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Curves of Foci of Conic Pencils in pseudo-Euclidean Plane

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ABSTRACT

In this article, it will be shown that the curve of foci of an order conic pencil in the pseudo-Euclidean plane is generally a bicircular curve of 6th order. In some cases, depending on a position of four base points of the pencil, this curve is of 5th, 4th or 3rd order and in some cases it is even a conic or only a line.

Key words: pseudo-Euclidean plane, conic sections, foci, conic pencil

MSC2010: 51A05; 51M15

Krivulje žarišta u pramenovima konika u pseudo-euklidskoj ravnini

SAŽETAK

U ovom članku pokazat će se da je krivulja žarišta pramena konika u pseudo-euklidskoj ravnini općenito bicirkularna krivulja šestog reda. U nekim slučajevima, u ovisnosti o položaju četiriju temeljnih točaka pramena, krivulja žarišta može biti petog, četvrtog ili trećeg reda, a može biti i konika ili samo pravac.

Ključne riječi: pseudo-euklidska ravnina, konike, žarišta, pramenovi konika

1 Introduction

A *pseudo-Euclidean plane* (*PE-plane*) is a real projective plane where the metric is induced by a real line a and two real points A_1 and A_2 incident with it, see [10]. It is one of nine plane geometries and according to [8], it is parabolic-hyperbolic plane. The affine model of the pseudo-Euclidean plane will be used, where the absolute line a is determined by the equation $x_0 = 0$ and the absolute points A_1, A_2 by the coordinates $(0,1,\pm 1)$, like in [4], [5], [6], [7], [11].

Further on, some basic, well-known definitions are given ([1], [2], [3], [7]).

Definition 1 *Points incident with the absolute line a are called isotropic points.*

Definition 2 *Lines incident with one of the absolute points A_1 or A_2 are called isotropic lines.*

Definition 3 *Foci of a conic are intersection points of its isotropic tangent lines.*

Definition 4 *Circular curve in PE-plane is a curve incident with at least one of two absolute points.*

Definition 5 *A curve k is said to be of (r,t)- type of circularity if the absolute point A_1 is the intersection of the curve k with the absolute line a of multiplicity r, and the absolute point A_2 is the intersection of the curve k with the absolute line a of multiplicity t. The sum $r+t$ is called degree of circularity.*

Definition 6 *The curve of order n is said to be entirely circular if $n = r+t$, i.e. the order of the curve equals the degree of circularity.*

Conics in pseudo-Euclidean plane are divided into ([7], [9]):

- hyperbola intersecting the absolute line in two real and distinct points
- ellipse intersecting the absolute line in a pair of conjugate-imaginary points
- parabola touching the absolute line

- special hyperbola intersecting the absolute line in two real and distinct points out of which one is an absolute point
- special parabola touching the absolute line in an absolute point
- circle intersecting the absolute line in both absolute points.

A conic has four foci which can be real and distinct, conjugate-imaginary, double real or even quadruple real. An ellipse has four real foci. A hyperbola may have four real or four conjugate-imaginary foci. A special hyperbola may have two double real or two double imaginary foci. A parabola has one non-isotropic focus, another one in the isotropic touching point of the parabola and the absolute line, and one focus in each absolute point. A special parabola has double focus in each absolute point. All four foci of a circle are in the same point - quadruple focus (which is also the center of the circle).

2 Curves of foci

A conic is uniquely determined by five of its points. Four points, called *base points* determine infinitely many conics which are called *an order pencil of conics*.
In the affine model of PE-plane, the coordinates of the points are determined with

$$x=\frac{x_1}{x_0},y=\frac{x_2}{x_0}, \tag{1}$$

the absolute line a with the equation $x_0 = 0$, and the absolute points A_1, A_2 with coordinates $(0,1,\pm1)$.
A conic is given by equation in homogeneous coordinates

$$a_{00}x_0^2+a_{11}x_1^2+a_{22}x_2^2+2a_{01}x_0x_1+2a_{02}x_0x_2+2a_{12}x_1x_2=0, \tag{2}$$

and in the affine coordinates

$$a_{00}+a_{11}x^2+a_{22}y^2+2a_{01}x+2a_{02}y+2a_{12}xy=0. \tag{3}$$

Some short calculations lead to the conclusions:

- c is a hyperbola iff $a_{12}^2-a_{11}a_{22}>0$,
- c is a parabola iff $a_{12}^2-a_{11}a_{22}=0$,
- c is an ellipse iff $a_{12}^2-a_{11}a_{22}<0$,
- c is a special hyperbola iff $a_{11}+a_{22}+2a_{12}=0$ or $a_{11}+a_{22}-2a_{12}=0$,
- c is a special parabola iff $a_{11}=a_{22}=-a_{12}$ or $a_{11}=a_{22}=a_{12}$,
- c is a circle iff $a_{12}=0$ and $a_{11}=-a_{22}$,

as it is shown in [6].
Let the conic pencil be given with the points $A(0,0)$, $B(1,0)$, $C(0,2)$ and $D(t_1,t_2)$. To avoid special cases, the point D should not be incident with lines AB, BC or AC, i.e. $t_1\neq 0$, $t_2\neq 0$ and $t_2\neq -2t_1+2$.
Substituting the coordinates of points A, B, C and D into (3) yields

$$a_{00}=0,$$

$$a_{00}+2a_{01}+a_{11}=0,$$
$$a_{00}+4a_{02}+4a_{22}=0,$$
$$a_{00}+2a_{01}t_1+a_{11}t_1^2+2a_{02}t_2+2a_{12}t_1t_2+a_{22}t_2^2=0.$$

So, the equation of the conic pencil is

$$a_{11}x^2+\frac{(a_{11}t_1-a_{11}t_1^2+2a_{22}t_2-a_{22}t_2^2)xy}{t_1t_2} +a_{22}y^2-a_{11}x-2a_{22}y=0. \tag{4}$$

Different form of the equation (4) is

$$a_{11}(x^2+\frac{1-t_1}{t_2}xy-x)+a_{22}(y^2+\frac{2-t_2}{t_1}xy-2y)=0.$$

It is obvious that the conic pencil is a linear combination of two degenerate conics of the pencil - the first one consists of lines AC and BD, and the second one of lines AB and CD. Introducing $\lambda=\frac{a_{22}}{a_{11}}$ into equation (4), the following equation for the conic pencil with the base points A, B, C and D is obtained

$$x^2+\frac{2\lambda xy}{t_1}+\frac{xy}{t_2}-\frac{t_1xy}{t_2}-\frac{\lambda t_2xy}{t_1}+\lambda y^2-x-2\lambda y=0, \tag{5}$$

where each conic of the pencil is uniquely defined by parameter $\lambda\in \mathrm{R}\cup\infty$.
In line coordinates, the pencil (5) has the following equation

$$-\lambda^2u^2+\lambda uv-\frac{v^2}{4}+\lambda u-\frac{2\lambda^2u}{t_1}-\frac{\lambda u}{t_2}+\frac{\lambda t_1u}{t_2}+\frac{\lambda^2t_2u}{t_1} +2\lambda v-\frac{\lambda v}{t_1}-\frac{v}{2t_2}+\frac{t_1v}{2t_2}+\frac{\lambda t_2v}{2t_1}+\frac{\lambda}{2}-\frac{\lambda^2}{t_1^2}+\frac{\lambda}{2t_1}-\frac{1}{4t_2^2} +\frac{t_1}{2t_2^2}-\frac{t_1^2}{4t_2^2}+\frac{\lambda}{t_2}-\frac{\lambda}{t_1t_2}+\frac{\lambda^2t_2}{t_1^2}-\frac{\lambda^2t_2^2}{4t_1^2}=0. \tag{6}$$

For an isotropic line $u_0x_0+u_1x_1+u_2x_2=0$ passing through the absolute point $A_1(0,1,1)$ the following must be true

$$u_1+u_2=0, \tag{7}$$

i.e. in affine coordinates

$$v=-u, \tag{8}$$

where $u=\frac{u_1}{u_0}$ and $v=\frac{u_2}{u_0}$. Thus, such line in affine coordinates has the equation $1+ux-uy=0$. From that equation follows:

$$u=\frac{1}{y-x}. \tag{9}$$

Analogously for an isotropic line passing through the absolute point $A_2(0,1,-1)$ the following is true

$$v=u \tag{10}$$

and

$$u=\frac{-1}{x+y}. \tag{11}$$

When v in (6) is replaced by using (8), and then u is replaced with (9), the following is obtained

$$\begin{aligned}&-\frac{1}{(y-x)^2}-\frac{4\lambda}{(y-x)^2}-\frac{4\lambda^2}{(y-x)^2}-\frac{2\lambda}{y-x}+\frac{4\lambda}{t_1(y-x)}\\&-\frac{8\lambda^2}{t_1(y-x)}+\frac{2}{t_2(y-x)}-\frac{4\lambda}{t_2(y-x)}-\frac{2t_1}{t_2(y-x)}\\&+\frac{4\lambda t_1}{t_2(y-x)}-\frac{2\lambda t_2}{t_1(y-x)}+\frac{4\lambda^2 t_2}{t_1(y-x)}-\lambda-\frac{4\lambda^2}{t_1^2}+\frac{2\lambda}{t_1}-\frac{1}{t_2^2}\\&+\frac{2t_1}{t_2^2}-\frac{t_1^2}{t_2^2}+\frac{4\lambda}{t_2}-\frac{4\lambda}{t_1t_2}+\frac{4\lambda^2 t_2}{t_1^2}-\frac{\lambda^2t_2^2}{t_1^2}=0.\end{aligned} \tag{12}$$

The intention is to isolate only those lines passing through the absolute point A_1 out of all lines from the pencil (6). Solving the equation (12) for λ, the following two solutions are obtained

$$\begin{aligned}\lambda_{1,2}=&-\frac{1}{t_2^2(-2x+t_2x+2y-t_2y+2t_1)^2}[2t_1t_2x^2-2t_1^2t_2x^2-t_1t_2^2x^2\\&-t_1^2t_2^2x^2-4t_1t_2xy+4t_1^2t_2xy+2t_1t_2^2xy+2t_1^2t_2^2xy+2t_1t_2y^2\\&-2t_1^2t_2y^2-t_1t_2^2y^2-t_1^2t_2^2y^2-2t_1^2t_2x+2t_1^3t_2x+2t_1t_2^2x-2t_1^2t_2^2x\\&-t_1t_2^3x+2t_1^2t_2y-2t_1^3t_2y-2t_1t_2^2y+2t_1^2t_2^2y+t_1t_2^3y+2t_1^2t_2^2\\&\pm 2\sqrt{-t_1^3t_2^3(-2+2t_1+t_2)(-1+x-y)(x-y)(2+x-y)(t_1-t_2-x+y)}].\end{aligned} \tag{13}$$

This corresponds to the fact that each isotropic line passing through A_1 is tangent line of two conics of the pencil, and those lines are precisely two lines defined by parameters λ_1 and λ_2. Analogously, separating from the whole pencil (6) only isotropic lines passing through the absolute point A_2 (by substituting v in (6) using (10), and then substituting u using (11)) and solving that equation for λ, the following is obtained

$$\begin{aligned}\lambda_{3,4}=&-\frac{1}{t_2^2(2t_1-2x+t_2x-2y+t_2y)^2}[2t_1t_2x^2-2t_1^2t_2x^2-t_1t_2^2x^2\\&-t_1^2t_2^2x^2+4t_1t_2xy-4t_1^2t_2xy-2t_1t_2^2xy-2t_1^2t_2^2xy+2t_1t_2y^2\\&-2t_1^2t_2y^2-t_1t_2^2y^2-t_1^2t_2^2y^2-2t_1^2t_2x+2t_1^3t_2x-2t_1t_2^2x+6t_1^2t_2^2x\\&+t_1t_2^3x-2t_1^2t_2y+2t_1^3t_2y-2t_1t_2^2y+6t_1^2t_2^2y+t_1t_2^3y-2t_1^2t_2^2\\&\pm 2\sqrt{-t_1^3t_2^3(-2+2t_1+t_2)(t_1+t_2-x-y)(-2+x+y)(-1+x+y)(x+y)}].\end{aligned} \tag{14}$$

The solutions are the parameters λ_3 and λ_4 which uniquely determine two conics of the pencil.

Foci of a conic are four intersections of its two tangent lines from the absolute point A_1 with its two tangent lines from the absolute point A_2. Hence, the conics determined by parameters λ_1 and λ_2 must be equal to the conics determined by parameters λ_3 and λ_4, i.e. $\lambda_1=\lambda_3$ or $\lambda_2=\lambda_3$ or $\lambda_1=\lambda_4$ or $\lambda_2=\lambda_4$ are valid. So, the equation of the curve of foci is obtained by using

$$(\lambda_1-\lambda_3)(\lambda_2-\lambda_3)(\lambda_1-\lambda_4)(\lambda_2-\lambda_4)=0. \tag{15}$$

Introducing (13) and (14) into (15), the equation of the curve of foci $F(x,y)=0$ is obtained, where

$$\begin{aligned}F(x,y)=&F_{60}(t_1,t_2)x^6+F_{51}(t_1,t_2)x^5y+F_{42}(t_1,t_2)x^4y^2\\&+F_{33}(t_1,t_2)x^3y^3+F_{24}(t_1,t_2)x^2y^4+F_{15}(t_1,t_2)xy^5\\&+F_{06}(t_1,t_2)y^6+F_{50}(t_1,t_2)x^5+F_{41}(t_1,t_2)x^4y+F_{32}(t_1,t_2)x^3y^2\\&+F_{14}(t_1,t_2)xy^4+F_{05}(t_1,t_2)y^5+F_{40}(t_1,t_2)x^4+F_{31}(t_1,t_2)x^3y\\&+F_{22}(t_1,t_2)x^2y^2+F_{13}(t_1,t_2)xy^3+F_{04}(t_1,t_2)y^4\\&+F_{30}(t_1,t_2)x^3+F_{21}(t_1,t_2)x^2y+F_{12}(t_1,t_2)xy^2+F_{03}(t_1,t_2)y^3\\&+F_{20}(t_1,t_2)x^2+F_{11}(t_1,t_2)xy+F_{02}(t_1,t_2)y^2+F_{10}(t_1,t_2)x\\&+F_{01}(t_1,t_2)y+F_{00}(t_1,t_2),\end{aligned} \tag{16}$$

where $F_{ij}(t_1,t_2)$ are polynomials in t_1, t_2 and i, $j=0,1,2,3,4,5,6$. For example,

$$\begin{aligned}F_{06}=&16t_1^4t_2^4+16t_1^3t_2^5+4t_1^2t_2^6-64t_1^4t_2^3-96t_1^3t_2^4-40t_1^2t_2^5\\&-4t_1t_2^6+64t_1^4t_2^2+192t_1^3t_2^3+128t_1^2t_2^4+24t_1t_2^5-128t_1^3t_2^2\\&-160t_1^2t_2^3-48t_1t_2^4+64t_1^2t_2^2+32t_1t_2^3.\end{aligned}$$

$F(x,y)$ is a polynomial of degree 6, so it is proved that the curve of foci is of order 6. The next goal is to calculate its intersection points with the absolute line a. In order to do that, since the absolute line has the equation $x_0=0$, it is necessary to write the polynomial $F(x,y)$ in homogeneous coordinates. The result is

$$4t_1t_2(-1+t_1)(-2+t_2)(-2+2t_1+t_2)(2t_1+t_2)(x_1-x_2)^2$$
$$(x_1+x_2)^2(-2t_2x_1^2+t_2^2x_1^2-2t_1t_2x_1x_2-t_1x_2^2+t_1^2x_2^2)=0. \tag{17}$$

From the equation (17), it is clear that the absolute points $A_1(0,1,1)$ and $A_2(0,1,-1)$ are double intersections of the curve of foci with the absolute line. It is easy to calculate that those points are not only double intersections, but also double points of the curve of foci. Hence, the curve of foci has type of circularity $(2,2)$. The curve of foci has two more intersections with the absolute line. They are obtained calculating the equation

$$-2t_2x_1^2+t_2^2x_1^2-2t_1t_2x_1x_2-t_1x_2^2+t_1^2x_2^2=0. \tag{18}$$

Those intersections are

$$X_{1,2}(0,\frac{t_1t_2\pm\sqrt{-2t_1t_2+2t_1^2t_2+t_1t_2^2}}{-2t_2+t_2^2},1). \tag{19}$$

It is easy to calculate that those points are precisely the points in which two parabolas of the pencil touch the absolute line. So, if one of two parabolas of the pencil is special parabola, then the curve of foci has type of circularity $(2,3)$ (or $(3,2)$). That is not a case in the Euclidean plane. If both parabolas of the pencil are special parabolas, then the curve of foci has type of circularity $(3,3)$.

Two examples of curves of foci are given in Figures 1 and 2. Six lines connecting four base points of pencils are also shown.

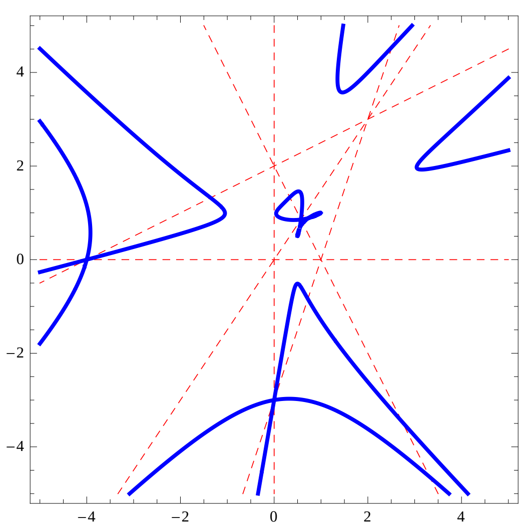

Figure 1: *Curve of foci for* $D(2,3)$

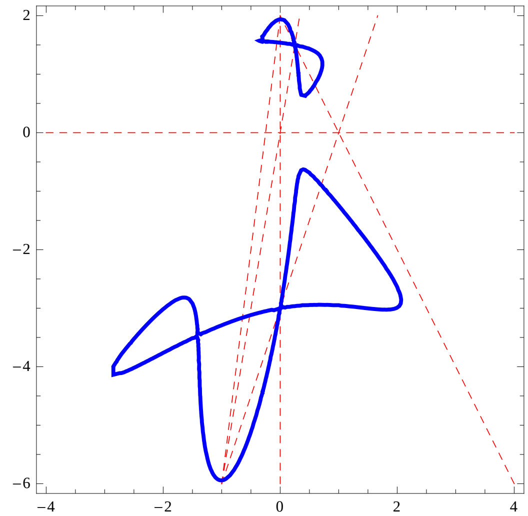

Figure 2: *Curve of foci for* $D(-1,-6)$

The study given above is for the pencil determined by four base points $A(0,0)$, $B(1,0)$, $C(0,2)$ i $D(t_1,t_2)$. In the same way, the investigation may be done more generally, having two base conics

$$c_1(x,y)=a_{00}+a_{11}x^2+a_{22}y^2+2a_{01}x+2a_{02}y+2a_{12}xy,$$
$$c_2(x,y)=b_{00}+b_{11}x^2+b_{22}y^2+2b_{01}x+2b_{02}y+2b_{12}xy.$$

Those two conics intersect in four points and it is assumed that those four points are base points of the pencil. Such conic pencil has the equation

$$c_1(x,y)+\lambda c_2(x,y)=a_{00}+\lambda b_{00}+(a_{11}+\lambda b_{11})x^2$$
$$+(a_{22}+\lambda b_{22})y^2+2(a_{01}+\lambda b_{01})x+2(a_{02}+\lambda b_{02})y$$
$$+2(a_{12}+\lambda b_{12})xy,$$

where $\lambda\in R\bigcup\infty$.

Repeating the process shown in the proof before, the polynomial of degree 6 (which is two long to show it here) is obtained

$$F(x,y)=F_{60}(a_{pr},b_{st})x^6+F_{51}(a_{pr},b_{st})x^5y+F_{42}(a_{pr},b_{st})x^4y^2$$
$$+F_{33}(a_{pr},b_{st})x^3y^3+F_{24}(a_{pr},b_{st})x^2y^4+F_{15}(a_{pr},b_{st})xy^5$$
$$+F_{06}(a_{pr},b_{st})y^6+F_{50}(a_{pr},b_{st})x^5+F_{41}(a_{pr},b_{st})x^4y$$
$$+F_{32}(a_{pr},b_{st})x^3y^2+F_{23}(a_{pr},b_{st})x^2y^3+F_{14}(a_{pr},b_{st})xy^4$$
$$+F_{05}(a_{pr},b_{st})y^5+F_{40}(a_{pr},b_{st})x^4+F_{31}(a_{pr},b_{st})x^3y$$
$$+F_{22}(a_{pr},b_{st})x^2y^2+F_{13}(a_{pr},b_{st})xy^3+F_{04}(a_{pr},b_{st})y^4$$
$$+F_{30}(a_{pr},b_{st})x^3+F_{21}(a_{pr},b_{st})x^2y+F_{12}(a_{pr},b_{st})xy^2$$
$$+F_{03}(a_{pr},b_{st})y^3+F_{20}(a_{pr},b_{st})x^2+F_{11}(a_{pr},b_{st})xy$$
$$+F_{02}(a_{pr},b_{st})y^2+F_{10}(a_{pr},b_{st})x+F_{01}(a_{pr},b_{st})y$$
$$+F_{00}(a_{pr},b_{st}),$$

where F_{ij} are polynomials in a_{pr}, b_{st}, i, $j = 0,1,2,3,4,5,6$ and pr, $st = 00,11,22,01,02,12$. For example,

$$F_{06} = a_{02}a_{11}^2a_{12}b_{02}b_{12}b_{22}^2 - a_{01}a_{11}a_{12}^2b_{02}b_{12}b_{22}^2 - a_{02}^2a_{11}a_{12}b_{11}b_{12}b_{22}^2 - a_{01}a_{02}a_{11}a_{22}b_{11}b_{12}b_{22}^2 + 2a_{01}^2a_{12}a_{22}b_{11}b_{12}b_{22}^2 - a_{01}a_{02}a_{11}a_{12}b_{12}^2b_{22}^2 + a_{01}^2a_{12}^2b_{12}^2b_{22}^2 - a_{02}a_{11}^2a_{12}b_{01}b_{22}^3 + a_{01}a_{11}a_{12}^2b_{01}b_{22}^3 + a_{01}a_{02}a_{11}a_{12}b_{11}b_{22}^3 - a_{01}^2a_{12}^2b_{11}b_{22}^3.$$

Therefore, the curve of foci is of order 6.

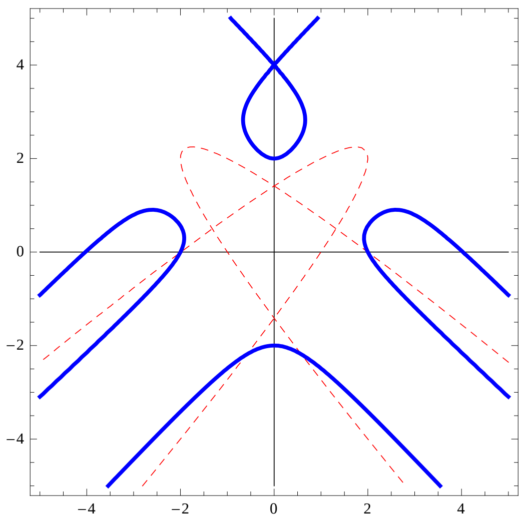

Figure 3: *Entirely circular curve of foci for a pencil containing two special parabolas*

Another example is shown in Figure 3. This is the example for entirely circular curve of foci containing two special parabolas

$$ps_1(x,y) = x^2 - 2xy + y^2 + x - 2,$$
$$ps_2(x,y) = x^2 + 2xy + y^2 - x - 2,$$

which are both shown in Figure 3 together with the curve of foci.
Hence, the following theorems are proved.

Theorem 1 *A curve of foci of all conics of an order conic pencil in PE-plane is a curve of order 6 with the type of circularity* $(2,2)$.

Theorem 2 *If one of two parabolas in the order conic pencil is a special parabola, the curve of foci has the type of circularity* $(3,2)$ *(or* $(2,3)$*).*

Theorem 3 *If both parabolas of the pencil are special parabolas, the curve of foci has the type of circularity* $(3,3)$*, i.e. it is entirely circular.*

3 Curves of foci of order less than 6 - examples

Depending on a position of four base points of the pencil, the curve of foci can be of order less than 6.

Example 1 *For base points* $A(0,0)$, $B(1,0)$, $C(0,2)$ *and* $D(1,3)$*, the line CD is isotropic. The curve of foci is of order 5,*

$$F(x,y) = -1620x^4y + 3240x^3y^2 + 1620x^2y^3 - 3240xy^4 + 2025x^4 - 9072x^3y + 1620x^2y^2 + 10368xy^3 + 1620y^4 + 6480x^3 - 5832x^2y - 12960xy^2 - 5184y^3 + 1944x^2 + 10368xy + 5184y^2 - 5184x - 2592y + 1296.$$

Example 2 *For base points* $A(0,0)$, $B(1,0)$, $C(0,2)$, $D(1,-2)$*, intersections of the lines AC and BD as well as of the lines AD and BC are isotropic points, i.e. the lines AC and BD as well as the lines AD and BC are parallel. The curve of foci is of order 4,*

$$F(x,y) = 2048x^3y + 5120x^2y^2 + 2048xy^3 - 3072x^2y - 5120xy^2 - 1024y^3 + 1024x^2 + 2048xy + 2304y^2 - 1024x - 512y + 256.$$

Example 3 *For base points A(0,0), B(1,1), C(0,3), D(1,2), the line AB is isotropic line through the absolute point* A_1*, the line CD is isotropic line through the absolute point* A_2 *and the lines AC and BD are parallel (i.e. they intersect in an isotropic point). The curve of foci is of order 3.*

$$F(x,y) = 16x^3 - 16xy^2 - 60x^2 + 48xy + 8y^2 + 36x - 24y - 9.$$

Example 4 *For base points* $A(0,0)$, $B(1,0)$, $C(0,1,t_1)$ *and* $D = A_2(0,1,-1)$ *where C and D are written in homogeneous coordinates, the curve of foci is*

$$F(x,y) = (2t_1x^2 + 2t_1^2x^2 + y^2 - t_1^2y^2 - 2t_1x - 2t_1^2x + t_1^2)^2,$$

i.e. the curve of foci is a conic. For $0 < t_1 < 1$ *it is an ellipse, and for* $t_1 < 0$ *and* $t_1 > 1$ *it is a hyperbola.*

Example 5 *For base points* $A(0,0)$, $B(1,1)$, $C(0,1,t_1)$ *and* $D = A_2(0,1,-1)$*, the line AB is isotropic and the curve of foci is a special hyperbola which does not exist in the Euclidean plane.*

Example 6 *For base points* $A(0,0)$, $B(1,0)$, $C = A_1(0,1,1)$ *and* $D = A_2(0,1,-1)$*, all conics in the pencil are circles and the curve of foci is a line* $x = \frac{1}{2}$.

4 Conclusion

In this article, it is proved that the curve of foci of a conic pencil in PE-plane is generally a curve of order 6 with the type of circularity $(2,2)$. It is also shown that, depending on the type of parabolas in the pencil, the curve of foci may have the type of circularity $(2,3)$ or $(3,2)$ which is not possible in the Euclidean plane. If both parabolas in the pencil are special parabolas, the curve of foci is entirely circular, i.e. its type of circularity is $(3,3)$. Some examples are shown that the curve of foci may be of order less than 6, but there are some more cases which do not happen in the Euclidean plane. They are possible themes for the next article.

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