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Applications of frames of subspaces in Richardson and Chebyshev methods for solving operator equations

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Abstract. This paper is concerned with the construction of two iterative methods by frames of subspaces for solving an operator equation on Hilbert spaces. We present two algorithms based on Richardson and Chebyshev methods, and investigate their convergence and optimality.

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1. Introduction and preliminaries

An iterative method for solving an operator equation starts from an initial approximation that is succesively improved until a sufficiently accurate solution is obtained. These methods are particularly useful for solving discretized elliptic self-adjoint partial differential equations. Hereof, the potential of frames in numerical analysis is an almost unexplored field. On the one hand, the redundancy of a frame can give freedom to implement further properties, which would be mutually exclusive in the Riesz bases case, such as high smoothness and local support. On the other hand, since we are working with a weaker concept, the concrete construction of a frame is usually much simpler than that of stable multiscale bases. Consequently, there is some hope that the frame approach may simplify the geometrical construction on bounded domains. To handle this emerging application of frames, new methods have to be developed. One starting point is first to build frames "locally" and then piece them together to obtain frames for the whole space. One advantage of this idea is that it would facilitate the construction of frames for special applications since we can first construct frames or choose already known frames for smaller spaces. In the second step, one could construct a frame for the whole space from them. This gives the concept of the frame of subspaces.

In this paper, we will use the frames of subspaces to get some approximated solutions for the operator equation

$$Lu = f, \tag{1}$$

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where $L : H \to H$ is a bounded, invertible and self-adjoint linear operator on a separable Hilbert space H. A natural approach to construct an approximate solution is to solve problem (1) on a finite dimensional subspace of H. Development of numerical methods for solving problem (1) by frames can be seen in [1, 8, 10].

First, we briefly recall the definitions and basic properties of frames and frames of subspaces. For more information we refer to the survey articles by Cassaza and Gitta Kutyniok [5] and the book by Christensen [7]. Throughout this paper, H denotes an arbitrary separable Hilbert space. Furthermore, all subspaces are assumed to be closed. Moreover, Λ and I denote a countable indexing set and the identity operator, respectively. Also, π_W denotes the orthogonal projection of H onto W, where W is a subspace of H.

Let $\Psi = (\psi_\lambda)_{\lambda\in\Lambda} \subset H$ be a frame for H. It means that there exist constants $0 < A_\Psi \leq B_\Psi < \infty$ such that

$$A_\Psi \|f\|_H^2 \leq \sum_{\lambda\in\Lambda} |\langle f, \psi_\lambda\rangle|^2 \leq B_\Psi \|f\|_H^2, \quad \forall f \in H. \tag{2}$$

For a frame Ψ, the operator $S : H \to H$ defined by

$$S(f) = \sum_{\lambda\in\Lambda} \langle f, \psi_\lambda\rangle \psi_\lambda,$$

is called the frame operator. It was shown in [7], for the frame $(\psi_\lambda)_{\lambda\in\Lambda}$, that S is a positive invertible operator satisfying $A_\Psi I_H \leq S \leq B_\Psi I_H$ and $B_\Psi^{-1} I_H \leq S^{-1} \leq A_\Psi^{-1} I_H$. Also, the sequence

$$\widetilde{\Psi} = (\widetilde{\psi}_\lambda)_{\lambda\in\Lambda} = (S^{-1}\psi_\lambda)_{\lambda\in\Lambda},$$

is a frame (called the canonical dual frame) for H with the bounds B_Ψ^{-1} and A_Ψ^{-1}. Every $f \in H$ has the expansion

$$f = \sum_{\lambda\in\Lambda} \langle f, \psi_\lambda\rangle \widetilde{\psi}_\lambda = \sum_{\lambda\in\Lambda} \langle f, \widetilde{\psi}_\lambda\rangle \psi_\lambda.$$

For an index set $\widetilde{\Lambda} \subset \Lambda$, $(\psi_\lambda)_{\lambda\in\widetilde{\Lambda}}$ is called a frame sequence if it is a frame for its closed span.

Now, let H be a separable Hilbert space and Λ a countable indexing set. For a family of weights $\{v_\lambda\}_{\lambda\in\Lambda}$, i.e. $v_\lambda > 0$ for all $\lambda \in \Lambda$, a family of subspaces $\{H_\lambda\}_{\lambda\in\Lambda}$ of a Hilbert space H are called a frame of subspaces with respect to $\{v_\lambda\}_{\lambda\in\Lambda}$ for H, if there exist constants $0 < A \leq B < \infty$ such that

$$A\|f\|^2 \leq \sum_{\lambda\in\Lambda} v_\lambda^2 \|\pi_{H_\lambda}(f)\|^2 \leq B\|f\|^2 \quad \forall f \in H, \tag{3}$$

where π_{H_λ} denotes the orthogonal projection onto the subspace H_λ.

The constants A and B are called the frame bounds of the frame of subspaces. If $A = B$, then the frame of subspaces $\{H_\lambda\}_{\lambda\in\Lambda}$ with respect to$\{v_\lambda\}_{\lambda\in\Lambda}$, is called an A-tight frame of subspaces. It is proved that the family $\{H_\lambda\}_{\lambda\in\Lambda}$ of the frame of subspaces is complete in the sense that $\overline{span}_{\lambda\in\Lambda}\{H_\lambda\} = H$, (see [5]).

The following theorem, shows how we able to string together frames for each of the subspaces H_λ to get a frame for H. (see [5]).

Theorem 1. *Let Λ be an index set, $v_\lambda > 0$ for each $\lambda \in \Lambda$, and let $\{\psi_{\lambda_i}\}_{i\in I_\Lambda}$ be a frame sequence in H with frame bounds A_λ and B_λ. Define $H_\lambda = \overline{span}_{i\in I_\Lambda}\{\psi_{\lambda i}\}$ for all $\lambda \in \Lambda$, and suppose that $0 < A = \inf_{\lambda\in\Lambda} A_\lambda \leq B = \sup_{\lambda\in\Lambda} B_\lambda < \infty$. Then $\{v_\lambda \psi_{i_\lambda}\}_{\lambda\in\Lambda, i\in I_\Lambda}$ is a frame for H if and only if $\{H_\lambda\}_{\lambda\in\Lambda}$ is a frame of subspaces with respect to $\{v_\lambda\}_{\lambda\in\Lambda}$ for H.*

Example 1. *Let H be a Sobolev space of order t on a domain $\Omega \subseteq \mathbb{R}^n$. According to [10], for some $\Gamma^D \subset \partial\Omega$, possibly $\Gamma^D = \phi$, we can construct a frame for the space*

$$\mathcal{H}^t = \begin{cases} H^t_{0,\Gamma^D}(\Omega), & t \geq 0, \\ (H^{-t}_{0,\Gamma^D}(\Omega))', & t < 0, \end{cases}$$

where $t \geq 0$,

$$H^t_{0,\Gamma^D}(\Omega) = \log_{H^t(\Omega)}\{u \in H^t(\Omega) \cap C^\infty(\Omega) : \sup u \cup \Gamma^D = \phi\}.$$

If we consider an open covering $\Omega = \cup_{i=1}^m \Omega_i$ and if $\{f_{ij}\}_{j\in I_i}$ is a frame for H_i^t, where

$$H_i^t = \begin{cases} H^t_{0,\Gamma_i^D}(\Omega_i), & t \geq 0, \\ (H^{-t}_{0,\Gamma_i^D}(\Omega_i))', & t < 0, \end{cases}$$

with

$$\Gamma_i^D = \begin{cases} \partial\Omega_i \cap (\Omega \cup \Gamma^D), & t \geq 0, \\ \partial\Omega_i \cap \Gamma^D, & t < 0, \end{cases}$$

then $\{f_{ij}\}_{i,j}$ is a frame for $\mathcal{H}^t$. If A, B are the bounds of this frame, then $\{H_i^t : i = 1, \ldots, m\}$ is a frame of subspaces with bounds $\frac{A}{B}$, 1.

As in the well-known frame situation, the frame operator $S_{H,v}$ for $\{H_\lambda\}_{\lambda\in\Lambda}$ and $\{v_\lambda\}_{\lambda\in\Lambda}$ is defined by

$$S_{H,v}(f) = \sum_{\lambda\in\Lambda} v^2 \pi_{H_\lambda}(f).$$

The frame operator $S_{H,v}$ for $\{H_\lambda\}_{\lambda\in\Lambda}$ and $\{v_\lambda\}_{\lambda\in\Lambda}$ is bounded, self-adjoint and invertible on H with $AI \leq S_{H,v} \leq BI$, where A and B are the bounds of the frame of subspaces. Further, the following reconstruction formula holds:

$$f = \sum_{\lambda\in\Lambda} v_\lambda^2 S_{H,v}^{-1} \pi_{H_\lambda}(f) \quad \forall f \in H.$$

In [5], it is proved that $\{S_{H,v}^{-1} H_\lambda\}_{\lambda\in\Lambda}$ is a frame with respect to $\{v_\lambda\}_{\lambda\in\Lambda}$.

Proposition 1. *Let $\{H_\lambda\}_{\lambda\in\Lambda}$ be a frame of subspaces with respect to $\{v_\lambda\}_{\lambda\in\Lambda}$, and let $L : H \to H$ be a bounded invertible operator on H. Then $\{L(H_\lambda)\}_{\lambda\in\Lambda}$ is a frame of subspaces with respect to $\{v_\lambda\}_{\lambda\in\Lambda}$.*

In this case, if u is the solution of equation (1) and S' is the frame operator of the frame of subspaces $\{LH_\lambda\}$, then

$$u = \sum_{\lambda\in\Lambda} v_\lambda^2 S'^{-1}_{H,v} \pi_{LH_\lambda} u.$$

Now since L is invertible, then $\pi_{LH_\lambda} = L\pi_{H_\lambda}L^{-1}$. Therefore

$$u = \sum_{\lambda\in\Lambda} v_\lambda^2 S'^{-1}_{H,v} L\pi_{H_\lambda} L^{-1}u = \sum_{\lambda\in\Lambda} v_\lambda^2 S'^{-1}_{H,v} L\pi_{H_\lambda} (L^{-1})^2 f.$$

It is difficult to compute L^{-1} and S'^{-1}. Our goal is to find a sequence u_i of an approximated solution it related to a frame of subspaces, such that it converges to the solution u of equation (1).

2. Richardson iterative method by using frames of subspaces

The most straightforward approach to an iterative solution of a linear system is to rewrite equation (1) as a linear fixed-point iteration. One way to do this is the Richardson iteration. The abstract method reads as follows:

write $Lu = f$ as

$$u = (I - L)u + f.$$

For given $u_0 \in H$, for $k \geq 0$ define

$$u_{k+1} = (I - L)u_k + f. \tag{4}$$

Since $Lu - f = 0$,

$$\begin{aligned} u_{k+1} - u &= (I - L)u_k + f - u - (f - Lu) \\ &= (I - L)u_k - u + Lu \\ &= (I - L)(u_k - u). \end{aligned}$$

Hence,

$$\|u_{k+1} - u\|_H \leq \|I - L\|_{H\to H} \|u_k - u\|_H,$$

so that (4) converges if

$$\|I - L\|_{H\to H} < 1.$$

It is occasionally possible to precondition (1) by multiplying both sides by a matrix B,

$$BLu = Bf,$$

so that the convergence rate of iterative methods is improved. This is a very effective technique for solving differential equations, integral equations, and related problems [2, 3]. We want to do the same by using frames of subspaces.

Let $\{H_\lambda\}_{\lambda\in\Lambda}$ be a frame of subspaces with respect to $\{v_\lambda\}_{\lambda\in\Lambda}$ for a separable Hilbert space H along with the frame operator $S_{H,v}$. By Proposition 1, $\{L(H_\lambda)\}_{\lambda\in\Lambda}$ also is a frame with respect to $\{v_\lambda\}_{\lambda\in\Lambda}$. We denote the frame operator for $\{L(H_\lambda)\}_{\lambda\in\Lambda}$ and $\{v_\lambda\}_{\lambda\in\Lambda}$, by $S'_{H,v}$. In addition, since L is bounded invertible, there exist two positive constants c_1 and c_2 such that

$$c_1\|u\|_H \leq \|Lu\|_H \leq c_2\|u\|_H, \quad \forall u \in H. \tag{5}$$

The following theorem represents an iterative method to approximate the solution u of equation (1). The solution and the convergence rate depend on the knowledge of the bounds of the frame of subspaces.

Theorem 2. *Let $\{H_\lambda\}_{\lambda\in\Lambda}$ be a frame of subspaces with respect to $\{v_\lambda\}_{\lambda\in\Lambda}$ for H with frame operator $S_{H,v}$ and let L be as given in (1). Taking $u_0 = 0$, for $k \geq 1$,*

$$u_k = u_{k-1} + \frac{2}{c_1^2 A + c_2^2 B} L S'_{H,v}(f - Lu_{k-1}),$$

where $S'_{H,v}$ is the frame operator for the frame of subspaces $\{L(H_\lambda)\}_{\lambda\in\Lambda}$ with respect to $\{v_\lambda\}_{\lambda\in\Lambda}$ with bounds A, B, and c_1, c_2 as given in (5). Then

$$\|u - u_k\|_H \leq \left(\frac{c_2^2 B - c_1^2 A}{c_1^2 A + c_2^2 B}\right)^k \|u\|_H.$$

In particular, the vectors u_k converge to u as $k \to \infty$.

Proof. By the definition of u_k we obtain

$$\begin{aligned}
u - u_k &= u - u_{k-1} - \frac{2}{c_1^2 A + c_2^2 B} L S'_{H,v}(f - Lu_{k-1}) \\
&= (I - \frac{2}{c_1^2 A + c_2^2 B} L S'_{H,v} L)(u - u_{k-1}) \\
&= (I - \frac{2}{c_1^2 A + c_2^2 B} L S'_{H,v} L)^2(u - u_{k-2}) \\
&= \cdots = (I - \frac{2}{c_1^2 A + c_2^2 B} L S'_{H,v} L)^k(u - u_0).
\end{aligned}$$

Therefore

$$\|u - u_k\|_H \leq \|I - \frac{2}{c_1^2 A + c_2^2 B} L S'_{H,v} L\|^k \|u\|_H. \tag{6}$$

On the other hand, for every $v \in H$ we have

$$\begin{aligned}
\langle (I - \frac{2}{c_1^2 A + c_2^2 B} L S'_{H,v} L)v, v\rangle &= \|v\|_H^2 - \frac{2}{c_1^2 A + c_2^2 B}\langle S'_{H,v} Lv, Lv\rangle \\
&= \|v\|_H^2 - \frac{2}{c_1^2 A + c_2^2 B}\langle \sum_{\lambda\in\Lambda} v_\lambda^2 \pi_{LH_\lambda}(Lv), Lv\rangle \\
&= \|v\|_H^2 - \frac{2}{c_1^2 A + c_2^2 B}\sum_{\lambda\in\Lambda} v_\lambda^2 \|\pi_{LH_\lambda}(Lv)\|_H^2 \\
&\leq \|v\|_H^2 - \frac{2A}{c_1^2 A + c_2^2 B}\|Lv\|_H^2 \\
&\leq \|v\|_H^2 - \frac{2A}{c_1^2 A + c_2^2 B} c_1^2 \|v\|_H^2 \\
&= (\frac{c_2^2 B - c_1^2 A}{c_1^2 A + c_2^2 B})\|v\|_H^2,
\end{aligned}$$

where in the first inequality we used the property of the lower bound of the frame of subspaces and in the second inequality we used the property of c_1 in (5). Similarly, we have

$$-(\frac{c_2^2 B - c_1^2 A}{c_1^2 A + c_2^2 B})\|v\|_H^2 \leq \langle (I - \frac{2}{c_1^2 A + c_2^2 B} L S'_{H,v} L)v, v\rangle,$$

so we conclude that

$$\|I - \frac{2}{c_1^2 A + c_2^2 B} L S'_{H,v} L\| \leq \frac{c_2^2 B - c_1^2 A}{c_1^2 A + c_2^2 B}. \tag{7}$$

Combining this inequality with (6) gives the result. □

3. Chebyshev method

Let $h_n = \sum_{k=1}^{n} d_{n_k} u_k$, where u_k is the same as given in Theorem 2, such that $\sum_{k=1}^{n} d_{n_k} = 1$, that is guaranteed if $u_k = u$ for all $1 \leq k \leq n$, then $h_n = u$. By the proof of Theorem 2 we have

$$u - u_k = (I - \frac{2}{c_1^2 A + c_2^2 B} L S^{'} L)^k (u - u_0),$$

so

$$\begin{aligned} u - h_n &= \sum_{k=1}^{n} d_{n_k} u - \sum_{k=1}^{n} d_{n_k} u_k = \sum_{k=1}^{n} d_{n_k} (u - u_k) \\ &= \sum_{k=1}^{n} d_{n_k} (I - \frac{2}{c_1^2 A + c_2^2 B} L S^{'} L)^k (u - u_0). \end{aligned}$$

Defining $R = I - \frac{2}{c_1^2 A + c_2^2 B} L S^{'} L$ and $Q_n(x) = \sum_{k=1}^{n} d_{n_k} x^k$, we obtain

$$u - h_n = Q_n(R)(u - u_0). \tag{8}$$

Proposition (7) and the spectral theorem deduce that

$$\begin{aligned} \|u - h_n\| &= \|Q_n(R)(u - u_0)\| \leq \|Q_n(R)\| \|u - u_0\| \\ &\leq \max_{|x| \leq \alpha_0} |Q_n(x)| \|u - u_0\|, \end{aligned} \tag{9}$$

where $\alpha_0 = \frac{c_2^2 B - c_1^2 A}{c_2^2 B + c_1^2 A}$.

The aim is to minimize this error. Therefore we try to find

$$\min_{Q_n(x) \in \pi_n} \max_{|x| \leq \alpha_0} |Q_n(x)|, \tag{10}$$

where π_n is the set of all polynomials of degrees no more than n such that $Q_n(1) = 1$. The minimax problem (10) is achieved by Chebyshev polynomials [6], satisfying the recurrence relation

$$C_0(x) = 1,\ C_1(x) = x,\ C_n(x) = 2x C_{n-1}(x) - C_{n-2}(x), \quad \forall n \geq 2. \tag{11}$$

In fact,

$$C_n(x) = \begin{cases} \cos(n \cos^{-1}(x)), & |x| \leq 1 \\ \cosh(\cosh^{-1}(x)) = \frac{1}{2}((x + \sqrt{x^2 - 1})^n + (x + \sqrt{x^2 - 1})^{-n}), & |x| \geq 1. \end{cases}$$

In this case, the following lemma holds [6].

Lemma 1. *Let π_n be the same as given before. For real numbers a, b such that $a < b < 1$, set*

$$P_n(x) = \frac{C_n(\frac{2x-a-b}{b-a})}{C_n(\frac{2-a-b}{b-a})}.$$

Then

$$\max_{a \le x \le b} |P_n(x)| \le \max_{a \le x \le b} |Q_n(x)|,$$

where $Q_n \in \pi_n$.

Furthermore,

$$max_{a \le x \le b} |P_n(x)| = \frac{1}{C_n(\frac{2-a-b}{b-a})}.$$

Setting $a = -\alpha_0$ and $b = \alpha_0$ deduce that

$$P_n(x) = \frac{C_n(\frac{2x+\alpha_0-\alpha_0}{\alpha_0+\alpha_0})}{C_n(\frac{2+\alpha_0-\alpha_0}{\alpha_0+\alpha_0})} = \frac{C_n(\frac{x}{\alpha_0})}{C_n(\frac{1}{\alpha_0})}. \tag{12}$$

By Lemma 1, $P_n(x) = \frac{C_n(\frac{x}{\alpha_0})}{C_n(\frac{1}{\alpha_0})}$ solves (10) and minimizes the error $\|u - h_n\|$ in (9).

Using (11) and (12),

$$\begin{aligned} C_n(\frac{1}{\alpha_0})P_n(x) &= C_n(\frac{x}{\alpha_0}) = \frac{2x}{\alpha_0}C_{n-1}(\frac{x}{\alpha_0}) - C_{n-2}(\frac{x}{\alpha_0}) \\ &= \frac{2x}{\alpha_0}C_{n-1}(\frac{1}{\alpha_0})P_{n-1}(x) - C_{n-2}(\frac{1}{\alpha_0})P_{n-2}(x), \end{aligned}$$

and replacing R instead of x yields:

$$C_n(\frac{1}{\alpha_0})P_n(R) = \frac{2R}{\alpha_0}C_{n-1}(\frac{1}{\alpha_0})P_{n-1}(R) - C_{n-2}(\frac{1}{\alpha_0})P_{n-2}(R).$$

Hence,

$$C_n(\frac{1}{\alpha_0})P_n(R)(u - u_0) = (\frac{2R}{\alpha_0}C_{n-1}(\frac{1}{\alpha_0})P_{n-1}(R) - C_{n-2}(\frac{1}{\alpha_0})P_{n-2}(R))(u - u_0).$$

Combining the last equation and (8) gives

$$C_n(\frac{1}{\alpha_0})(u - h_n) = \frac{2}{\alpha_0}C_{n-1}(\frac{1}{\alpha_0})R(u - h_{n-1}) - C_{n-2}(\frac{1}{\alpha_0})(u - h_{n-2}). \tag{13}$$

Now, since $Ru = u - \frac{2}{c_1^2A+c_2^2B}LS^{'}Lu$, by using (13) we have

$$C_n(\frac{1}{\alpha_0})(u-h_n) = \frac{2}{\alpha_0}C_{n-1}(\frac{1}{\alpha_0})(I - \frac{2}{c_1^2A + c_2^2B}LS^{'}L)(u-h_{n-1}) - C_{n-2}(\frac{1}{\alpha_0})(u-h_{n-2}),$$

that is

$$\begin{aligned}
C_n(\frac{1}{\alpha_0})u - C_n(\frac{1}{\alpha_0})h_n &= \frac{2}{\alpha_0}C_{n-1}(\frac{1}{\alpha_0})u \\
&+\frac{2}{\alpha_0}C_{n-1}(\frac{1}{\alpha_0})(-h_{n-1} - \frac{2}{c_1^2A+c_2^2B}LS^{'}L(u-h_{n-1})) \\
&-C_{n-2}(\frac{1}{\alpha_0})u + C_{n-2}(\frac{1}{\alpha_0})h_{n-2}.
\end{aligned}$$

Finally, the definition of C_n in (11) implies that

$$\begin{aligned}
C_n(\frac{1}{\alpha_0})h_n &= \frac{2}{\alpha_0}C_{n-1}(\frac{1}{\alpha_0})(h_{n-1} + \frac{2}{c_1^2A+c_2^2B}LS^{'}L(u-h_{n-1})) \\
&-C_{n-2}(\frac{1}{\alpha_0})h_{n-2}. \qquad (14)
\end{aligned}$$

By defining $\beta_n = \frac{\frac{2}{\alpha_0}C_{n-1}(\frac{1}{\alpha_0})}{C_n(\frac{1}{\alpha_0})}$ and (11), we observe that

$$-\frac{C_{n-2}(\frac{1}{\alpha_0})}{C_n(\frac{1}{\alpha_0})} = 1-\beta_n.$$

Now by (11) and (14) we can write

$$\begin{aligned}
h_n &= \frac{2}{\alpha_0}\frac{C_{n-1}(\frac{1}{\alpha_0})}{C_n(\frac{1}{\alpha_0})}(h_{n-1} + \frac{2}{c_1^2A+c_2^2B}LS^{'}L(u-h_{n-1})) - \frac{C_{n-2}(\frac{1}{\alpha_0})}{C_n(\frac{1}{\alpha_0})}h_{n-2} \\
&= \frac{2}{\alpha_0}\frac{C_{n-1}(\frac{1}{\alpha_0})}{C_n(\frac{1}{\alpha_0})}(h_{n-1} + \frac{2}{c_1^2A+c_2^2B}LS^{'}L(u-h_{n-1})) \\
&\quad -\frac{\frac{2}{\alpha_0}C_{n-1}(\frac{1}{\alpha_0}) - C_n(\frac{1}{\alpha_0})}{C_n(\frac{1}{\alpha_0})}h_{n-2} \\
&= \beta_n(h_{n-1} + \frac{2}{c_1^2A+c_2^2B}LS^{'}(f-Lh_{n-1}))(1-\beta_n)h_{n-2} \\
&= \beta_n(h_{n-1} + \frac{2}{c_1^2A+c_2^2B}LS^{'}(f-Lh_{n-1})) - \beta_n h_{n-2} + h_{n-2}.
\end{aligned}$$

Therefore

$$h_n = \beta_n(h_{n-1} - h_{n-2} + \frac{2}{c_1^2A+c_2^2B}LS^{'}(f-Lh_{n-1})) + h_{n-2}.$$

More precisely, by (11) we obtain

$$\beta_n = (\frac{\alpha_0 C_n(\frac{1}{\alpha_0})}{2C_{n-1}(\frac{1}{\alpha_0})})^{-1} = (\frac{\alpha_0}{2}\frac{\frac{2}{\alpha_0}C_{n-1}(\frac{1}{\alpha_0}) - C_{n-2}(\frac{1}{\alpha_0})}{C_{n-1}(\frac{1}{\alpha_0})})^{-1} = (1-\frac{\alpha_0^2}{4}\beta_{n-1})^{-1}.$$

Now, based on the preceding statement, we present the following iterative method to give an approximated solution to equation (1). Suppose that $\{H_\lambda\}_{\lambda\in\Lambda}$ is a frame of subspaces for H and $S^{'}$ denotes the frame operator of the frame of subspaces $\{LH_\lambda\}_{\lambda\in\Lambda}$ with bounds A and B. Also consider c_1 and c_2 as given in (5) and let ϵ be a positive number.

Algorithm $[L,\epsilon,A,B,c_1,c_2]\to u_\epsilon$

(i) Let $\alpha_0=\frac{c_2^2B-c_1^2A}{c_2^2B+c_1^2A}$, $\sigma=\frac{c_2\sqrt{B}-c_1\sqrt{A}}{c_2\sqrt{B}+c_1\sqrt{A}}$

(ii) $h_0:=0$, $h_1:=\frac{2}{c_1^2A+c_2^2B}LS^{'}f$, $\beta_1=2$, $n=1$

(iii) While $\frac{2\sigma^n}{1+\sigma^{2n}}\frac{\|f\|}{c_1}>\epsilon$

(1) $n:=n+1$

(2) $\beta_n=(1-\frac{\alpha_0^2}{4}\beta_{n-1})^{-1}$

(3) $h_n=\beta_n(h_{n-1}-h_{n-2}+\frac{2}{c_1^2A+c_2^2B}LS^{'}(f-Lh_{n-1}))+h_{n-2},\quad n\geq 2$

(iv) $u_\epsilon:=h_n$.

We note that $\sigma<1$ and this implies that the algorithm terminates in finite iteration. Convergence of the above algorithm is proved in the following theorem.

Theorem 3. *The approximated solution h_n in the **Algorithm** $[L,\epsilon,A,B,c_1,c_2]$ satisfies*

$$\|u-h_n\|\leq\frac{2\sigma^n}{1+\sigma^{2n}}\frac{\|f\|}{c_1}.$$

Proof. First, we note that by definition $C_n(x)$

$$\begin{aligned}
C_n(\frac{1}{\alpha_0}) &= C_n(\frac{c_2^2B+c_1^2A}{c_2^2B-c_1^2A})\\
&=\frac{1}{2}((\frac{c_2^2B+c_1^2A}{c_2^2B-c_1^2A}+\sqrt{\frac{(c_2^2B+c_1^2A)^2}{(c_2^2B-c_1^2A)^2}-1})^n+\frac{1}{(\frac{c_2^2B+c_1^2A}{c_2^2B-c_1^2A}+\sqrt{\frac{(c_2^2B+c_1^2A)^2}{(c_2^2B-c_1^2A)^2}-1})^n})\\
&=\frac{1}{2}((\frac{c_2^2B+c_1^2A}{c_2^2B-c_1^2A}+\sqrt{\frac{4(c_2^2B)(c_1^2A)}{(c_2^2B-c_1^2A)^2}})^n+\frac{1}{(\frac{c_2^2B+c_1^2A}{c_2^2B-c_1^2A}+\sqrt{\frac{4(c_2^2B)(c_1^2A)}{(c_2^2B-c_1^2A)^2}})^n})\\
&=\frac{1}{2}((\frac{(\sqrt{c_2^2B}+\sqrt{c_1^2A})^2}{c_2^2B-c_1^2A})^n+\frac{1}{(\frac{(\sqrt{c_2^2B}+\sqrt{c_1^2A})^2}{c_2^2B-c_1^2A})^n})\\
&=\frac{1}{2}((\frac{c_2\sqrt{B}+c_1\sqrt{A}}{c_2\sqrt{B}-c_1\sqrt{A}})^n+\frac{1}{(\frac{c_2\sqrt{B}+c_1\sqrt{A}}{c_2\sqrt{B}-c_1\sqrt{A}})^n})\\
&=\frac{1}{2}(\frac{1}{\sigma^n}+\sigma^n)=\frac{1+\sigma^{2n}}{2\sigma^n}.
\end{aligned}$$

Now, by Lemma 1 and inequalities 9 and 5

$$\|u-h_n\|\leq\frac{1}{C_n(\frac{1}{\alpha_0})}\|u\|=(C_n(\frac{1}{\alpha_0}))^{-1}\|u\|=\frac{2\sigma^n}{1+\sigma^{2n}}\|u\|\leq\frac{2\sigma^n}{1+\sigma^{2n}}\frac{\|f\|}{c_1}.$$

This proves the theorem. $\square$

Similarly to the Richardson iterative method (Theorem 2), this algorithm depends on the knowledge of the frame bounds and the guaranteed speed of convergence also depends thereon. But this algorithm designs an iterative method that guarantees a faster convergence than the Richardson method, specially when B is much larger than A.

4. Numerical experiments

In this section, we present two examples to confirm the theoretical results given in the previous sections.

Example 2. *Consider the boundary value problem*

$$\begin{cases} -\ddot{u} = f & in\ \Omega = (0,1), \\ u(0) = u(1) = 0, \end{cases}$$

on the space $H = span\{x^i(1-x) : \quad i = 1, 2, \ldots, 20\}$*. The function* f *is chosen such that* $u(x) = 4x^3(1-x) - 3x^4(1-x)$ *is the exact solution.*

Now, we consider the frame of subspaces $\{W_1, W_2\}$ *for* H *such that*

$$W_1 = span\{x^i(1-x) : i = 1, 2, ..., 10\}, \quad W_2 = span\{x^i(1-x) : i = 10, 11..., 20\}.$$

The value σ *is derived* 0.81 *that enables the algorithm to converge at limited iterations. Table 1 shows the error* $\|u - \bar{u}\|_{L^2([0,1])}$*, where* $\bar{u}$ *denotes the approximated solution given by the Chebyshev method. As we seen, after 53 iterations in 67 seconds, the proposed method would converge.*

n	2	15	30	53	CPU(sec.)
$\|u - \bar{u}\|_{L^2([0,1])}$	1.36	0.81	0.012	0.001	67

Table 1: *L2-norm of the error between the exact and approximated solutions*

Example 3. *Consider the boundary value problem*

$$\begin{cases} -\ddot{u} + 2u = f & in\ \Omega = (0,1), \\ u(0) = u(1) = 0, \end{cases}$$

on the space $H = span\{sin(i\pi x) : \quad i = 1, 2, \ldots, 40\}$*. The function* f *is chosen such that* $u(x) = 3sin(2\pi x) - sin(8\pi x)$ *is the exact solution.*

Now, we consider the frame of subspaces $\{W_1, W_2\}$ *for* H *such that*

$$W_1 = span\{sin(i\pi x) : i = 1, 2, ..., 20\}, \quad W_2 = span\{sin(i\pi x) : i = 20, 21..., 40\}.$$

The value σ *is derived* 0.91 *that enables the algorithm to converge at limited iterations. Table 2 shows the error* $\|u - \bar{u}\|_{L^2([0,1])}$*, where* $\bar{u}$ *denotes the approximated solution given by the Chebyshev method. As we seen, after 98 iterations in 126 seconds, the proposed method would converge.*

n	2	20	40	98	CPU(sec.)
$\|u-\bar{u}\|_{L^2([0,1])}$	1.25	0.91	0.014	0.001	126

Table 2: *L2-norm of the error between the exact and approximated solutions*

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