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On certain surfaces in the isotropic 4-space

Muhittin Evren Aydin[1,*] and Ion Mihai[2]

[1] *Department of Mathematics, Faculty of Science, Firat University, Elazig, 23 200, Turkey*
[2] *Department of Mathematics, Faculty of Mathematics and Computer Science, University of Bucharest, Bucharest, 010 014, Romania*

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Abstract. The isotropic space is a special ambient space obtained from the Euclidean space by substituting the usual Euclidean distance with the isotropic distance. In the present paper, we establish a method to calculate the second fundamental form of surfaces in the isotropic 4-space. Further, we classify some surfaces (spherical product surfaces and Aminov surfaces) in the isotropic 4-space with vanishing curvatures.

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1. Preliminaries

Let $\mathbb{R}^{n+1}$ be the Euclidean $(n+1)$-space, i.e., the Cartesian $(n+1)$-space endowed with the Euclidean metric. We will denote the Euclidean scalar product and the induced norm on $\mathbb{R}^{n+1}$ by $\langle \cdot, \cdot \rangle$ and $\|\cdot\|$, respectively.

The isotropic $(n+1)$-space $\mathbb{I}^{n+1}$ introduced by H. Sachs [22] is the product of $\mathbb{R}^n$ and the isotropic line equipped with a degenerate parabolic distance metric. It is derived from $\mathbb{R}^{n+1}$ by substituting the usual Euclidean distance with the isotropic distance.

The group of motions of $\mathbb{I}^{n+1}$ is given by the matrix

$$\begin{bmatrix} A & 0 \\ B & 1 \end{bmatrix},$$

where A is an orthogonal $(n,n)-$matrix, $\det A = 1$, B a real $(1,n)-$matrix.

Consider the points $\mathbf{p} = (p, p_{n+1})$ and $\mathbf{q} = (q, q_{n+1})$ in $\mathbb{I}^{n+1}$, with $p = (p_1, \ldots, p_n)$, $q = (q_1, \ldots, q_n)$. Thus the *isotropic distance* (*i-distance*) of two points $\mathbf{p} = (p, p_{n+1})$ and $\mathbf{q} = (q, q_{n+1})$ is defined as

$$\|\mathbf{p} - \mathbf{q}\|_i = \|p - q\| = \sqrt{\sum_{j=1}^{n} (q_j - p_j)^2}. \tag{1}$$

*Corresponding author. *Email addresses:* meaydin@firat.edu.tr (M. E. Aydin), imihai@fmi.unibuc.ro (I. Mihai)

The i-metric (1) is degenerate along the lines in x_{n+1}-direction, and these lines are called *isotropic lines*. k-planes containing an isotropic line are called *isotropic* $k-$*planes*. Other planes are *non-isotropic*.

A surface M^2 immersed in $\mathbb{I}^{n+1}$ is called *admissible* if it has no isotropic tangent planes.

Isotropic scalar product (*i-scalar product*) "$\cdot$" of vectors $\mathbf{u} = (u, u_{n+1})$ and $\mathbf{v} = (v, v_{n+1})$ in $\mathbb{I}^{n+1}$ for $u = (u_1, \ldots, u_n)$ and $v = (v_1, \ldots, v_n)$ is given by

$$\mathbf{u} \cdot \mathbf{v} = \begin{cases} \langle u, v \rangle & \text{, if at least one of } u_i \text{ or } v_i \text{ is nonzero, } i = \overline{1,n}, \\ u_{n+1}v_{n+1} & \text{, if } u_i = 0 = v_i \text{ for all } i = \overline{1,n}. \end{cases} \tag{2}$$

We call vectors of the form $\mathbf{u} = (0, u_{n+1})$ in $\mathbb{I}^{n+1}$, $0 = \left(\underbrace{0, \ldots, 0}_{n-tuple} \right)$, $u_{n+1} \neq 0$, *isotropic vectors* and ones of the form $\mathbf{u} = (u \neq 0, u_{n+1})$ *non-isotropic vectors*. With respect to the i-scalar product (2), all isotropic vectors are orthogonal to non-isotropic ones. Morever, two non-isotropic vectors $\mathbf{u}, \mathbf{v}$ in $\mathbb{I}^{n+1}$ are orthogonal if $\mathbf{u} \cdot \mathbf{v} = 0$.

In particular, the isotropic 3-space $\mathbb{I}^3$ is a Cayley–Klein space defined from a 3-dimensional projective space $P\left(\mathbb{R}^3\right)$ with the absolute figure which is an ordered triple (ω, f_1, f_2), where ω is a plane in $P\left(\mathbb{R}^3\right)$ and f_1, f_2 are two complex-conjugate straight lines in ω (see [23]). The homogeneous coordinates in $P\left(\mathbb{R}^3\right)$ are introduced in such a way that the absolute plane ω is given by $X_0 = 0$ and the absolute lines f_1, f_2 by $X_0 = X_1 + iX_2 = 0$, $X_0 = X_1 - iX_2 = 0$. The intersection point $F(0:0:0:1)$ of these two lines is called the absolute point. The group of motions of $\mathbb{I}^3$ is a six-parameter group given in the affine coordinates $x_1 = \frac{X_1}{X_0}$, $x_2 = \frac{X_2}{X_0}$, $x_3 = \frac{X_3}{X_0}$ by

$$(x_1, x_2, x_3) \longmapsto (x_1', x_2', x_3') : \begin{cases} x_1' = a + x_1 \cos\phi - x_2 \sin\phi, \\ x_2' = b + x_1 \sin\phi + x_2 \cos\phi, \\ x_3' = c + dx_1 + ex_2 + x_3, \end{cases} \tag{3}$$

where $a, b, c, d, e, \phi \in \mathbb{R}$.

Such affine transformations are called *isotropic congruence transformations* or *i-motions*. It can be easily seen from (3) that i-motions are indeed composed of an Euclidean motion in the x_1x_2-plane (i.e. translation and rotation) and an affine shear transformation in x_3-direction.

Consider the points $\mathbf{x} = (x_1, x_2, x_3)$ and $\mathbf{y} = (y_1, y_2, y_3)$. The projection in the x_3-direction onto $\mathbb{R}^2$, $(x_1, x_2, x_3) \longmapsto (x_1, x_2, 0)$, is called the *top view*. In the sequel, many of metric properties in isotropic geometry (invariants under (3)) are Euclidean invariants in the top view such as i-distance.

Planes, circles and spheres. There are two types of planes in $\mathbb{I}^3$ ([17]-[19]).

(1) *Non-isotropic planes* are planes non-parallel to the x_3-direction. In these planes we basically have a Euclidean metric. This is not the one we are used to, since we have to make the usual Euclidean measurements in the top view. An *i-circle* (of *elliptic type*) in a non-isotropic plane P is an ellipse, whose top view is a Euclidean

circle. Such an i-circle with center $\mathbf{m} \in P$ and radius r is the set of all points $\mathbf{x} \in P$ with $\|\mathbf{x} - \mathbf{m}\|_i = r$.

(2) *Isotropic planes* are planes parallel to the x_3-axis. There, $\mathbb{I}^3$ induces an isotropic metric. An *isotropic circle* (of *parabolic type*) is a parabola with an x_3-parallel axis and thus it lies in an isotropic plane

There are also two types of *isotropic spheres*. An *i-sphere of the cylindrical type* is the set of all points $\mathbf{x} \in \mathbb{I}^3$ with $\|\mathbf{x} - \mathbf{m}\|_i = r$. Speaking in a Euclidean way, such a sphere is a right circular cylinder with x_3-parallel rulings; its top view is the Euclidean circle with center $\mathbf{m}$ and radius r. A more interesting and important type of spheres are the *i-spheres of parabolic type,*

$$x_3 = \frac{A}{2}\left(x_1^2 + x_2^2\right) + Bx_1 + Cx_2 + D, \quad A \neq 0.$$

From an Euclidean perspective, they are paraboloids of revolution with the x_3-parallel axis. The intersections of these i-spheres with planes P are i-circles. If P is nonisotropic, then the intersection is an i-circle of elliptic type. If P is isotropic, the intersection curve is an i-circle of parabolic type.

For an admissible surface M^2 the coefficients g_{11}, g_{12}, g_{22} of its first fundamental form are calculated with respect to the induced metric.

The normal field of M^2 in $\mathbb{I}^3$ is always the isotropic vector $(0, 0, 1)$ since it is perpendicular to all tangent vectors to M^2. The coefficients h_{11}, h_{12}, h_{22} of the second fundamental form of M^2 are calculated with respect to the normal field of M^2.

The *relative curvature* (the so-called *isotropic Gaussian curvature*) and the *isotropic mean curvature* of M^2 in $\mathbb{I}^3$ are defined by

$$K = \frac{\det(h_{ij})}{\det(g_{ij})}, \quad H = \frac{g_{11}h_{22} - 2g_{12}h_{12} + g_{22}h_{11}}{2\det(g_{ij})}.$$

For the formulas of relative and isotropic mean curvatures of a surface M^2 with codimension 2 in $\mathbb{I}^4$, see Section 2. Also, more details on $\mathbb{I}^{n+1}$ can be found in [1, 10, 11], [14]-[16], [20]-[24].

On the other hand, isotropic geometry naturally appears when properties of functions shall be geometrically visualized and interpreted via their graph surfaces [17]. One of the remarkable applications of isotropic geometry is pertinent to Image Processing and has been presented in [12]. Another one by H. Pottmann and Y. Liu is the study of discrete surfaces in isotropic geometry with applications in architectural design [18].

More recently, B.Y. Chen et al. [8, 9] studied production models in microeconomics via isotropic geometry.

One of the present authors [2, 3] classified the translation and homothetical hypersurfaces in $\mathbb{I}^{n+1}$ with constant curvature.

In this paper, we introduce a method to calculate the second fundamental form of the surfaces with codimension 2 in $\mathbb{I}^4$. Moreover, we classify spherical product surfaces and Aminov surfaces in $\mathbb{I}^4$ with vanishing curvatures.

2. Surfaces in isotropic 4-space

Let M^2 be a surface immersed in $\mathbb{I}^4$ and $D \subseteq \mathbb{R}^2$ an open domain. Then we parametrize the surface M^2 by mapping

$$\mathbf{x} : D \subseteq \mathbb{R}^2 \longrightarrow \mathbb{I}^4, \ \ (u_1, u_2) \longmapsto \mathbf{x}(u_1, u_2) := (\ x_1(u_1, u_2), x_2(u_1, u_2), \\ x_3(u_1, u_2), x_4(u_1, u_2)),$$

where x_i, $i = \overline{1,4}$, are smooth real-valued functions on D.

Throughout this paper, we only consider admissible surfaces.

As usual, the pair $\left\{ \mathbf{x}_{u_1} := \frac{\partial \mathbf{x}}{u_1}, \ \ \mathbf{x}_{u_2} := \frac{\partial \mathbf{x}}{u_2} \right\}$ is a basis of T_pM^2, $p \in M^2$. Hence we have

$$\mathfrak{g} := \sum_{i,j=1}^{2} \mathfrak{g}_{ij} du_i du_j, \ \mathfrak{g}_{ij} := \mathbf{x}_{u_i} \cdot \mathbf{x}_{u_j}, \ i, j = 1, 2,$$

where $\mathfrak{g}$ is the metric tensor on T_pM^2 induced from the i-scalar product on $\mathbb{I}^4$. Denote $W_1 := \sqrt{\det(\mathfrak{g}_{ij})}$.

Now, let $\alpha = (\alpha_i)$, $\beta = (\beta_i)$, $\gamma = (\gamma_i)$ be vectors in $\mathbb{I}^4$. Then we can define a cross product on $\mathbb{I}^4$ by

$$\alpha \times \beta \times \gamma := \begin{vmatrix} e_1 & e_2 & e_3 & 0 \\ \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \\ \beta_1 & \beta_2 & \beta_3 & \beta_4 \\ \gamma_1 & \gamma_2 & \gamma_3 & \gamma_4 \end{vmatrix},$$

for $e_i = (\delta_{i1}, \delta_{i2}, \delta_{i3}, \delta_{i4})$, $i = 1, \ldots, 4$, where δ is Kronocker delta. It is easy to check that

$$(\alpha \times \beta \times \gamma) \cdot \xi = \det\left(\alpha, \beta, \gamma, \bar{\xi}\right),$$

where $\bar{\xi}$ denotes the projection of ξ on the Euclidean (x_1, x_2, x_3)-space.

Therefore the normal space of M^2 in $\mathbb{I}^4$ is spanned by the vectors $\{N_1, N_2\}$,

$$N_1 := (0, 0, 0, 1),$$

which is completely a unit isotropic vector, and

$$N_2 := \frac{1}{W_1} \mathbf{x}_{u_1} \times \mathbf{x}_{u_2} \times N_1.$$

The second fundamental form of M^2 in $\mathbb{I}^4$ has the components

$$h_{ij}^1 := \det\left(\mathbf{x}_{u_i u_j}, \mathbf{x}_{u_1}, \mathbf{x}_{u_2}, N_2\right), \ h_{ij}^2 := \mathbf{x}_{u_i u_j} \cdot N_2.$$

A surface in $\mathbb{I}^4$ for which the second fundamental form vanishes is called *totally geodesic.*

The *relative curvature* (a counterpart to Gaussian curvature) of M^2 in $\mathbb{I}^4$ is defined by

$$G := \frac{1}{W_1^2} \sum_{r=1}^{2} \left[h_{11}^r h_{22}^r - \left(h_{12}^r\right)^2 \right] \tag{4}$$

and the *isotropic mean curvature field* by

$$\overrightarrow{H} := \frac{1}{2W_1^2}\sum_{r=1}^{2}\left[\mathfrak{g}_{11}h_{22}^r - 2\mathfrak{g}_{12}h_{12}^r + \mathfrak{g}_{22}h_{11}^r\right]N_r. \tag{5}$$

A surface M^2 in $\mathbb{I}^4$ is called *isotropic minimal* (resp. *isotropic flat*) if $\overrightarrow{H} \equiv 0$ (resp. $G \equiv 0$).

3. Aminov surfaces in isotropic 4-space

Let r be a nonzero smooth real-valued function on an open interval $I \subset \mathbb{R}$. Then we consider a surface M^2 in $\mathbb{I}^4$ given by

$$\mathbf{x} : I \times [0, 2\pi) \longrightarrow \mathbb{I}^4,\ \ (u, v) \longmapsto \mathbf{x}(u, v) = (u, v, r(u)\cos v, r(u)\sin v).$$

Such surfaces are called *Aminov surfaces* [7].

The basis $\{\mathbf{x}_u, \mathbf{x}_v\}$ of the tangent space of M^2 is

$$\mathbf{x}_u = (1, 0, r'\cos v, r'\sin v) \ \text{ and } \mathbf{x}_v = (0, 1, -r\sin v, r\cos v), \tag{6}$$

where $r' = \frac{dr}{du}$. From (6), we have

$$\mathfrak{g}_{11} = 1 + (r'\cos v)^2,\ \mathfrak{g}_{12} = -rr'\cos v\sin v,\ \mathfrak{g}_{22} = 1 + (r\sin v)^2 \tag{7}$$

and $W_1^2 = 1 + (r'\cos v)^2 + (r\sin v)^2$.

The basis vectors of the normal space of M^2 are

$$N_1 = (0, 0, 0, 1) \ \text{ and } N_2 = \frac{1}{W_1}(-r'\cos v, r\sin v, 1, 0).$$

The components of the second fundamental form are

$$\begin{cases} h_{11}^1 = -\frac{r''\sin v}{W_1}\left(1 + r^2\right),\ h_{12}^1 = -\frac{r'\cos v}{W_1}\left(1 + (r')^2\right),\ h_{22}^1 = \frac{r\sin v}{W_1}\left(1 + r^2\right) \\ h_{11}^2 = \frac{r''}{W_1}\cos v,\ h_{12}^2 = -\frac{1}{W_1}(r'\sin v),\ h_{22}^2 = -\frac{1}{W_1}(r\cos v) \end{cases}. \tag{8}$$

Theorem 1. *The isotropic flat Aminov surfaces in $\mathbb{I}^4$ are only generalized cylinders over circular helices from the Euclidean perspective.*

Proof. Let M^2 be a flat isotropic Aminov surface. Then, from (4), it follows that

$$h_{11}^1h_{22}^1 - \left(h_{12}^1\right)^2 + h_{11}^2h_{22}^2 - \left(h_{12}^2\right)^2 = 0. \tag{9}$$

Substituting (8) into (9) we have

$$\left(rr''\left(1 + r^2\right)^2 + (r')^2\right)\sin^2 v + \left(\left(r'\left(1 + (r')^2\right)\right)^2 + rr''\right)\cos^2 v = 0,$$

which implies that

$$\begin{cases} rr''\left(1+r^2\right)^2+\left(r'\right)^2=0, \\ rr''+\left(r'\left(1+\left(r'\right)^2\right)\right)^2=0. \end{cases} \tag{10}$$

If r is not a constant, from (10) we get

$$\left(r'\right)^2\left(\left(1+\left(r'\right)^2\right)^2-\frac{1}{\left(1+r^2\right)^2}\right)=0,$$

or equivalently,

$$r^2+\left(r'\right)^2+\left(rr'\right)^2=0.$$

This is a contradiction and thus we derive r is a constant.

Therefore we obtain

$$\mathbf{x}\left(u,v\right)=\left(u,0,0,0\right)+\left(0,v,\lambda\cos v,\lambda\sin v\right),$$

which completes the proof. □

Theorem 2. *There does not exist an isotropic minimal Aminov surface in* $\mathbb{I}^4$.

Proof. Consider an isotropic minimal Aminov surface in $\mathbb{I}^4$. It follows from (5) that

$$\mathfrak{g}_{11}h_{22}^l-2\mathfrak{g}_{12}h_{12}^l+\mathfrak{g}_{22}h_{11}^l=0,\ \ l=1,2. \tag{11}$$

Taking $l=2$ in (11) and using (8), we have

$$-\left(1+\left(r'\cos v\right)^2\right)\left(r\cos v\right)-2r\left(r'\right)^2\cos v\sin^2 v+\left(1+\left(r\sin v\right)^2\right)r''\cos v=0. \tag{12}$$

For $v=0$ in (12) we have

$$-r\left(1+\left(r'\right)^2\right)+r''=0. \tag{13}$$

Now dividing (12) by $\cos v$ and taking a partial derivative of (12) with respect to v gives that

$$\left(-\left(r'\right)^2+rr''\right)\sin 2v=0$$

or

$$-\left(r'\right)^2+rr''=0. \tag{14}$$

On the other hand, for $l=1$ in (11), we get

$$\begin{aligned}\left(1+\left(r'\cos v\right)^2\right)\left(1+\left(r^2\right)\right)r-2r\left(r'\right)^2\left(1+\left(r'\right)^2\right)\cos^2 v\\ -\left(1+\left(r\sin v\right)^2\right)r''\left(1+r^2\right)=0.\end{aligned} \tag{15}$$

Taking partial derivative of (15) with respect to v and dividing by $\sin 2v$ gives

$$\left(r'\right)^2 r\left(1+r^2\right)-2r\left(r'\right)^2\left(1+\left(r'\right)^2\right)+r^2r''\left(1+r^2\right)=0. \tag{16}$$

By substituting $rr'' = (r')^2$ in (16), we derive

$$2r(r')^2\left\{r^2 - (r')^2\right\} = 0. \tag{17}$$

If $r' = 0$, then by (13) we have $r = 0$. This is not possible and thus by (17) we conclude

$$r^2 = (r')^2. \tag{18}$$

Substituting (18) in (14) one gets $r = r''$ and from (13) we obtain $r' = 0$. This yields a contradiction and thereby the proof is completed. □

4. Spherical product surfaces in isotropic 4-space

The tight embeddings of product spaces were investigated by N. H. Kuiper (see [13]) who introduced a different tight embedding in the $(n_1 + n_2 - 1)$ −dimensional Euclidean space $\mathbb{R}^{n_1+n_2-1}$ as follows. Let

$$c_1 : M^m \longrightarrow \mathbb{R}^{n_1},$$
$$c_1(u_1, \dots, u_m) = (f_1(u_1, \dots, u_m), \dots, f_{n_1}(u_1, \dots, u_m))$$

be a tight embedding of an $m-$dimensional manifold M^m satisfying the Morse equality and

$$c_2 : \mathbb{S}^{n_2-1} \longrightarrow \mathbb{R}^{n_2},$$
$$c_1(v_1, \dots, v_{n_2-1}) = (g_1(v_1, \dots, v_{n_2-1}), \dots, g_{n_2}(v_1, \dots, v_{n_2-1}))$$

the standard embedding of the $(n_2 - 1)$ −sphere in $\mathbb{R}^{n_2}$, where $u = (u_1, \dots, u_m)$ and $v = (v_1, \dots, v_{n_2-1})$ are the local coordinate systems on M^m and $\mathbb{S}^{n_2-1}$, respectively. Then a new *tight embedding* is given by

$$\mathbf{x} = c_1 \otimes c_2 : M^m \times \mathbb{S}^{n_2-1} \longrightarrow \mathbb{R}^{n_1+n_2-1},$$
$$(u, v) \longmapsto (f_1(u), \dots, f_{n_1-1}(u), f_{n_1}(u) g_1(v), \dots, f_{n_1}(u) g_{n_2}(v)).$$

Such embeddings are obtained from c_1 by rotating $\mathbb{R}^{n_1}$ about $\mathbb{R}^{n_1-1}$ in $\mathbb{R}^{n_1+n_2-1}$.

B. Bulca et al. [5, 6] called such embeddings *rotational embeddings* and considered spherical product surfaces in Euclidean spaces, which are a special type of rotational embeddings as taking $m = 1, n_1 = 2, 3$ and $n_2 = 2$ in the above definition.

The surfaces of revolution in $\mathbb{R}^3$ can be considered as simplest models of spherical product surfaces as well as the quadrics and the superquadrics [4].

Now, let us consider an isotropic 3-space curve and an isotropic plane curve, respectively,

$$c_1(u) = (u, f_1(u), f_2(u)) \text{ and } c_2(v) = (v, g(v))$$

for nonzero smooth functions f_1, f_2 and g.

Then the *spherical product surface* $\left(M^2, c_1 \otimes c_2\right)$ of two curves c_1 and c_2 in $\mathbb{I}^4$ is defined by

$$\mathbf{x} := c_1 \otimes c_2 : \mathbb{R}^2 \longrightarrow \mathbb{I}^4, \ \ (u, v) \longmapsto (u, f_1(u), f_2(u) v, f_2(u) g(v)). \tag{19}$$

We call the curves c_1 and c_2 the *generating curves* of the surface.

Note that such surfaces given by (19) are always admissible.

The tangent space of $\left(M^2, c_1 \otimes c_2\right)$ is spanned by

$$\mathbf{x}_u = (1, f_1', f_2'v, f_2'g) \text{ and } \mathbf{x}_v = f_2\,(0,0,1,g'),$$

where $f_i' = \frac{\partial f_i}{\partial u}$, $i = 1,2$ and $g' = \frac{\partial g}{\partial v}$.

The induced metric g on M^2 from $\mathbb{I}^4$ has the components

$$\mathfrak{g}_{11} = 1 + (f_1')^2 + (f_2'v)^2,\ \mathfrak{g}_{12} = f_2 f_2' v,\ \mathfrak{g}_{22} = (f_2)^2 \tag{20}$$

and $W_1^2 = \det(\mathfrak{g}_{ij}) = (f_2)^2 \left(1 + (f_1')^2\right)$.

The orthonormal basis of the normal space of $\left(M^2, c_1 \otimes c_2\right)$ is

$$N_1 = (0,0,0,1) \text{ and } N_2 = \frac{1}{\sqrt{1 + (f_1')^2}}\,(f_1', -1, 0, 0).$$

Thereby, the nonzero components of the second fundamental form are

$$\begin{aligned} h_{11}^1 &= f_2\,(g'v - g)\sqrt{1 + (f_1')^2}\left(f_2'' - f_2'\frac{f_1' f_1''}{1 + (f_1')^2}\right), \\ h_{22}^1 &= -(f_2)^2\sqrt{1 + (f_1')^2}\,g'', \\ h_{11}^2 &= -\frac{f_1''}{\sqrt{1 + (f_1')^2}}. \end{aligned} \tag{21}$$

The following results classify spherical product surfaces in $\mathbb{I}^4$ with vanishing curvature.

Theorem 3. *Let $\left(M^2, c_1 \otimes c_2\right)$ be an isotropic flat spherical product surface in $\mathbb{I}^4$. Then either it is a non-isotropic plane or one of the following holds:*

(i) c_1 is a planar curve in $\mathbb{I}^3$ lying in the non-isotropic plane $z = const.$;

(ii) c_1 is a line in $\mathbb{I}^3$;

(iii) c_1 is a curve in $\mathbb{I}^3$ of the form

$$c_1(u) = \left(u, f_1(u), \lambda \int \sqrt{1 + (f_1')^2}\,du + \xi\right),\ \lambda, \xi \in \mathbb{R},\ \lambda \neq 0;$$

(iv) c_2 is a line in $\mathbb{I}^2$.

Proof. Let us assume that the spherical product surface $\left(M^2, c_1 \otimes c_2\right)$ is isotropic flat. Then, from (4) and (21), we have

$$f_2^3 g''\,(g'v - g)\left(f_2'' - f_2'\frac{f_1' f_1''}{1 + (f_1')^2}\right)\left(1 + (f_1')^2\right) = 0. \tag{22}$$

It immediately implies that either g is a linear function (which implies the statement (iv) of the theorem) or

$$f_2'' - f_2' \frac{f_1' f_1''}{1 + (f_1')^2} = 0. \tag{23}$$

For equation (23) we have three cases:

Case 1. f_2 is constant. In this case, the generating curve c_1 is a planar curve in $\mathbb{I}^3$ lying in the non-isotropic plane $z = const$. This gives statement (i) of the theorem.

Case 2. f_1 and f_2 are linear. Thus c_1 is a line in $\mathbb{I}^3$, which implies statement (ii).

Case 3. f_1 and f_2 are non-linear. After solving (23), we derive

$$f_2 = \lambda \int \sqrt{1 + (f_1')^2} du + \xi, \ \lambda \neq 0, \xi \in \mathbb{R},$$

which completes the proof. □

Theorem 4. *There does not exist an isotropic minimal spherical product surface in $\mathbb{I}^4$, except totally geodesic ones.*

Proof. Suppose that $(M^2, c_1 \otimes c_2)$ is an isotropic minimal spherical product surface in $\mathbb{I}^4$. By taking $r = 2$ and $r = 1$ in (5), respectively, and considering (21), we get

$$\frac{(f_2)^2 f_1''}{\sqrt{1 + (f_1')^2}} = 0 \tag{24}$$

and

$$f_2 (g'v - g) \left(f_2'' - f_2' \frac{f_1' f_1''}{1 + (f_1')^2} \right) - g'' \left(1 + (f_1')^2 + (f_2' v)^2 \right) = 0. \tag{25}$$

It follows from (24) that f_1 is a linear function. By considering it into (25), we derive

$$(f_2 f_2'') (g'v - g) - g'' \left(1 + a^2 + (f_2' v)^2 \right) = 0, \ a \in \mathbb{R}. \tag{26}$$

For (26), we have to distinguish two cases:

Case 1. g is linear. We have again two cases:

Case 1.1. $g(v) = av$ is a solution for (26). It yields from (21) that $(M^2, c_1 \otimes c_2)$ is totally geodesic.

Case 1.2. $g(v) = av + b$, $a, b \neq 0$. (26) gives that f_2 is linear and it follows from (21) that $(M^2, c_1 \otimes c_2)$ is again totally geodesic.

Case 2. g is non-linear. There exist two cases depending on the function f_2:

Case 2.1. f_2 is linear, $f_2(u) = cu + d$. By (26) we derive

$$g''\left(1 + a^2 + c^2 v^2\right) = 0,$$

which is not possible.

Case 2.2. f_2 is non-linear. Equation (4.8) can then be rewritten as

$$\frac{g'v - g}{g''} - \frac{1 + a^2}{f_2 f_2''} - \frac{(f_2')^2}{f_2 f_2''} v^2 = 0 \tag{27}$$

After taking partial derivative of (4.9) with respect to u, we deduce

$$\left(\frac{1 + a^2}{f_2 f_2''}\right)' + \left(\frac{(f_2')^2}{f_2 f_2''}\right)' v^2 = 0,$$

which yields a contradiction since f_2 is a non-linear function and v is an independent variable.

Therefore the proof is completed. □

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References

[1] M. Akar, S. Yuce, N. Kuruoglu, *One-parameter planar motion on the Galilean plane*, Int. Electron. J. Geom. **6**(2013), 79–88.

[2] M. E. Aydin, *A generalization of translation surfaces with constant curvature in the isotropic space*, J. Geom, doi: 10.1007/s00022-015-0292-0.

[3] M. E. Aydin, A. O. Ogrenmis, *Homothetical and translation hypersurfaces with constant curvature in the isotropic space*, B.S.G. Proceedings **23**(2016), 1–10.

[4] A. H. Barr, *Superquadrics and angle-preserving transformations*, IEEE Comput. Graph. Appl. **1**(1998), 11–23.

[5] B. Bulca, K. Arslan, B. (Kilic) Bayram, G. Ozturk, *Spherical product surfaces in* $\mathbb{E}^4$, An. St. Univ. Ovidius Constanta **20**(2012), 41–54.

[6] B. Bulca, K. Arslan, B. (Kilic) Bayram, G. Ozturk, H. Ugail, *On spherical product surfaces in* $\mathbb{E}^3$, IEEE Computer Society, 2009, Int. Conference on CYBERWORLDS.

[7] B. Bulca, K. Arslan, *Surfaces given with the Monge patch in* $\mathbb{E}^4$, J. Math. Phys. Anal. Geom. **9**(2013), 435–447.

[8] B.-Y. Chen, S. Decu, L. Verstraelen, *Notes on isotropic geometry of production models* , Kragujevac J. Math. **38**(2014), 23–33.

[9] S. Decu, L. Verstraelen, *A note on the isotropical geometry of production surfaces*, Kragujevac J. Math. **37**(2013), 217–220.

[10] Z. Erjavec, B. Divjak, D. Horvat, *The general solutions of Frenet's system in the equiform geometry of the Galilean, pseudo-Galilean, simple isotropic and double isotropic space*, Int. Math. Forum **6**(2011), 837–856.

[11] I. Kamenarović, *Associated curves on ruled surfaces in the isotropic space* $I_3^{(1)}$, Glas. Mat. Ser. III **29**(1994), 363–370.

[12] J. J. Koenderink, A. van Doorn, *Image processing done right*, Lecture Notes in Comput. Sci. **2350**(2002), 158–172.

[13] N. H. Kuiper, *Minimal total absolute curvature for immersions*, Invent. Math. **10**(1970), 209–238.

[14] A. Onishchick, R. Sulanke, *Projective and Cayley-Klein Geometries*, Springer, Berlin, 2006.

[15] H. B. Oztekin, S. Tatlipinar, *On some curves in Galilean plane and 3-dimensional Galilean space*, J. Dyn. Syst. Geom. Theor. **10**(2012), 189–196.

[16] B. Pavković, *An interpretation of the relative curvatures for surfaces in the isotropic space*, Glas. Mat. Ser. III **15**(1980), 149–152.

[17] H. Pottmann, K. Opitz, *Curvature analysis and visualization for functions defined on Euclidean spaces or surfaces*, Comput. Aided Geom. Design **11**(1994), 655–674.

[18] H. Pottmann, Y. Liu, *Discrete surfaces of isotropic geometry with applications in architecture*, Lecture Notes in Comput. Sci. **4647**(2007), 341–363.

[19] H. Pottmann, P. Grohs, N. J. Mitra, *Laguerre minimal surfaces, isotropic geometry and linear elasticity*, Adv. Comput. Math. **31**(2009), 391–419.

[20] H. Sachs, *Isotrope Geometrie des Raumes*, Vieweg Verlag, Braunschweig, 1990.

[21] H. Sachs, *Ebene Isotrope Geometrie*, Vieweg Verlag, Braunschweig, 1987.

[22] H. Sachs, *Zur Geometrie der Hypersphären in n-dimensionalen einfach isotropen Raum*, J. Reine Angew. Math. **298**(1978), 199–217.

[23] Ž. M. Šipuš, *Translation surfaces of constant curvatures in a simply isotropic space*, Period. Math. Hung. **68**(2014), 160–175.

[24] Ž. M. Šipuš, B. Divjak, *Curves in n-dimensional k-isotropic space*, Glas. Mat. Ser. III **33**(1998), 267–286.