}, m^{(34)}\right\},$$
$$g_2^1 : S_2^1 \equiv \left\{m^{(31)}, m^{(41)}, m^{(34)}\right\},$$
$$g_2^4 : S_2^4 \equiv \left\{m^{(31)}, m^{(14)}, m^{(43)}\right\},$$
$$g_2^3 : S_2^3 \equiv \left\{m^{(13)}, m^{(41)}, m^{(43)}\right\},$$

we see that the centroids g_2^2 and g_2^1 are the only options for S. If we choose $i_2 = 2$, then the set

$$\left\{m^{(34)}, m^{(13)}, m^{(24)} m^{(14)}, m^{(23)}\right\} \subseteq S_m$$

which forces $i_3 = 3$, and $i_4 = 4$, as $S_3^3 \equiv \left\{m^{(12)}, m^{(14)}, m^{(24)}\right\}$ and $S_4^4 \equiv \left\{m^{(12)}, m^{(13)}, m^{(23)}\right\}$. Else if $i_2 = 1$, then the set

$$\left\{m^{(34)}, m^{(31)}, m^{(24)}, m^{(41)}, m^{(23)}\right\} \subseteq S_m$$

which forces $i_3 = i_4 = 1$, as $S_3^1 \equiv \left\{m^{(21)}, m^{(31)}, m^{(24)}\right\}$ and $S_4^1 \equiv \left\{m^{(21)}, m^{(31)}, m^{(23)}\right\}$. Therefore there are two distinct midpoint consistent sets of subtriangle centroids which contain the centroid g_1^1. The above method can be used for any choice of $i_1 \in \{1,2,3,4\}$. Hence there are in total exactly eight distinct sets of subtriangle centroids which are midpoint consistent which can be constructed in this manner. Each of these sets are distinct and this construction in fact covers all the cases of midpoint consistent sets. □

Corollary 1 *The complete list of midpoint consistent sets is given below:*

$S:$	$S_m:$
$S_\iota \equiv \{g_1^1, g_2^2, g_3^3, g_4^4\}$	$\left\{m^{(12)}, m^{(34)}, m^{(13)}, m^{(24)}, m^{(14)}, m^{(23)}\right\}$
$S_\alpha \equiv \{g_1^2, g_2^1, g_3^4, g_4^3\}$	$\left\{m^{(12)}, m^{(34)}, m^{(31)}, m^{(42)}, m^{(41)}, m^{(32)}\right\}$
$S_\beta \equiv \{g_1^3, g_2^4, g_3^1, g_4^2\}$	$\left\{m^{(21)}, m^{(43)}, m^{(13)}, m^{(24)}, m^{(41)}, m^{(32)}\right\}$
$S_\gamma \equiv \{g_1^4, g_2^3, g_3^2, g_4^1\}$	$\left\{m^{(21)}, m^{(43)}, m^{(31)}, m^{(42)}, m^{(14)}, m^{(23)}\right\}$
$S_1 \equiv \{g_1^1, g_2^1, g_3^1, g_4^1\}$	$\left\{m^{(21)}, m^{(34)}, m^{(31)}, m^{(24)}, m^{(41)}, m^{(23)}\right\}$
$S_2 \equiv \{g_1^2, g_2^2, g_3^2, g_4^2\}$	$\left\{m^{(21)}, m^{(34)}, m^{(13)}, m^{(42)}, m^{(14)}, m^{(32)}\right\}$
$S_3 \equiv \{g_1^3, g_2^3, g_3^3, g_4^3\}$	$\left\{m^{(12)}, m^{(43)}, m^{(31)}, m^{(24)}, m^{(14)}, m^{(32)}\right\}$
$S_4 \equiv \{g_1^4, g_2^4, g_3^4, g_4^4\}$	$\left\{m^{(12)}, m^{(43)}, m^{(13)}, m^{(42)}, m^{(41)}, m^{(23)}\right\}$.

These midpoint consistent sets of subtriangle centroids are quite pleasant. Indeed the labels almost chose themselves. The S_α, S_β, and S_γ sets represent the α, β, and γ pairings of the subtriangle centroids respectively, while the S_ι set represents the ι pairing or *identity pairing* of subtriangle centroids. The S_1, S_2, S_3 and S_4 sets contain the subtriangle centroids associated with the points a_1, a_2, a_3 and a_4 respectively.

It turns out that these sets of midpoint consistent subtriangle centroids behave in a very similar way as do quadrangle barycentres in the Euclidean case.

7.2 Quadrangle centroids

Define the **g-lines** to be the join $a_i g_i$ of the points a_i with the centroids g_i of the corresponding subtriangle $\triangle_i$. Thus a quadrangle has sixteen distinct g-lines, four passing through each point a_i.

Theorem 6 (Quadrangle centroids) *The four g-lines associated with a midpoint consistent set of subtriangle centroids are concurrent, producing eight distinct **quadrangle centroids** q.*

Proof. Consider the midpoint consistent set of subtriangle centroids

$S_\iota = \{g_1^1, g_2^2, g_3^3, g_4^4\}$.

The set S_ι has the four associated g-lines

$$a_1g_1^1 = \langle \sigma_{23}(\sigma_{24}+\sigma_{34}) : -\sigma_{24}(\sigma_{23}+\sigma_{34}) : \sigma_{34}(\sigma_{24}-\sigma_{23})\rangle,$$
$$a_2g_2^2 = \langle \sigma_{14}(\sigma_{13}+\sigma_{34}) : -\sigma_{13}(\sigma_{14}+\sigma_{34}) : \sigma_{34}(\sigma_{14}-\sigma_{13})\rangle,$$
$$a_3g_3^3 = \langle \sigma_{14}(\sigma_{12}+\sigma_{24}) : \sigma_{24}(\sigma_{12}+\sigma_{14}) : \sigma_{12}(\sigma_{24}-\sigma_{14})\rangle,$$
$$a_4g_4^4 = \langle \sigma_{23}(\sigma_{12}+\sigma_{13}) : \sigma_{13}(\sigma_{12}+\sigma_{23}) : \sigma_{12}(\sigma_{23}-\sigma_{13})\rangle.$$

Now let $q_\iota \equiv \left(a_1g_1^1\right)\left(a_2g_2^2\right)$, then we find that

$$\begin{aligned}
q_\iota &= \left\langle \begin{matrix} \sigma_{23}(\sigma_{24}+\sigma_{34}) : \\ -\sigma_{24}(\sigma_{23}+\sigma_{34}) : \\ \sigma_{34}(\sigma_{24}-\sigma_{23}) \end{matrix} \right\rangle \times \left\langle \begin{matrix} \sigma_{14}(\sigma_{13}+\sigma_{34}) : \\ -\sigma_{13}(\sigma_{14}+\sigma_{34}) : \\ \sigma_{34}(\sigma_{14}-\sigma_{13}) \end{matrix} \right\rangle \\
&= \begin{bmatrix} \sigma_{24}\sigma_{34}(\sigma_{13}-\sigma_{14})(\sigma_{23}+\sigma_{34}) \\ \quad -\sigma_{13}\sigma_{34}(\sigma_{23}-\sigma_{24})(\sigma_{14}+\sigma_{34}) : \\ \sigma_{23}\sigma_{34}(\sigma_{13}-\sigma_{14})(\sigma_{24}+\sigma_{34}) \\ \quad -\sigma_{14}\sigma_{34}(\sigma_{23}-\sigma_{24})(\sigma_{13}+\sigma_{34}) : \\ \sigma_{14}\sigma_{24}(\sigma_{23}+\sigma_{34})(\sigma_{13}+\sigma_{34}) \\ \quad -\sigma_{13}\sigma_{23}(\sigma_{24}+\sigma_{34})(\sigma_{14}+\sigma_{34}) \end{bmatrix} \\
&= \begin{bmatrix} -\sigma_{34}(\eta(\sigma_{13}-\sigma_{14}-\sigma_{23}+\sigma_{24}-2\sigma_{34})+\sigma_3+\sigma_4) : \\ -\sigma_{34}(\eta(\sigma_{13}-\sigma_{14}-\sigma_{23}+\sigma_{24}+2\sigma_{34})-\sigma_3-\sigma_4) : \\ -\sigma_{34}(\eta(\sigma_{13}-\sigma_{14}+\sigma_{23}-\sigma_{24})+\sigma_3-\sigma_4) \end{bmatrix}
\end{aligned}$$

where $\eta = \sigma_{12}\sigma_{34} = \sigma_{13}\sigma_{24} = \sigma_{14}\sigma_{23}$. Now as $\sigma_{34} \neq 0$ we can use the fact that $\eta = \sigma_{12}\sigma_{34}$ and divide by $(\sigma_{34})^2$ to get the representation

$$q_\iota = \begin{bmatrix} \sigma_{12}(\sigma_{13}-\sigma_{14}-\sigma_{23}+\sigma_{24}-2\sigma_{34})+\sigma_{13}\sigma_{23}+\sigma_{14}\sigma_{24} : \\ \sigma_{12}(\sigma_{13}-\sigma_{14}-\sigma_{23}+\sigma_{24}+2\sigma_{34})-\sigma_{13}\sigma_{23}-\sigma_{14}\sigma_{24} : \\ \sigma_{12}(\sigma_{13}-\sigma_{14}+\sigma_{23}-\sigma_{24})+\sigma_{13}\sigma_{23}-\sigma_{14}\sigma_{24} \end{bmatrix}.$$

Recalling that A_3 and A_4 have representations $\sigma_{12}/(\sigma_{13}\sigma_{23})$ and $\sigma_{12}/(\sigma_{14}\sigma_{24})$ respectively, divide by σ_{12} to get the alternate expression

$$q_\iota = \begin{bmatrix} \sigma_{13}-\sigma_{14}-\sigma_{23}+\sigma_{24}-2\sigma_{34}+1/A_3+1/A_4 : \\ \sigma_{13}-\sigma_{14}-\sigma_{23}+\sigma_{24}+2\sigma_{34}-1/A_3-1/A_4 : \\ \sigma_{13}-\sigma_{14}+\sigma_{23}-\sigma_{24}+1/A_3-1/A_4 \end{bmatrix}.$$

It is then a calculation that q_ι is also incident with the g-lines $a_3g_3^3$ and $a_4g_4^4$, so that the four g-lines are concurrent at the point q_ι. The representation of q_ι above seems to be more associated with the points a_3 and a_4 when looking at the subscripts of the sigma values, and in particular the terms A_3 and A_4 are involved in the equation. This is due in part to the two g-lines we chose at the start, and so since there are six ways to choose two from four there are also six equal representations for the centroid q_ι. The five remaining representations of q_ι are

$$\left(a_3g_3^3\right)\left(a_4g_4^4\right) = \begin{bmatrix} \sigma_{13}-\sigma_{14}-\sigma_{23}+\sigma_{24}-2\sigma_{12}+1/A_1+1/A_2 : \\ -\sigma_{13}+\sigma_{14}+\sigma_{23}-\sigma_{24}-2\sigma_{12}+1/A_1+1/A_2 : \\ \sigma_{13}+\sigma_{14}-\sigma_{23}-\sigma_{24}+1/A_1-1/A_2 \end{bmatrix},$$

$$\left(a_1g_1^1\right)\left(a_3g_3^3\right) = \begin{bmatrix} \sigma_{12}+\sigma_{14}+\sigma_{23}+\sigma_{34}-2\sigma_{24}-1/A_2-1/A_4 : \\ \sigma_{12}+\sigma_{14}-\sigma_{23}-\sigma_{34}-1/A_2+1/A_4 : \\ \sigma_{12}+\sigma_{14}+\sigma_{23}+\sigma_{34}+2\sigma_{24}+1/A_2+1/A_4 \end{bmatrix},$$

$$\left(a_2g_2^2\right)\left(a_4g_4^4\right) = \begin{bmatrix} -\sigma_{12}-\sigma_{14}-\sigma_{23}-\sigma_{34}+2\sigma_{13}+1/A_1+1/A_3 : \\ -\sigma_{12}+\sigma_{14}-\sigma_{23}+\sigma_{34}+1/A_1-1/A_3 : \\ \sigma_{12}+\sigma_{14}+\sigma_{23}+\sigma_{34}+2\sigma_{13}+1/A_1+1/A_3 \end{bmatrix},$$

$$\left(a_1g_1^1\right)\left(a_4g_4^4\right) = \begin{bmatrix} \sigma_{12}+\sigma_{13}-\sigma_{24}-\sigma_{34}-1/A_2+1/A_3 : \\ \sigma_{12}+\sigma_{13}+\sigma_{24}+\sigma_{34}-2\sigma_{23}-1/A_2-1/A_3 : \\ \sigma_{12}+\sigma_{13}+\sigma_{24}+\sigma_{34}+2\sigma_{23}+1/A_2+1/A_3 \end{bmatrix},$$

$$\left(a_2g_2^2\right)\left(a_3g_3^3\right) = \begin{bmatrix} -\sigma_{12}+\sigma_{13}-\sigma_{24}+\sigma_{34}+1/A_1-1/A_4 : \\ -\sigma_{12}-\sigma_{13}-\sigma_{24}-\sigma_{34}+2\sigma_{14}+1/A_1+1/A_4 : \\ \sigma_{12}+\sigma_{13}+\sigma_{24}+\sigma_{34}+2\sigma_{14}+1/A_1+1/A_4 \end{bmatrix}.$$

These can be transformed from one to another through an appropriate multiplication of a non-zero scalar.

We can similarly do this for each of the other midpoint consistent sets of subtriangle centroids producing eight distinct quadrangle centroids. □

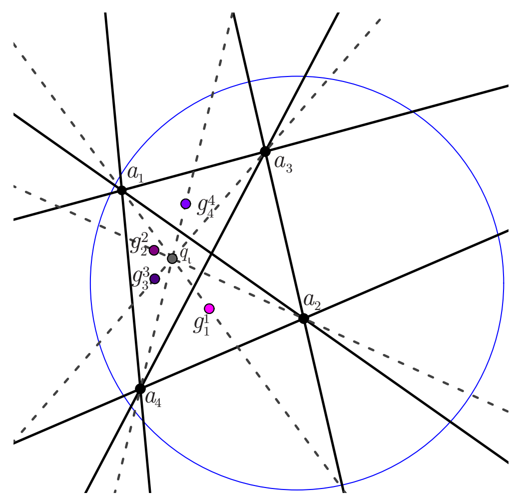

Figure 17: *Concurrent g-lines associated with a midpoint consistent set of subtriangle centroids*

The six different representations of a quadrangle centroid give an equal weighting between all the four points of the quadrangle. Despite this, we will break this symmetry to save space, and write all the quadrangle centroids in the representation derived from the meet of the g-lines $a_1g_1^{i_1}$ and $a_2g_2^{i_2}$ which are associated with the points a_1 and a_2. These representations have a bias associated with the points a_3 and a_4.

The eight distinct quadrangle centroids are then

$$q_\iota \equiv \begin{bmatrix} \sigma_{13}-\sigma_{14}-\sigma_{23}+\sigma_{24}-2\sigma_{34}+1/A_3+1/A_4 : \\ \sigma_{13}-\sigma_{14}-\sigma_{23}+\sigma_{24}+2\sigma_{34}-1/A_3-1/A_4 : \\ \sigma_{13}-\sigma_{14}+\sigma_{23}-\sigma_{24}+1/A_3-1/A_4 \end{bmatrix},$$

$$q_\alpha \equiv \begin{bmatrix} \sigma_{13}-\sigma_{14}-\sigma_{23}+\sigma_{24}+2\sigma_{34}-1/A_3-1/A_4 : \\ \sigma_{13}-\sigma_{14}-\sigma_{23}+\sigma_{24}-2\sigma_{34}+1/A_3+1/A_4 : \\ \sigma_{13}-\sigma_{14}+\sigma_{23}-\sigma_{24}-1/A_3+1/A_4 \end{bmatrix},$$

$$q_\beta \equiv \begin{bmatrix} \sigma_{13}+\sigma_{14}+\sigma_{23}+\sigma_{24}+2\sigma_{34}+1/A_3+1/A_4 : \\ \sigma_{13}+\sigma_{14}+\sigma_{23}+\sigma_{24}-2\sigma_{34}-1/A_3-1/A_4 : \\ \sigma_{13}+\sigma_{14}-\sigma_{23}-\sigma_{24}+A_3-A_4 \end{bmatrix},$$

$$q_\gamma \equiv \begin{bmatrix} \sigma_{13}+\sigma_{14}+\sigma_{23}+\sigma_{24}-2\sigma_{34}-1/A_3-1/A_4 : \\ \sigma_{13}+\sigma_{14}+\sigma_{23}+\sigma_{24}+2\sigma_{34}+1/A_3+1/A_4 : \\ \sigma_{13}+\sigma_{14}-\sigma_{23}-\sigma_{24}-1/A_3+1/A_4 \end{bmatrix},$$

$$q_1 \equiv \begin{bmatrix} \sigma_{13}-\sigma_{14}+\sigma_{23}-\sigma_{24}+2\sigma_{34}-1/A_3-1/A_4 : \\ \sigma_{13}-\sigma_{14}+\sigma_{23}-\sigma_{24}-2\sigma_{34}+1/A_3+1/A_4 : \\ \sigma_{13}-\sigma_{14}-\sigma_{23}+\sigma_{24}-1/A_3+1/A_4 \end{bmatrix},$$

$$q_2 \equiv \begin{bmatrix} \sigma_{13}-\sigma_{14}+\sigma_{23}-\sigma_{24}-2\sigma_{34}+1/A_3+1/A_4 : \\ \sigma_{13}-\sigma_{14}+\sigma_{23}-\sigma_{24}+2\sigma_{34}-1/A_3-1/A_4 : \\ \sigma_{13}-\sigma_{14}-\sigma_{23}+\sigma_{24}+1/A_3-1/A_4 \end{bmatrix},$$

$$q_3 \equiv \begin{bmatrix} \sigma_{13}+\sigma_{14}-\sigma_{23}-\sigma_{24}-2\sigma_{34}-1/A_3-1/A_4 : \\ \sigma_{13}+\sigma_{14}-\sigma_{23}-\sigma_{24}+2\sigma_{34}+1/A_3+1/A_4 : \\ \sigma_{13}+\sigma_{14}+\sigma_{23}+\sigma_{24}-1/A_3+1/A_4 \end{bmatrix},$$

$$q_4 \equiv \begin{bmatrix} \sigma_{13}+\sigma_{14}-\sigma_{23}-\sigma_{24}+2\sigma_{34}+1/A_3+1/A_4 : \\ \sigma_{13}+\sigma_{14}-\sigma_{23}-\sigma_{24}-2\sigma_{34}-1/A_3-1/A_4 : \\ \sigma_{13}+\sigma_{14}+\sigma_{23}+\sigma_{24}+1/A_3-1/A_4 \end{bmatrix}.$$

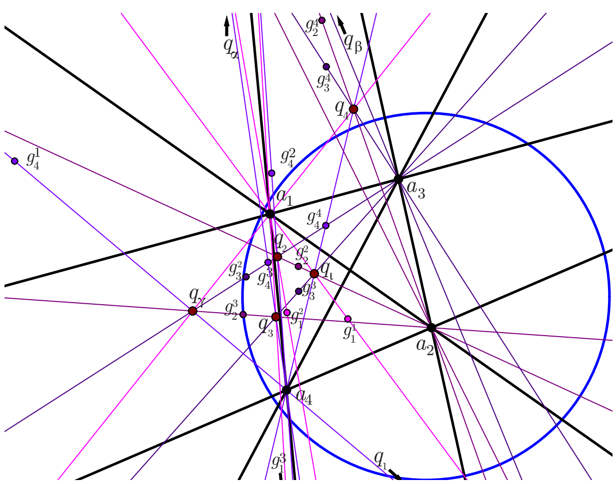

Figure 18: *The quadrangle centroids produced by g-lines*

From the list of midpoint consistent sets of subtriangle centroids and from the Quadrangle Centroid theorem we see that each g-line passes through exactly two quadrangle centroids. Moreover each g-line that passes through each of the quadrangle centroids $q_\iota, q_\alpha, q_\beta$ and q_γ is incident with one of the quadrangle centroids q_1, q_2, q_3, and q_4 and *not* any of the other quadrangle centroids $q_\iota, q_\alpha, q_\beta$ and q_γ. This is a symmetric relation, and so it is useful to consider the eight centroids as two quadrangles

$$\square_a \equiv \overline{q_\iota q_\alpha q_\beta q_\gamma} = \{q_\iota, q_\alpha, q_\beta, q_\gamma\}$$

and

$$\square_b \equiv \overline{q_1 q_2 q_3 q_4} = \{q_1, q_2, q_3, q_4\}.$$

We record this in a theorem.

Theorem 7 (Quadrangle centroid perspectivities) *The three quadrangles $\square$, $\square_a$ and $\square_b$ are pair-wise perspective in four ways, where the points of perspectivity are exactly the points of the third quadrangle.*

This is seen in Figure 19. The grey lines, which show the perspectivities, are exactly the g-lines of the quadrangle $\square = \overline{a_1 a_2 a_3 a_4}$.

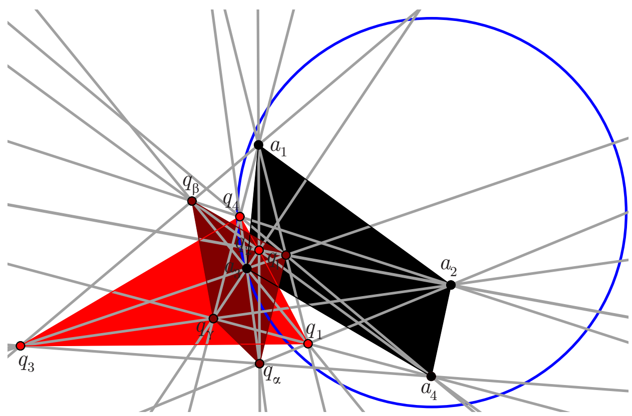

Figure 19: *Three perspective quadrangles $\square$, $\square_a$ and $\square_b$*

8 Bimedian Lines

Just as in the Euclidean case, there is more than one way to find the barycentric centroid of a quadrangle; we can also look at meets of the bimedian lines of the quadrangle.

A **bimedian line** $B_{\{ij,k\ell\}}$ is the join of two midpoints $m^{(ij)}$ and $m^{(k\ell)}$ from opposite sides $\overline{a_i a_j}$ and $\overline{a_k a_\ell}$ of the quadrangle, where $\{1,2,3,4\} = \{i,j,k,\ell\}$. By calculation, the bimedian lines $B_{\{ij,k\ell\}}$ of the quadrangle $\overline{a_1 a_2 a_3 a_4}$ are given as follows- note the pleasant linear aspect of these expressions.

The bimedian lines corresponding to the α opposite sides are:

$$B_{\{12,34\}} \equiv \langle \sigma_{13}-\sigma_{24} : \sigma_{23}-\sigma_{14} : \sigma_{23}+\sigma_{14}-\sigma_{13}-\sigma_{24} \rangle,$$
$$B_{\{12,43\}} \equiv \langle \sigma_{13}+\sigma_{24} : \sigma_{23}+\sigma_{14} : \sigma_{23}-\sigma_{14}-\sigma_{13}+\sigma_{24} \rangle,$$
$$B_{\{21,34\}} \equiv \langle \sigma_{13}+\sigma_{24} : -\sigma_{23}-\sigma_{14} : \sigma_{14}-\sigma_{23}-\sigma_{13}+\sigma_{24} \rangle,$$
$$B_{\{21,43\}} \equiv \langle \sigma_{24}-\sigma_{13} : \sigma_{23}-\sigma_{14} : \sigma_{23}+\sigma_{14}+\sigma_{13}+\sigma_{24} \rangle.$$

The bimedian lines corresponding to the β opposite sides are:

$$B_{\{13,24\}} \equiv \langle \sigma_{34}-\sigma_{12} : \sigma_{14}+\sigma_{34}+\sigma_{23}+\sigma_{12} : \sigma_{23}-\sigma_{14}\rangle,$$
$$B_{\{13,42\}} \equiv \langle \sigma_{34}+\sigma_{12} : \sigma_{14}-\sigma_{23}+\sigma_{34}-\sigma_{12} : -\sigma_{14}-\sigma_{23}\rangle,$$
$$B_{\{31,24\}} \equiv \langle \sigma_{34}+\sigma_{12} : \sigma_{23}-\sigma_{14}+\sigma_{34}-\sigma_{12} : \sigma_{14}+\sigma_{23}\rangle,$$
$$B_{\{31,42\}} \equiv \langle \sigma_{12}-\sigma_{34} : \sigma_{23}+\sigma_{14}-\sigma_{34}-\sigma_{12} : \sigma_{23}-\sigma_{14}\rangle.$$

The bimedian lines corresponding to the γ opposite sides are:

$$B_{\{14,23\}} \equiv \langle \sigma_{12}+\sigma_{34}+\sigma_{13}+\sigma_{24} : \sigma_{34}-\sigma_{12} : \sigma_{24}-\sigma_{13}\rangle,$$
$$B_{\{14,32\}} \equiv \langle \sigma_{12}-\sigma_{34}+\sigma_{24}-\sigma_{13} : -\sigma_{34}-\sigma_{12} : \sigma_{24}+\sigma_{13}\rangle,$$
$$B_{\{41,23\}} \equiv \langle \sigma_{34}-\sigma_{12}+\sigma_{24}-\sigma_{13} : \sigma_{34}+\sigma_{12} : \sigma_{24}+\sigma_{13}\rangle,$$
$$B_{\{41,32\}} \equiv \langle \sigma_{34}+\sigma_{12}-\sigma_{24}-\sigma_{13} : \sigma_{34}-\sigma_{12} : \sigma_{13}-\sigma_{24}\rangle.$$

Theorem 8 (Quadrangle bimedian centroids) *The bimedian lines $B_{\{ij,k\ell\}}$ of the quadrangle are concurrent three at a time at the quadrangle centroids.*

Proof. Given that we have equations of all the bimedian lines and the quadrangle centroids, this is a calculation involving the various sigma relations. □

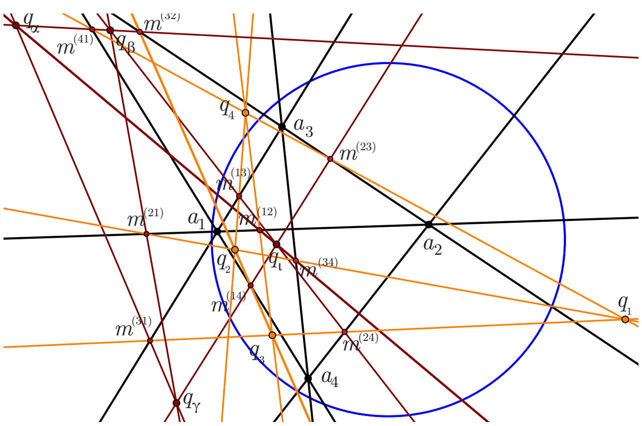

Figure 20: *The bimedian lines of a quadrangle determine the quadrangle centroids*

To record which triples of bimedian lines are concurrent, we once again can look at midpoint consistent sets of subtriangle centroids, and then look at the bimedian lines induced from the union of the associated midpoints.

Returning to the example previous we have the midpoint consistent set of subtriangle centroids $S_\iota = \{g_1^1, g_2^2, g_3^3, g_4^4\}$, where the union of associated midpoints is $\left\{m^{(12)}, m^{(34)}, m^{(13)}, m^{(24)}, m^{(14)}, m^{(23)}\right\}$. By definition of a midpoint consistent set, this union of the sets of associated midpoints contains precisely one midpoint for every side. Hence the set naturally corresponds to the bimedian lines $B_{\{12,34\}}$, $B_{\{13,24\}}$, and $B_{\{14,23\}}$. These are in fact concurrent at the point q_ι.

As each subtriangle centroid is in precisely two midpoint consistent sets, we get that each bimedian line is incident with precisely two quadrangle centroids, namely if a bimedian is incident with a quadrangle centroid from $\square_a$ then it is *not* incident with a quadrangle centroid from $\square_b$. So the twelve bimedian lines of the quadrangle $\overline{a_1a_2a_3a_4}$ are *exactly the lines of the sides of the quadrangles $\square_a$ and $\square_b$.*

More precisely the relations are

$$q_\iota q_\alpha = B_{\{12,34\}}, \qquad q_\beta q_\gamma = B_{\{21,43\}},$$
$$q_\iota q_\beta = B_{\{13,24\}}, \qquad q_\alpha q_\gamma = B_{\{31,42\}},$$
$$q_\iota q_\gamma = B_{\{14,23\}}, \qquad q_\alpha q_\beta = B_{\{41,32\}}$$

and

$$q_1q_2 = B_{\{21,34\}}, \qquad q_3q_4 = B_{\{12,43\}},$$
$$q_1q_3 = B_{\{31,24\}}, \qquad q_2q_4 = B_{\{13,42\}},$$
$$q_1q_4 = B_{\{14,32\}}, \qquad q_2q_3 = B_{\{41,23\}}.$$

We can once again recognise the α, β and γ pairings in these relations. For example in $\square_a$ the line $q_\iota q_\alpha$ is a bimedian line from the α opposite sides, while the line $q_\alpha q_\gamma$ is a bimedian line from the β opposite sides. The sides of $\square_a$ correspond to the bimedian lines constructed from midpoints with the same orientation. In contrast the sides of $\square_b$ correspond to the bimedian lines constructed from the midpoints with opposite orientations.

9 Diagonal triangles and perspectivities

We have observed that from the quadrangle $\overline{a_1a_2a_3a_4}$ we can find eight centroids, which separate into two distinct quadrangles, and that these three quadrangles have a threefold perspective relation. There is in addition a strong correlation between diagonal triangles of these three quadrangles which manifests itself algebraically in a very elegant way.

Recall that the diagonal triangle of the quadrangle $\overline{a_1a_2a_3a_4}$ is given by (1). Now from the equations of the bimedian lines, we can determine that the diagonal triangles of the quadrangles $\square_a$ and $\square_b$ are given by the points

$$d_\alpha^a \equiv B_{\{12,34\}}B_{\{21,43\}} = \begin{bmatrix} (\sigma_{13}+\sigma_{24})(\sigma_{14}-\sigma_{23}): \\ (\sigma_{13}-\sigma_{24})(\sigma_{14}+\sigma_{23}): \\ (\sigma_{13}-\sigma_{24})(\sigma_{14}-\sigma_{23}) \end{bmatrix},$$

$$d_\beta^a \equiv B_{\{13,24\}}B_{\{31,42\}} = \begin{bmatrix} (\sigma_{12}+\sigma_{34})(\sigma_{14}-\sigma_{23}): \\ (\sigma_{12}-\sigma_{34})(\sigma_{14}-\sigma_{23}): \\ (\sigma_{12}-\sigma_{34})(\sigma_{14}+\sigma_{23}) \end{bmatrix},$$

$$d_\gamma^a \equiv B_{\{14,23\}}B_{\{41,32\}} = \begin{bmatrix} (\sigma_{12}-\sigma_{34})(\sigma_{13}-\sigma_{24}): \\ (\sigma_{12}+\sigma_{34})(\sigma_{13}-\sigma_{24}): \\ (\sigma_{12}-\sigma_{34})(\sigma_{13}+\sigma_{24}) \end{bmatrix}$$

and

$$d_\alpha^b \equiv B_{\{21,34\}}B_{\{12,43\}} = \begin{bmatrix} (\sigma_{13}-\sigma_{24})(\sigma_{14}+\sigma_{23}): \\ (\sigma_{13}+\sigma_{24})(\sigma_{14}-\sigma_{23}): \\ (\sigma_{13}+\sigma_{24})(\sigma_{14}+\sigma_{23}) \end{bmatrix},$$

$$d_\beta^b \equiv B_{\{31,24\}}B_{\{13,42\}} = \begin{bmatrix} (\sigma_{12}-\sigma_{34})(\sigma_{14}+\sigma_{23}): \\ (\sigma_{12}+\sigma_{34})(\sigma_{14}+\sigma_{23}): \\ (\sigma_{12}+\sigma_{34})(\sigma_{14}-\sigma_{23}) \end{bmatrix},$$

$$d_\gamma^b \equiv B_{\{41,23\}}B_{\{14,32\}} = \begin{bmatrix} (\sigma_{12}+\sigma_{34})(\sigma_{13}+\sigma_{24}): \\ (\sigma_{12}-\sigma_{34})(\sigma_{13}+\sigma_{24}): \\ (\sigma_{12}+\sigma_{34})(\sigma_{13}-\sigma_{24}) \end{bmatrix}.$$

There is no abuse of notation as the α, β, and γ diagonal points are constructed from the bimedian lines of the α, β, and γ opposite sides respectively.

We start with this first theorem concerned with the collinearity of diagonal points from different diagonal triangles.

Theorem 9 *The diagonal points, d, d^a and d^b are collinear, on six distinct lines, called **d-lines**.*

Proof. By computations we see that the following diagonal points from the quadrangles $\square, \square_a$ and $\square_b$ respectively are collinear on the lines

$\langle \sigma_{34}-\sigma_{12} : \sigma_{12}+\sigma_{34} : 0\rangle$ through $d_\alpha, d_\beta^a, d_\gamma^b$,

$\langle \sigma_{12}+\sigma_{34} : \sigma_{34}-\sigma_{12} : 0\rangle$ through $d_\alpha, d_\gamma^a, d_\beta^b$,

$\langle \sigma_{13}+\sigma_{24} : 0 : \sigma_{24}-\sigma_{13}\rangle$ through $d_\beta, d_\gamma^a, d_\alpha^b$,

$\langle \sigma_{24}-\sigma_{13} : 0 : \sigma_{13}+\sigma_{24}\rangle$ through $d_\beta, d_\alpha^a, d_\gamma^b$,

$\langle 0 : \sigma_{23}-\sigma_{14} : \sigma_{14}+\sigma_{23}\rangle$ through $d_\gamma, d_\alpha^a, d_\beta^b$,

$\langle 0 : \sigma_{14}+\sigma_{23} : \sigma_{23}-\sigma_{14}\rangle$ through $d_\gamma, d_\alpha^a, d_\beta^b$. □

It is quite pleasant that the equations of these six lines reduce to something so elementary, in our view.

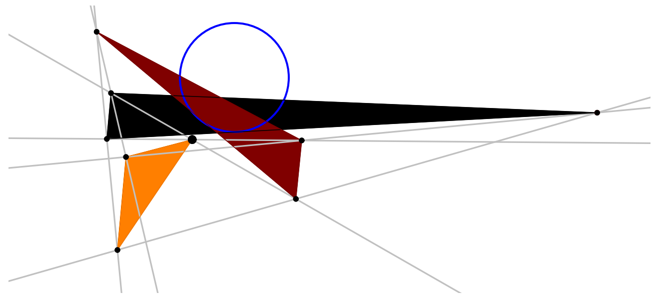

Figure 21: *Collinearity of diagonal triangles*

It is apparent that there is a strong relation between these three triangles, this is emphasized in the next couple of theorems which are concerned with perspectivity.

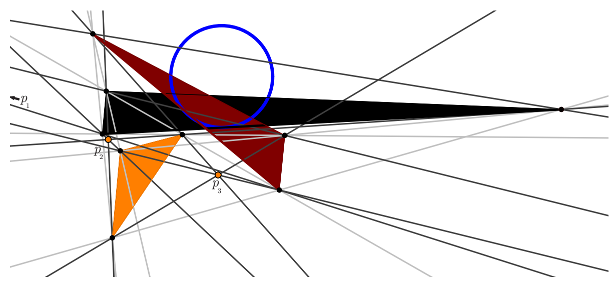

Figure 22: *The diagonal triangles are pairwise perspective.*

Theorem 10 *The three diagonal triangles of the quadrangles $\square, \square_a$ and $\square_b$ are pair-wise perspective.*

Proof. We can see this by computing the lines through corresponding diagonal points, for example

$$d_\alpha d_\alpha^a = \langle(\sigma_{13}-\sigma_{24})(\sigma_{14}+\sigma_{23}) : (\sigma_{13}+\sigma_{24})(\sigma_{23}-\sigma_{14}) : 0\rangle,$$
$$d_\beta d_\beta^a = \langle(\sigma_{12}-\sigma_{34})(\sigma_{14}+\sigma_{23}) : 0 : (\sigma_{12}+\sigma_{34})(\sigma_{23}-\sigma_{14})\rangle,$$
$$d_\gamma d_\gamma^a = \langle 0 : (\sigma_{12}-\sigma_{34})(\sigma_{13}+\sigma_{24}) : (\sigma_{12}+\sigma_{34})(\sigma_{24}-\sigma_{13})\rangle$$

are all concurrent at the point

$$p_1 \equiv \begin{bmatrix} (\sigma_{12}+\sigma_{34})(\sigma_{13}+\sigma_{24})(\sigma_{14}-\sigma_{23}): \\ (\sigma_{12}+\sigma_{34})(\sigma_{13}-\sigma_{24})(\sigma_{14}+\sigma_{23}): \\ (\sigma_{12}-\sigma_{34})(\sigma_{13}+\sigma_{24})(\sigma_{14}+\sigma_{23}) \end{bmatrix}.$$

Similarly the lines $d_\alpha d_\alpha^b$, $d_\beta d_\beta^b$, and $d_\gamma d_\gamma^b$ and the lines $d_\alpha^a d_\alpha^b$, $d_\beta^a d_\beta^b$, and $d_\gamma^a d_\gamma^b$ are concurrent respectively at the points

$$p_2 \equiv \begin{bmatrix} (\sigma_{12}-\sigma_{34})(\sigma_{13}-\sigma_{24})(\sigma_{14}+\sigma_{23}): \\ (\sigma_{12}-\sigma_{34})(\sigma_{13}+\sigma_{24})(\sigma_{14}-\sigma_{23}): \\ (\sigma_{12}+\sigma_{34})(\sigma_{13}-\sigma_{24})(\sigma_{14}-\sigma_{23}) \end{bmatrix}$$

and

$$p_3 \equiv \begin{bmatrix} \sigma_{234}-\sigma_{134}-\sigma_{124}+\sigma_{123}: \\ \sigma_{234}-\sigma_{134}+\sigma_{124}-\sigma_{123}: \\ \sigma_{234}+\sigma_{134}-\sigma_{124}-\sigma_{123} \end{bmatrix}. \qquad \square$$

The points that correspond in the different diagonal triangles are the α, β, and γ diagonal points respectively. We also have that these points of perspectivity are collinear, and we can find an attractive equation for this common line.

Theorem 11 *The points of perspectivity p_1,p_2 and p_3 for each pair of diagonal triangles are collinear.*

Proof. The points p_1,p_2 and p_3 lie on the line

$$\begin{aligned} L_1 &\equiv \left\langle \begin{array}{c} \left(\sigma_{14}^2-\sigma_{23}^2\right)\left(\sigma_{12}^2-\sigma_{13}^2-\sigma_{24}^2+\sigma_{34}^2\right): \\ \left(\sigma_{24}^2-\sigma_{13}^2\right)\left(\sigma_{12}^2-\sigma_{14}^2-\sigma_{23}^2+\sigma_{34}^2\right): \\ \left(\sigma_{12}^2-\sigma_{34}^2\right)\left(\sigma_{13}^2-\sigma_{14}^2-\sigma_{23}^2+\sigma_{24}^2\right) \end{array} \right\rangle \\ &= \left\langle \begin{array}{c} (A_2A_3-A_1A_4)(A_3A_4-A_2A_4-A_1A_3+A_1A_2): \\ (A_1A_3-A_2A_4)(A_3A_4-A_2A_3-A_1A_4+A_1A_2): \\ (A_3A_4-A_1A_2)(A_2A_4-A_2A_3-A_1A_4+A_1A_3) \end{array} \right\rangle \\ &= \left\langle \begin{array}{c} \left(d^2-f^2\right)(ag+bg+cg-2df): \\ \left(g^2-d^2\right)(af+bf+cf-2dg): \\ \left(f^2-g^2\right)(ad+bd+cd-2fg) \end{array} \right\rangle. \qquad \square \end{aligned}$$

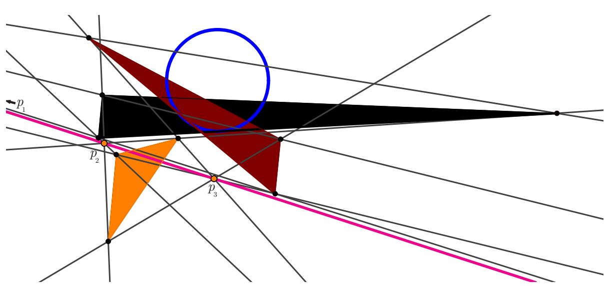

Figure 23: *A common Desargues line*

As discussed in [11], a corollary of Desargues' theorem is that if three triangles are pair-wise perspective and their points of perspectivity are collinear, then the corresponding sides of the three triangles are concurrent and hence the Desargues line is shared. We can verify this in this particular situation as follows

Corollary 2 *The perspective diagonal triangles share the same Desargues line which is*

$$L_2 \equiv \langle df:dg:fg\rangle.$$

Proof. The lines $d_\alpha d_\beta$, $d_\alpha^a d_\beta^a$, and $d_\alpha^b d_\beta^b$ are all concurrent at the point

$$\begin{aligned} &[0:-\sigma_4+\sigma_3-\sigma_2+\sigma_1:\sigma_4+\sigma_3-\sigma_2-\sigma_1] \\ &=[0:f:-d] \end{aligned}$$

while the lines $d_\alpha d_\gamma$, $d_\alpha^a d_\gamma^a$, and $d_\alpha^b d_\gamma^b$ are concurrent at the point

$$\begin{aligned} &[\sigma_4-\sigma_3-\sigma_2+\sigma_1:0:\sigma_4+\sigma_3-\sigma_2-\sigma_1] \\ &=[g:0:-d] \end{aligned}$$

and the lines $d_\beta d_\gamma$, $d_\beta^a d_\gamma^a$, and $d_\beta^b d_\gamma^b$ are concurrent at the point

$$\begin{aligned} &[\sigma_4-\sigma_3-\sigma_2+\sigma_1:\sigma_4-\sigma_3+\sigma_2-\sigma_1:0] \\ &=[g:-f:0]. \end{aligned}$$

These give the same Desargues line

$$L_2 \equiv \langle df:dg:fg\rangle$$

for each pair of perspective triangles. $\square$

The dual of L_2 is the point

$$L_2^\perp = L_2^T\mathbf{B} = \begin{bmatrix} (bf-dg)(cd-fg): \\ (ag-df)(cd-fg): \\ (ag-df)(bf-dg) \end{bmatrix}.$$

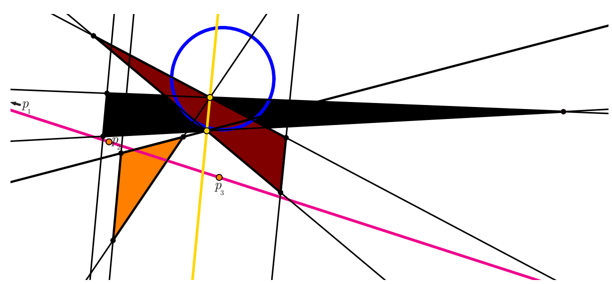

Figure 24: *Shared Desargues line (yellow) of the perspective triangles*

The situation here is richer than that of Desargues' corollary, as the points of the triangles are also collinear in threes.

The final representation of the formulas in the last two theorems deserve a mention. These are only dependent on variables of the general bilinear form and are independent of *sigma values*. This suggests that these lines have other roles to play in the geometry of the hyperbolic quadrangle, independent of centroid considerations. It would also be interesting of course to have synthetic projective arguments for these results that we have described algebraically.

10 Connections with desmic systems and tetrahedra, and final remarks

Both the books of Sommerville [13] and Coolidge [5], after constructing a projective metric, discuss how a side has two midpoints, a three-point system has four centroids, and finally that a four-point system has eight centroids. Their construction of the centroids for a four-point takes place in a three-dimensional space and is referred to as a *desmic system*. A desmic system is concerned with 'strongly' related tetrahedra, where these relations are the same as those described in the Quadrangle centroids subsection of this chapter. Namely each centroid lies on a line joining one point with the center of the other three (g-lines,) and that the produced tetrahedra are pair-wise perspective in four different ways, where the points of perspectivity are exactly the points of the other tetrahedra. They also state that the corresponding planes of perspective tetrahedra of a desmic system intersect at four lines which are coplanar, the four planes (one for each point of perspectivity) are the faces of the third tetrahedra. This last fact is unseen in the planar quadrangle case, as the third dimension is needed for this type of perspectivity. Yet it might be possible to view the planar quadrangle case that we go through above as some sort of projection of this desmic system onto a plane.

A major point of departure with the planar case is that tetrahedra *do not have an analogue of a diagonal triangle*. So the relations we described between the diagonal triangles of the quadrangles are unique and separate from what is described in these books. Furthermore, since classically only the interior points of the absolute were used for basic geometry these eight centroids were not visible, and thus the relations unseen.

The observant reader might have noticed that at no point do we prove that the sets of centroids $\{q_\iota, q_\alpha, q_\beta, q_\gamma\}$ and $\{q_1, q_2, q_3, q_4\}$ are in fact quadrangles. They are quadrangles except when the points of the original quadrangle lie on a hyperbolic circle.

If this happens it means that the subtriangles share a circumcenter, or equivalently they share a circumline. Recall that a circumline is the line through three collinear midpoints of a triangle. By a simple counting exercise, the number of collinear midpoints must be six. This is for the number of collinear midpoints must be more than 3, as different subtriangles have different midpoints, and less than 7 as there are six sides and seven or more collinear midpoints would imply that the points of the quadrangle are collinear. Finally the number of collinear midpoints must be a multiple of 3, giving the only possible choice as 6.

Thus, the three bimedian lines produced from these midpoints coincide, producing three collinear quadrangle centroids. In this case the diagonal triangle degenerates to three collinear points, coinciding with the three collinear quadrangle centroid points. All the remaining theorems still hold but one of the perspective triangles is degenerate, as in Figure 25.

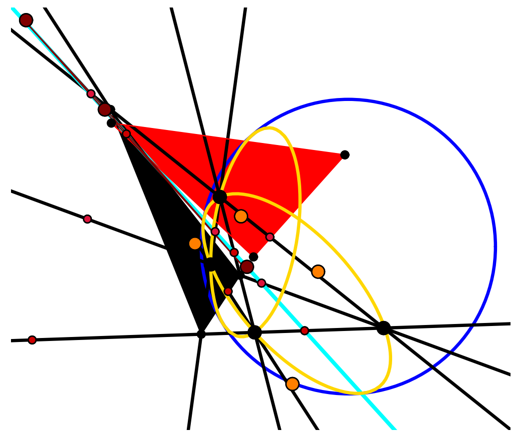

Figure 25: *Three quadrangle centroids are collinear on the blue line.*

If furthermore the points of the quadrangle lie on two circles, then the situation completely degenerates as two sets of six midpoints, and three centroids are collinear. This configuration results in two midpoints coinciding, and furthermore reduces to seven quadrangle centroids. The diagonal theorems in this case do not hold, as one diagonal triangle degenerates into two points, as seen in Figure 26.

Finally, apart from these theorems relating to the quadrangle centroids, there appear to be many other notable relations between subtriangle centroids and some structures concerning the circumlines of the subtriangles. And more generally we can look at other quadrangle structures that are analogous to the triangle centre investigations in [14], [17] and [20] in hyperbolic geometry.

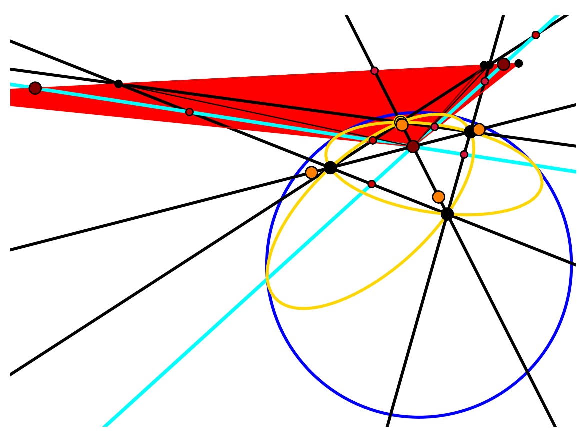

Figure 26: *Two quadrangle centroids coincide at the intersection of the blue lines.*

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