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On crossed modules in modified categories of interest

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Abstract. We introduce some algebraic structures such as singularity, commutators and central extension in modified categories of interest. Additionally, we introduce the cat^1-objects and internal categories with their connection to crossed modules in these categories, which gives rise to unification of many notions about (pre)crossed modules in various algebras of categories.

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1. Introduction

Categories of interest were introduced in order to study properties of different algebraic categories and different algebras simultaneously. The idea comes from P. G. Higgins [25] and the definition is due to M. Barr and G. Orzech [39]. Many categories of algebraic structures are main examples of these categories (see [13, 17, 39, 33, 34, 35]). The categories of crossed modules and precrossed modules in the category of groups, respectively, are equivalent to categories of interest as well, in the sense of [11, 14]. Nevertheless, the cat^1-Lie (associative, Leibniz, etc.) algebras are not categories of interest. Consequently, in [5], Y. Boyacı et al. introduce and study a new type of category of interest; namely, a category which satisfies all axioms of a category of groups with operations stated in [40], except the one, which is replaced by a new axiom; this category also satisfies two additional axioms introduced in [39] for categories of interest. They called this category "Modified Category of Interest", which will be denoted by MCI from now on. The examples are mainly those categories, which are equivalent to the categories of crossed modules and precrossed modules in the categories of Lie algebras, Leibniz algebras, associative and associative commutative algebras. For more examples, see [3, 6, 7, 9, 12, 16, 18, 21, 22, 31, 36, 40].

Crossed modules were introduced by J.H.C Whitehead in [41] as a model of homotopy 3-types and used to classify higher dimensional cohomology groups in [42]. Since then, the whole property adapted to many algebras. The notions of crossed modules were defined on various algebras such as (associative) commutative algebras, Lie algebras, Leibniz algebras, Lie-Rinehart algebras in [3, 6, 7, 9, 12, 16, 18, 21, 22, 31, 36, 40]. The definition of crossed modules in modified categories of interest unifies all these definitions. As a different model of homotopy types, Loday defined cat^1-groups in [32]. The categories of cat^1-groups and crossed modules are naturally equivalent and this result was adapted to many algebras as well. The notions of cat^1-algebras were introduced in [23].

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In this paper our main purpose is to unify the notions of center, singularity, commutator and central extensions in various categories of (pre)crossed modules (see [1, 5, 9, 38]). For this, first we introduce the notions of center, singularity and central extensions in modified categories of interest. Inspired by the equivalence between the categories of (pre)cat^1-groups and (pre)crossed modules, we introduce the notion of (pre)cat^1-objects and their connection to crossed modules in modified categories of interest. Then applying those definitions to (pre)cat^1-objects, we get unification of many notions related to (pre)crossed modules in different types of categories. Additionally, we show that our definitions coincide with those given in [24, 28, 26].

The paper is organized as follows: In Section 2, we recall the notion of MCI and some related structures with basic properties. In Section 3, we introduce the notions of a (pre)cat^1-object and the internal category in an arbitrary modified category of interest $\mathbb{C}$ with their connection to crossed modules in $\mathbb{C}$. Then we introduce singularity, commutators and central extensions in MCI. In Section 4, as an application of Section 3 we get a (pre)crossed module version of the introduced notions. Finally, in the last section, we conclude by some generalizations to internal category objects and crossed complexes which were indicated by referee in her/his report.

2. Preliminaries

We will recall the notions of MCI and the main constructions from [5], which are modified versions of those given in [14, 21, 39].

Let $\mathbb{C}$ be a category of groups a set of operations Ω and with a set of identities $\mathbb{E}$, such that $\mathbb{E}$ includes group identities and the following conditions hold. If Ω_i is the set of i-ary operations in Ω, then:

(a) $\Omega = \Omega_0 \cup \Omega_1 \cup \Omega_2$;

(b) group operations (written additively : $0, -, +$) are elements of Ω_0, Ω_1 and Ω_2, respectively. Let $\Omega_2' = \Omega_2 \setminus \{+\}$, $\Omega_1' = \Omega_1 \setminus \{-\}$. Assume that if $* \in \Omega_2$, then Ω_2' contains $*^\circ$ defined by $x *^\circ y = y * x$ and assume $\Omega_0 = \{0\}$;

(c) for each $* \in \Omega_2'$, $\mathbb{E}$ includes the identity $x * (y + z) = x * y + x * z$;

(d) for each $\omega \in \Omega_1'$ and $* \in \Omega_2'$, $\mathbb{E}$ includes the identities $\omega(x + y) = \omega(x) + \omega(y)$ and either the identity $\omega(x * y) = \omega(x) * \omega(y)$ or the identity $\omega(x * y) = \omega(x) * y$.

Let C be an object of $\mathbb{C}$ and $x_1, x_2, x_3 \in C$:

Axiom 1. *$x_1 + (x_2 * x_3) = (x_2 * x_3) + x_1$, for each $* \in \Omega_2'$.*

Axiom 2. *For each ordered pair $(*, \overline{*}) \in \Omega_2' \times \Omega_2'$ there is a word W such that*

$$(x_1 * x_2)\overline{*}x_3 = W(x_1(x_2x_3), x_1(x_3x_2), (x_2x_3)x_1,$$
$$(x_3x_2)x_1, x_2(x_1x_3), x_2(x_3x_1), (x_1x_3)x_2, (x_3x_1)x_2),$$

where each juxtaposition represents an operation in Ω_2'.

Definition 1 (see [5])**.** *A category of groups with operations $\mathbb{C}$ satisfying conditions $(a) - (d)$, Axiom 1 and Axiom 2, is called a modified category of interest.*

The difference of this definition from the original one of the category of interest is the identity $\omega(x) * \omega(y) = \omega(x * y)$, which is $\omega(x) * y = \omega(x * y)$ in the definition of the category of interest.

Example 1. *The categories* $\mathbf{Cat^1}$**-Ass**, $\mathbf{Cat^1}$**-Lie**, $\mathbf{Cat^1}$**-Leibniz**, $\mathbf{PreCat^1}$**-Ass**, $\mathbf{PreCat^1}$**-Lie** *and* $\mathbf{PreCat^1}$**-Leibniz** *are modified categories of interest, which are not categories of interest. Also, the category of commutative Von Neumann regular rings is isomorphic to the category of commutative rings with a unary operation* $(\ )^*$ *satisfying two axioms defined in [4], which is a modified category of interest.*

Notation 1. *From now on,* $\mathbb{C}$ *will denote an arbitrary modified category of interest.*

Let $B \in \mathbb{C}$. A subobject of B is called an ideal if it is the kernel of some morphism. Then A is an ideal of B if and only if A is a normal subgroup of B and $a * b \in A$, for all $a \in A$, $b \in B$ and $* \in \Omega_2'$.

For $A, B \in \mathbb{C}$ we say that we have a set of actions of B on A whenever there is a map $f_* : A \times B \longrightarrow A$, for each $* \in \Omega_2$. A split extension of B by A induces an action of B on A corresponding to the operations in $\mathbb{C}$. For a given split extension

$$0 \longrightarrow A \xrightarrow{\ i\ } E \xrightarrow{\ p\ } B \longrightarrow 0\ ,$$

we have

$$b \cdot a = s(b) + a - s(b),$$
$$b * a = s(b) * a,$$

for all $b \in B$, $a \in A$ and $* \in {\Omega_2}'$. Actions defined by previous equations are called derived actions of B on A.

Given an action of B on A, a semi-direct product $A \rtimes B$ is a universal algebra, whose underlying set is $A \times B$ and the operations are defined by

$$\begin{aligned} \omega(a,b) &= (\omega(a), \omega(b)), \\ (a',b') + (a,b) &= (a' + b' \cdot a, b' + b), \\ (a',b') * (a,b) &= (a' * a + a' * b + b' * a, b' * b), \end{aligned}$$

for all $a, a' \in A$, $b, b' \in B$. See [5], for details.

Example 2. *A dialgebra (or diassociative algebra) over a field* $\mathbb{K}$ *introduced in [34] is a* $\mathbb{K}$*-vector space defined with two* $\mathbb{K}$*-linear maps:*

$$\dashv, \vdash : A \otimes A \to A,$$

such that

$$\begin{aligned} (x \dashv y) \dashv z &= x \dashv (y \vdash z), \\ (x \dashv y) \dashv z &= x \dashv (y \dashv z), \\ (x \vdash y) \dashv z &= x \vdash (y \dashv z), \\ (x \dashv y) \vdash z &= x \vdash (y \vdash z), \\ (x \vdash y) \vdash z &= x \vdash (y \vdash z), \end{aligned}$$

for all $x, y, z \in A$.

Let A and B be two dialgebras. A dialgebra action of B on A is defined with four bilinear maps:

$$\triangleright_{\vdash}, \triangleright_{\dashv} \colon B \times A \to A$$
$$\triangleleft_{\vdash}, \triangleleft_{\dashv} \colon A \times B \to A$$

satisfying the required 30 axioms. (For details about these axioms, see [7])

The semi-direct product $A \rtimes B$ is the dialgebra whose underlying set is $A \times B$ with usual scalar multiplication, component-wise addition and binary operations defined by

$$(a,b) \dashv (a',b') = (a \dashv a' + b \triangleright_{\dashv} a' + a \triangleleft_{\dashv} b', b \dashv b'),$$
$$(a,b) \vdash (a',b') = (a \vdash a' + b \triangleright_{\vdash} a' + a \triangleleft_{\vdash} b', b \vdash b'),$$

for $a, a' \in A$ and $b, b' \in B$.

Theorem 2 (see [5]). *An action of B on A is a derived action if and only if $A \rtimes B$ is an object of $\mathbb{C}$.*

Proposition 1 (see [5]). *A set of actions of B on A in $\mathbb{C}_G$ is a set of derived actions if and only if it satisfies the following conditions:*

1. $0 \cdot a = a$,
2. $b \cdot (a_1 + a_2) = b \cdot a_1 + b \cdot a_2$,
3. $(b_1 + b_2) \cdot a = b_1 \cdot (b_2 \cdot a)$,
4. $b * (a_1 + a_2) = b * a_1 + b * a_2$,
5. $(b_1 + b_2) * a = b_1 * a + b_2 * a$,
6. $(b_1 * b_2) \cdot (a_1 * a_2) = a_1 * a_2$,
7. $(b_1 * b_2) \cdot (a * b) = a * b$,
8. $a_1 * (b \cdot a_2) = a_1 * a_2$,
9. $b * (b_1 \cdot a) = b * a$,
10. $\omega(b \cdot a) = \omega(b) \cdot \omega(a)$,
11. $\omega(a * b) = \omega(a) * \omega(b)$,
12. $x * y + z * t = z * t + x * y$,

for each $\omega \in \Omega_1'$, $* \in {\Omega_2}'$, b, b_1, $b_2 \in B$, $a, a_1, a_2 \in A$ and for $x, y, z, t \in A \cup B$ whenever each side of 12 has sense.

Definition 2 (see [5]). *Let $A \in \mathbb{C}$. The center of A is*

$$Z(A) = \{z \in A \mid a + z = z + a,\ a + \omega(z) = \omega(z) + a,\ a * z = 0,\ a * \omega(z) = 0, \text{ for all } a \in A,\ \omega \in \Omega_1 \text{ and } * \in {\Omega_2}'\}.$$

On the other hand, if A is an ideal of B, then the centralizer of A in B is the ideal

$$Z(B,A) = \{b \in B \mid a + b = b + a, a + \omega(b) = \omega(b) + a, a * b = 0,\ a * \omega(b) = 0, \text{ for all } a \in A,\ \omega \in \Omega_1 \text{ and } * \in {\Omega_2}'\}.$$

A precrossed module in $\mathbb{C}$ is a triple (C_1, C_0, ∂), where $C_0, C_1 \in \mathbb{C}$, C_0 has a derived action on C_1 and $\partial : C_1 \longrightarrow C_0$ is a morphism in $\mathbb{C}$ satisfying

a) $\partial(c_0 \cdot c_1) = c_0 + \partial(c_1) - c_0$,

b) $\partial(c_0 * c_1) = c_0 * \partial(c_1)$,

for all $c_0 \in C_0$, $c_1 \in C_1$, and $* \in {\Omega_2}'$. In addition, if

c) $\partial(c_1) \cdot c_1' = c_1 + c_1' - c_1$,

d) $\partial(c_1) * c_1' = c_1 * c_1'$,

for all c_1, $c_1' \in C_1$, and $* \in {\Omega_2}'$, then the triple (C_1, C_0, ∂) is called a crossed module in $\mathbb{C}$.

Definition 3. *A morphism between two (pre)crossed modules $(C_1, C_0, \partial) \longrightarrow (C_1', C_0', \partial')$ is a pair (μ_1, μ_0) of morphisms $\mu_0 : C_0 \longrightarrow C_0'$, $\mu_1 : C_1 \longrightarrow C_1'$, such that*

a) $\mu_0 \partial = \partial' \mu_1$,

b) $\mu_1(c_0 \cdot c_1) = \mu_0(c_0) \cdot \mu_1(c_1)$,

c) $\mu_1(c_0 * c_1) = \mu_0(c_0) * \mu_1(c_1)$,

for all $c_0 \in C_0$, $c_1 \in C_1$ and $* \in {\Omega_2}'$.

Consequently, we have categories **PXMod**$(\mathbb{C})$ of precrossed modules and **XMod**$(\mathbb{C})$ of crossed modules.

Example 3. *A crossed module in the category of dialgebras is a homomorphism $\partial : D_1 \longrightarrow D_0$ with an action of D_0 on D_1 such that*

1) $\partial(d_0 \rhd_{\dashv} d_1) = d_0 \dashv \partial(d_1)$,

$\partial(d_0 \rhd_{\vdash} d_1) = d_0 \vdash \partial(d_1)$,

$\partial(d_1 \lhd_{\dashv} d_0) = \partial(d_1) \dashv d_0$,

$\partial(d_1 \lhd_{\vdash} d_0) = \partial(d_1) \vdash d_0$,

2) $\partial(d_1) \rhd_{\dashv} d_1' = d_1 \dashv d_1' = d_1 \lhd_{\dashv} \partial(d_1')$,

$\partial(d_1) \rhd_{\vdash} d_1' = d_1 \vdash d_1' = d_1 \lhd_{\vdash} \partial(d_1')$,

for all $d_1, d_1' \in D_1$, $d_0 \in D_0$. The definition is equivalent to the definition given in [7].

Example 4. *Let $\partial : D_1 \longrightarrow D_0$ and $\partial' : D_1' \longrightarrow D_0'$ be crossed modules of dialgebras. The pair (μ_1, μ_0) consists of dialgebra homomorphisms $\mu_1 : D_1 \longrightarrow D_1'$, $\mu_0 : D_0 \longrightarrow D_0'$ which satisfies $\partial' \mu_1 = \mu_0 \partial$ and*

$$\mu_1(d_0 \rhd_{\vdash} d_1) = \mu_0(d_0) \rhd_{\vdash} \mu_1(d_1),$$

$$\mu_1(d_1 \lhd_{\dashv} d_0) = \mu_1(d_1) \lhd_{\dashv} \mu_0(d_0),$$

$$\mu_1(d_0 \rhd_{\dashv} d_1) = \mu_0(d_0) \rhd_{\dashv} \mu_1(d_1),$$

$$\mu_1(d_1 \lhd_{\vdash} d_0) = \mu_1(d_1) \lhd_{\vdash} \mu_0(d_0),$$

for all $d_1 \in D_1$ and $d_0 \in D_0$ is called a morphism between $\partial : D_1 \longrightarrow D_0$ and $\partial' : D_1' \longrightarrow D_0'$.

Definition 4. *Let (C_1, C_0, μ) be a (pre)crossed module in $\mathbb{C}$. A (pre)crossed module (C_1', C_0', μ') is a (pre)crossed submodule of (C_1, C_0, μ) if C_1' and C_0' are subobjects of C_1, C_0, respectively, $\mu' = \mu|_{C_1'}$ and the action of C_0' on C_1' is induced by the action of C_0 on C_1. Additionally, if C_0' and C_1' are ideals of C_0 and C_1, respectively, $c_0 * c_1' \in C_1'$, $c_0' * c_1 \in C_1'$, $c_0 \cdot c_1' \in C_1'$, $c_0' \cdot c_1 - c_1 \in C_1'$, for all $c_1 \in C_1$, $c_0 \in C_0$, $c_1' \in C_1'$, $c_0' \in C_0'$, then (C_1', C_0', μ') is called a crossed ideal of (C_1, C_0, μ).*

Equivalently, (C_1', C_0', μ') is a crossed ideal of (C_1, C_0, μ) if and only if (C_1', C_0', μ') is the kernel of some morphism.

3. Some algebraic structures in MCI

In this section, first we introduce the notion of (pre)cat^1-objects in a modified category of interest $\mathbb{C}$ and construct the corresponding category (**Pre**)**Cat^1**($\mathbb{C}$) of (pre)cat^1-objects with natural equivalence with the category (**P**)**Xmod**($\mathbb{C}$) of (pre) crossed modules in $\mathbb{C}$. Then we introduce the notions of singularity, commutator and central extensions in $\mathbb{C}$. We also show that the notion of central extension introduced in Definition 9 coincides with the definition of centrality, in terms of [27].

3.1. (Pre)Cat^1- objects in MCI

Definition 5. *A precat^1-object in $\mathbb{C}$ is a triple (C, ω_0, ω_1), where $C \in \mathbb{C}$ and $\omega_0, \omega_1 : C \longrightarrow C$, are morphisms in $\mathbb{C}$ which satisfy*

1) $\omega_0\omega_1 = \omega_1$, $\omega_1\omega_0 = \omega_0$.

In addition, if

2) $x * y = 0$, $x + y - x - y = 0$,

for all $* \in {\Omega_2}'$ and $x \in ker\omega_0$, $y \in ker\omega_1$, then the triple (C, ω_0, ω_1) is called a cat^1-object in $\mathbb{C}$.

Consider the category, whose objects are cat^1-objects and morphisms are $\mathbb{C}$-morphisms compatible with the maps ω_0 and ω_1. We will denote this category by **Cat^1**($\mathbb{C}$).
We also have the category **PreCat^1**($\mathbb{C}$) of precat^1-objects, in the same manner.

Example 5. *Let $\mathbb{C}$ be the category of Leibniz algebras. Then a cat^1-Leibniz algebra is a triple (L, ω_0, ω_1), consisting of a Leibniz algebra L and Leibniz algebra homomorphisms $\omega_0, \omega_1 : L \longrightarrow L$ such that*

1) $\omega_0\omega_1 = \omega_1, \quad \omega_1\omega_0 = \omega_0$,

2) $[x, y] = 0 = [y, x]$,

for all $x \in ker\omega_0$, $y \in ker\omega_1$.

Example 6. *A cat^1-dialgebra is a triple (D, ω_0, ω_1) consisting of a dialgebra D and homomorphisms $\omega_0, \omega_1 : D \longrightarrow D$ such that*

1) $\omega_0\omega_1 = \omega_1, \quad \omega_1\omega_0 = \omega_0$, *2)* $x \dashv y = 0 = y \dashv x$, $x \vdash y = 0 = y \vdash x$,

for all $x \in ker\omega_0$, $y \in ker\omega_1$.

Proposition 2. *The categories* $\mathbf{XMod}(\mathbb{C})$ *and* $\mathbf{Cat^1}(\mathbb{C})$ *are canonically equivalent.*

Proof. Let (C_1, C_0, ∂) be a crossed module in $\mathbb{C}$. Consider the corresponding semi-direct product $C_1 \rtimes C_0$ induced from the action of C_0 on C_1. By Theorem 2, $C_1 \rtimes C_0 \in \mathbb{C}$. It is obvious that maps $\omega_0 : C_1 \rtimes C_0 \longrightarrow C_1 \rtimes C_0$, $\omega_1 : C_1 \rtimes C_0 \longrightarrow C_1 \rtimes C_0$ defined by $\omega_0(c_1, c_0) = (0, c_0)$, $\omega_1(c_1, c_0) = (0, \partial(c_1) + c_0)$, for all $(c_1, c_0) \in C_1 \times C_0$ are $\mathbb{C}$-morphisms. On the other hand, since

$$\omega_0\omega_1(c_1, c_0) = \omega_0(0, \partial(c_1) + c_0) = (0, \partial(c_1) + c_0) = \omega_1(c_1, c_0)$$

and

$$\omega_1\omega_0(c_1, c_0) = \omega_1(0, c_0) = (0, c_0) = \omega_0(c_1, c_0),$$

for all $(c_1, c_0) \in C_1 \times C_0$, we have $\omega_0\omega_1 = \omega_1$, $\omega_1\omega_0 = \omega_0$. Let $(c_1, c_0) \in ker\omega_0$ and $(\overline{c_1}, \overline{c_0}) \in ker\omega_1$. Then we have $c_0 = 0$ and $\partial(\overline{c_1}) + \overline{c_0} = 0$. Consequently,

$$\begin{aligned}
(c_1, c_0) + (\overline{c_1}, \overline{c_0}) &= (c_1 + c_0.\overline{c_1}, c_0 + \overline{c_0}) \\
&= (c_1 + \overline{c_1}, \overline{c_0}) \\
&= (\overline{c_1} - \overline{c_1} + c_1 + \overline{c_1}, \overline{c_0}) \\
&= (\overline{c_1} + (-\partial(\overline{c_1})) \cdot c_1, \overline{c_0}) \\
&= (\overline{c_1} + \overline{c_0} \cdot c_1, \overline{c_0} + c_0) \\
&= (\overline{c_1}, \overline{c_0}) + (c_1, c_0)
\end{aligned}$$

and

$$\begin{aligned}
(c_1, c_0) * (\overline{c_1}, \overline{c_0}) &= (c_1 * \overline{c_1} + c_1 * \overline{c_0} + c_0 * \overline{c_1}, c_0 * \overline{c_0}) \\
&= (c_1 * \overline{c_1} + c_1 * \overline{c_0} + 0 * \overline{c_1}, 0 * \overline{c_0}) \\
&= (c_1 * (\partial(\overline{c_1})) + c_1 * \overline{c_0}, 0) \\
&= (c_1 * (\partial(\overline{c_1}) + \overline{c_0}), 0) \\
&= (c_1 * 0, 0) \\
&= (0, 0),
\end{aligned}$$

as required. So we have the functor $\mathfrak{C} : \mathbf{XMod}(\mathbb{C}) \longrightarrow \mathbf{Cat^1}(\mathbb{C})$.

Conversely, given a cat^1-object (C, ω_0, ω_1) in $\mathbb{C}$. Consider the morphism $\partial : C_1 \longrightarrow C_0$, where $C_1 = ker\omega_0$, $C_0 = Im\omega_0$ and $\partial = \omega_1 |_{\ker \omega_0}$. Define the dot action of C_0 on C_1 by $c_0 \cdot c_1 = c_0 + c_1 - c_0$ and star actions by $c_0 * c_1$, for $c_0 \in C_0$, $c_1 \in C_1$, $* \in \Omega_2'$. We claim that (C_1, C_0, ∂) is a crossed module in $\mathbb{C}$ with these actions.

By a direct calculation we have $\omega_0(c_1) = 0$ and there exist $c \in C$ such that $\omega_0(c) = c_0$, for all $c_0 \in C_0$, $c_1 \in C_1$.

i) For all $c_0 \in C_0$, $c_1 \in C_1$, we have

$$\begin{aligned}\partial(c_0.c_1) &= \omega_1(c_0 + c_1 - c_0)\\ &= \omega_1(\omega_0(c) + c_1 - \omega_0(c))\\ &= \omega_1\omega_0(c) + \omega_1(c_1) - \omega_1\omega_0(c)\\ &= \omega_0(c) + \omega_1(c_1) - \omega_0(c)\\ &= c_0 + \partial(c_1) - c_0.\end{aligned}$$

ii) For all $c_0 \in C_0$, $c_1 \in C_1$, we have

$$\begin{aligned}\partial(c_0 * c_1) &= \omega_1(\omega_0(c) * c_1)\\ &= \omega_1\omega_0(c) * \omega_1(c_1)\\ &= \omega_0(c) * \omega_1(c_1)\\ &= c_0 * \partial(c_1).\end{aligned}$$

iii) Since $\omega_1\omega_1 = \omega_1\omega_0\omega_1 = \omega_0\omega_1 = \omega_1$, we have $\omega_1(c_1 - \partial(c_1)) = 0$, which means $(c_1 - \partial(c_1)) \in ker\omega_1$ and $(c_1 - \partial(c_1)) + c_1' - (c_1 - \partial(c_1)) - c_1' = 0$, for all $c_1' \in C_1$. Then

$$\begin{aligned}\partial(c_1).c_1' &= \partial(c_1) + c_1' - \partial(c_1)\\ &= c_1 - c_1 + \partial(c_1) + c_1' - \partial(c_1)\\ &= c_1 + c_1' - c_1,\end{aligned}$$

for all $c_1, c_1' \in C_1$, as required.

iv) By a calculation similar to (iii) we have $\partial(c_1 * c_1') = \partial(c_1) * c_1' = c_1 * c_1'$, for all $c_1, c_1' \in C_1$, $* \in \Omega_2'$. Consequently, we have the functor $\mathcal{X} : \mathbf{Cat^1}(\mathbb{C}) \longrightarrow \mathbf{XMod}(\mathbb{C})$. The functors $\mathfrak{C}$ and $\mathfrak{X}$ give rise to a natural equivalence between $\mathbf{XMod}(\mathbb{C})$ and $\mathbf{Cat^1}(\mathbb{C})$.

The correspondence and functoriality for the morphisms are straightforward. □

Similarly, we have the natural equivalence between $\mathbf{Precat^1}(\mathbb{C})$ and $\mathbf{PXMod}(\mathbb{C})$.

3.2. Internal category in MCI

As a more general setting, in this subsection we recall the definition of an internal category in a modified category of interest. Then we give the relation between the category of internal categories and that of cat^1-objects and crossed modules in $\mathbb{C}$

Let $\boldsymbol{C}$ be a category with finite limits. We recall the definition of an internal category [30] An internal category C in $\boldsymbol{C}$ consists of:

(a) a pair of objects $C_0,\ C_1$;

(b) four morphisms $C_1 \xrightarrow{d_0} C_0$, $C_1 \xrightarrow{d_1} C_0$, $C_0 \xrightarrow{i} C_1$, and $C_1 \times_{C_0} C_1 \xrightarrow{m} C_1$, such that $d_0 i = d_1 i = 1_{C_0}$, $d_0 m = d_0\pi_2$, $d_1 m = d_1\pi_1$, $m(1 \times m) = m(m \times 1) : C_1 \times_{C_0} C_1 \times_{C_0} C_1 \to C_1$, and $m(1 \times i) = m(i \times 1) = 1_{C_1}$. Here and below, $C_1 \times_{C_0} C_1$ denotes the pullback

$$\begin{array}{ccc} C_1 \times_{C_0} C_1 & \xrightarrow{\pi_2} & C_1 \\ \downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle d_1} \\ C_1 & \longrightarrow & C_1 \end{array}$$

Let $C = (C_0, C_1, d_0, d_1, i, m)$ and $C' = (C'_0, C'_1, d'_0, d'_1, i', m')$ be internal categories and $F = (F_0, F_1) : C \longrightarrow C'$ and the diagrams

$$\begin{array}{ccc} d_0, d_1 : C_1 & \overset{i}{\underset{\longrightarrow}{\overset{\longleftarrow}{\longrightarrow}}} & C_0 \\ \downarrow{\scriptstyle F_1} & & \downarrow{\scriptstyle F_0} \\ d_0, d_1 : C'_1 & \overset{i'}{\underset{\longrightarrow}{\overset{\longleftarrow}{\longrightarrow}}} & C'_0 \end{array} \qquad \begin{array}{ccc} C_1 \times_{C_0} C_1 & \xrightarrow{m} & C_1 \\ \downarrow{\scriptstyle (F_1, F_1)} & & \downarrow{\scriptstyle F_1} \\ C'_1 \times_{C'_0} C'_1 & \xrightarrow{m'} & C'_1 \end{array} \tag{1}$$

are commutative.

Denote by $CAT(\boldsymbol{C})$ the category of internal categories and functors in $\boldsymbol{C}$.

Remark 1. *Let $\mathbb{C}$ be a modified category of interest and $(C_0, C_1, d_0, d_1, i, m)$ an internal category in $\mathbb{C}$. We have the split exact extension*

$$kerd_0 \longrightarrow C_1 \underset{i}{\overset{d_0}{\rightleftarrows}} C_0 \ ,$$

from which we get the actions of C_0 on $kerd_0$ defined by

$$r \cdot c = i(r) + c - i(r),$$
$$r * c = (i(r)) * c,$$
$$c * r = c * (i(r)),$$

for all $r \in C_0$, $c \in kerd_0$. Consequently, we have the semi-direct product $kerd_0 \rtimes C_0$, which is also an object in $\mathbb{C}$. Additionally, $\partial = d_1|_{kerd_0} : kerd_0 \to C_0$ satisfies

(i) $\partial(r \cdot c) = r + \partial(c) - r$;

(ii) $\partial(c) \cdot c' = c + c' - c$;

(iii) $\partial(c) * r = c * r$;

(iv) $\partial(c * c') = \partial(c) * c'$

for all $r \in C_0$, $c, c' \in kerd_0$ and $* \in \Omega'_2$. Consequently, $(kerd_0, C_0, \partial)$ is a crossed module in $\mathbb{C}$.

Inverse formulas are left to the reader.

Corollary 1. *Let $CAT(\mathbb{C})$ be the category of internal categories in $\mathbb{C}$. Then categories $CAT(\mathbb{C})$,* **Xmod**$(\mathbb{C})$ *and* $\mathbf{Cat^1}(\mathbb{C})$ *are equivalent.*

Proof. Follows from Remark 1 and Proposition 2. □

3.3. Singularity, commutators and central extensions

In this section, we introduce the notions of singularity, commutators and central extensions in MCI.

3.3.1. Singularity and commutators

Definition 6. *An object C in $\mathbb{C}$, which coincides with its center, is called singular.*

Example 7. *Let A be a dialgebra. Then the center $Z(A)$ of A is the set*

$$\{z \in A \,|\, a \dashv z = 0 = z \dashv a,\ a \vdash z = 0 = z \vdash a,\ \textit{for all } a \in A\}.$$

Consequently, A is singular if $a \dashv a' = 0 = a \vdash a'$, for all $a, a' \in A$.

Example 8. *Consider a cat^1-group (G, ω_0, ω_1). Then (G, ω_0, ω_1) is singular if $g + g' = g' + g$, $g + \omega_i(g') = \omega_i(g') + g$, for all $g, g' \in G$, $i = 0, 1$.*

Definition 7. *Let $A \in \mathbb{C}$ and $S \subseteq A$. The smallest ideal containing S is called the ideal generated by S and denoted by $< S >$.*

Definition 8. *Let $A \in \mathbb{C}$ and B, C be ideals of A. Then the ideal generated by the set:*

$$\{b + c - b - c,\ b * c,\ b + \omega(c) - b - \omega(c),\ c + \omega(b) - c - \omega(b),\ b * \omega(c),\ c * \omega(b) \,|\ b \in B, c \in C\}$$

will be called the commutator object of B and C.

Let $A \in \mathbb{C}$. The ideal generated by the set:

$$\{x + y - x - y, x + \omega(y) - x - \omega(y), x * y, x * \omega(y) \,|\, x, y \in A, * \in \Omega_2'\}$$

is called the commutator of A and denoted by $[A, A]$. Also, $A/[A, A]$ will be called the singularization of A.

Example 9. *Let D be a dialgebra. The commutator of D is the ideal generated by the set $\{a \dashv b, b \vdash a \,|\, a, b \in D\}$. Additionally, the singularization of D is*

$$D/\left\langle a \dashv b, b \vdash a; a, b \in D \right\rangle.$$

Proposition 3. *An object $C \in \mathbb{C}$ is singular if and only if $[C, C] = 0$.*

Proof. Direct checking. □

Remark 2. *The definition of commutators in $\mathbb{C}$ coincides with Huq's commutator [26] and the relative commutator (see [24]) with the Birkhoff subcategory $\mathbf{Ab}(\mathbb{C})$ of singular objects in $\mathbb{C}$.*

Theorem 3. *For any object $A \in \mathbb{C}$, the commutator ideal $[A, A]$ is the unique smallest ideal I for which A/I is singular.*

Proof. Direct checking. □

Denote the full subcategory consists of all singular objects in $\mathbb{C}$ by $\mathbf{Ab}(\mathbb{C})$. We have the functor $\mathfrak{Sing} : \mathbb{C} \longrightarrow \mathbf{Ab}(\mathbb{C})$, which takes any object C to its singularization $C/[C, C]$. Additionally, we have the functor $\mathfrak{inc.} : \mathbf{Ab}(\mathbb{C}) \longrightarrow \mathbb{C}$, which is the inclusion of the Birkhoff variety $\mathbf{Ab}(\mathbb{C})$ in $\mathbb{C}$. Consequently we have the adjunction "$\mathfrak{Sing} \dashv \mathfrak{inc.}$" , which can be diagrammed by

$$\mathbb{C} \underset{\mathfrak{inc.}}{\overset{\mathfrak{Sing}}{\rightleftarrows}} \mathbf{Ab}(\mathbb{C}) \ .$$

3.3.2. Central extensions

Definition 9. *Let $C \in \mathbb{C}$ and $A \in \mathbf{Ab}(\mathbb{C})$. A central extension of C by A is an extension*

$$E: \ A \rightarrowtail B \twoheadrightarrow C$$

such that A is a subobject of $Z(B)$.

Janelidze and Kelly [27] introduced the central extension in an exact category relative to an "admissible" subcategory. From [29], any modified category of interest $\mathbb{C}$ is Barr exact Mal'tsev category and so any Birkhoff subcategory of $\mathbb{C}$ is admissible, which gives rise to consideration the categorical theory of central extensions in $\mathbb{C}$.

An extension $f : A \longrightarrow B$ is called trivial in terms of [27] if the diagram

$$\begin{array}{ccc} A & \longrightarrow & \mathfrak{Sing}(A) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \mathfrak{Sing}(f)} \\ B & \longrightarrow & \mathfrak{Sing}(B) \end{array}$$

is a pullback, where the horizontal morphisms are given by the unit of the adjunction. An extension is called central in terms of [27] if there exists an extension $\rho : E \longrightarrow B$ of B such that in the pullback

$$\begin{array}{ccc} E \times_B A & \xrightarrow{\pi_2} & A \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle f} \\ E & \xrightarrow[\rho]{} & B \end{array}$$

the morphism π_1 is a trivial extension.

Proposition 4. *Definition 9 coincides with the definition of centrality given in [27]. (Here, we consider the category $\mathbb{C}$ and the admissible subcategory $\mathbf{Ab}(\mathbb{C})$.*

Proof. Let

$$A \rightarrowtail B \twoheadrightarrow C$$

be an extension in $\mathbb{C}$ with $A \subset Z(B)$. Consider the pullback diagram

$$\begin{array}{ccc} B \times_C B & \xrightarrow{\pi_2} & B \\ {\scriptstyle \pi_1}\downarrow & & \downarrow \\ B & \longrightarrow & C \end{array}$$

By a direct calculation, the diagram

$$\begin{array}{ccc} B \times_C B & \longrightarrow & \mathfrak{Sing}(B \times_C B) \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle \mathfrak{Sing}(\pi_1)} \\ C & \longrightarrow & \mathfrak{Sing}(C) \end{array}$$

is a pullback, that is, there exists an isomorphism between $B \times_C B$ and the fiber product

$$C \times_{\mathfrak{Sing}(C)} \mathfrak{Sing}(B \times_C B)$$

defined by $(b, b') \longmapsto (b, \overline{(b, b')})$. So the morphism $\pi_1 : B \times_C B \longrightarrow C$ is a trivial extension from which we get the centrality in terms of [27].

Conversely, given an extension

$$A \rightarrowtail B \xrightarrow{\vartheta_B} C$$

in $\mathbb{C}$, which is central in terms of [27]. Then there exists an extension $E \xrightarrow{\vartheta_E} C$ such that in the pullback

$$\begin{array}{ccc} E \times_C B & \xrightarrow{\pi_2} & B \\ {\scriptstyle \pi_1}\downarrow & & \downarrow \\ E & \longrightarrow & C \end{array}$$

the morphism $\pi_1 : E \times_C B \longrightarrow C$ is a trivial extension; in other words, the diagram

$$\begin{array}{ccc} E \times_C B & \xrightarrow{\pi_2} & \mathfrak{Sing}(E \times_C B) \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle \mathfrak{Sing}(\pi_1)} \\ E & \longrightarrow & \mathfrak{Sing}(E) \end{array}$$

is a pullback. The kernel of π_1 is the injection $A \rightarrowtail E \times_C B$ and the kernel of $\mathfrak{Sing}(\pi_1)$ is the injection $\sigma : A \longrightarrow \mathfrak{Sing}(E \underset{C}{\times} B)$ defined by $\sigma(a) = \overline{(0, a)}$, where $\overline{(0, a)}$ denotes the related coset. We want to show that $A \subset Z(B)$. For this, we need to show $b + a = a + b$, $b + \omega(a) = \omega(a) + b$, $b * a = 0$, $b * \omega(a) = 0$ for all $a \in A, b \in B, \omega \in \Omega_1, * \in \Omega_2'$. For all $b \in B$ there exists $e \in E$ such that $\varphi_B(b) = \varphi_E(e)$. Since

$$\begin{aligned} \sigma(b + a - b - a) &= \overline{(0, b + a - b - a)} \\ &= \overline{(0, b)} + \overline{(e, a)} - \overline{(0, b)} - \overline{(e, a)} \\ &= \overline{(0, b) - (0, b)} + \overline{(e, a) - (e, a)} \\ &= \overline{(0, 0)}, \end{aligned}$$

we have $b + a - b - a = 0$. By similar calculations we get that $A \subseteq Z(B)$, as required. □

4. Applications to (pre)crossed modules in MCI

In this section, we introduce the notions of center, singularity and central extension of (pre)crossed modules in modified categories of interest. For this, we were inspired by the equivalence of the categories (**Pre**)$\mathbf{Cat^1}(\mathbb{C})$ of (pre)cat^1-objects and (**P**)**Xmod**($\mathbb{C}$) of (pre)crossed modules. In the case of precrossed modules of groups (Lie algebras), the notions give the definitions of centers, singularity and central extensions [1, 18, 19, 37, 38].

4.1. Center and singularity of precrossed modules in MCI

Let (C_1, C_0, ∂) a precrossed module and $(C_1 \rtimes C_0, \omega_0, \omega_1)$ be the corresponding precat^1-object. The center $Z(C_1 \rtimes C_0, \omega_0, \omega_1)$ of $(C_1 \rtimes C_0, \omega_0, \omega_1)$ is the ideal

$$\begin{aligned} Z(C_1 \rtimes C_0, \omega_0, \omega_1) = \{(z_1, z_0) \in C_1 \rtimes C_0 \mid\ & z_1 + z_0 \cdot c_1 = c_1 + c_0 \cdot z_1,\ z_1 + c_1 = c_1 + z_1, \\ & c_1 = z_0 \cdot c_1,\ c_1 = \partial(z_1) \cdot c_1,\ c_0 + \partial(z_1) = \partial(z_1) + c_0, \\ & (c_1 * z_0) + (c_0 * z_1) + (c_1 * z_0) = 0,\ (c_1 * z_1) = 0,\ (c_1 * z_0) = 0, \\ & (c_1 * \partial(z_1) = 0,\ \partial(c_0 * z_1) = 0,\ \text{for all } (c_1, c_0) \in C_1 \rtimes C_0, * \in {\Omega_2}'\} \end{aligned} .$$

The image $\mathcal{X}(Z(C_1 \rtimes C_0, \omega_0, \omega_1))$ is the precrossed ideal $(Z_1, Z_0, \partial\,|)$ of (C_1, C_0, ∂), where

$$\begin{aligned} Z_1 = \{\ & z_1 \in C_1 \mid z_1 + c_1 = c_1 + z_1, c_1 \cdot (\partial(z_1)) = c_1, \\ & c_0 + \partial(z_1) = \partial(z_1) + c_0,\ z_1 = c_0 \cdot z_1,\ c_1 * z_1 = 0, \\ & c_1 * (\partial(z_1)) = 0,\ c_0 * z_1 = 0,\ \text{for all } c_1 \in C_1, c_0 \in C_0, * \in {\Omega_2}'\}, \end{aligned}$$

and

$$\begin{aligned} Z_0 = \{& z_0 \in C_0 \mid z_0 \cdot c_1 = c_1,\ z_0 + c_0 = c_0 + z_0, \\ & c_1 * z_0 = 0,\ c_0 * z_0 = 0,\ \text{for all } c_0 \in C_0, c_1 \in C_1, * \in {\Omega_2}'\}. \end{aligned}$$

If (C_1, C_0, ∂) is a crossed module, then

$$\begin{aligned} Z_1 = \{\ & z_1 \in C_1 \mid z_1 + c_1 = c_1 + z_1, c_0 + \partial(z_1) = \partial(z_1) + c_0, c_0 \cdot z_1 = z_1, \\ & c_1 * z_1 = 0, c_0 * z_1 = 0,\ \text{for all } c_0 \in C_0, c_1 \in C_1, * \in {\Omega_2}'\}, \\ Z_0 = \{\ & z_0 \in C_0 | z_0 \cdot c_1 = c_1, z_0 + c_0 = c_0 + z_0, c_1 * z_0 = 0, \\ & c_0 * z_0 = 0,\ \text{for all } c_0 \in C_0, c_1 \in C_1, * \in {\Omega_2}'\}. \end{aligned}$$

Definition 10. (Z_1, Z_0, ∂) *will be called the center of* (C_1, C_0, ∂).

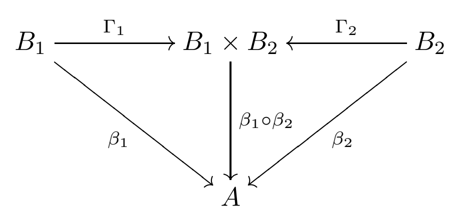
We will denote the center of (C_1, C_0, ∂) by $Z(C_1, C_0, \partial)$.

The notions of commuting morphisms and central objects were defined by Huq [26] in the categories with zero objects, products and coproducts, whose morphisms have images. From these properties following the existence of injections $\Gamma_i : B_i \longrightarrow B_1 \times B_2, i = 1, 2$ in the direct product in such a category, we have the following.

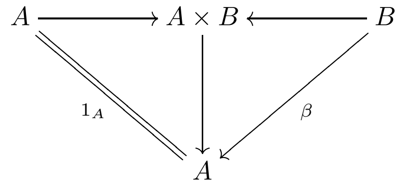
Definition 11 (see [26])**.** *Two coterminal morphisms* $\beta_1 : B_1 \longrightarrow A$ *and* $\beta_2 : B_2 \longrightarrow A$ *are said to commute if there exists a morphism*

$$\beta_1 \circ \beta_2 : B_1 \times B_2 \longrightarrow A$$

making the diagram



commutative, where Γ_i*,* $i = 1, 2$ *denotes the injection of the direct product. In particular, a morphism* $\beta : B \longrightarrow A$ *is said to be central if the identity morphism on* A *commutes with* β*, i.e., if it makes the diagram*

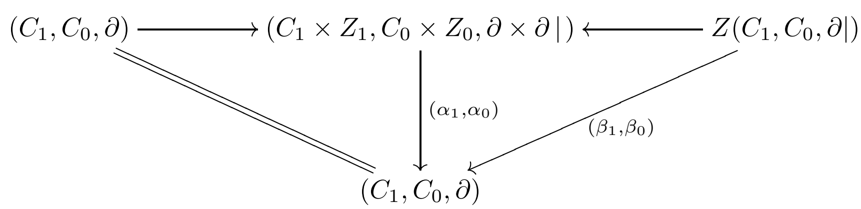


commutative. Additionally, if we have a monomorphism $\beta : B \longrightarrow A$*, then it is said that* B *is a central subobject of* A*.*

Definition 12 (see [26])**.** *The center of an object is the maximal central subobject relative to the order relation that exists on the set of monomorphisms.*

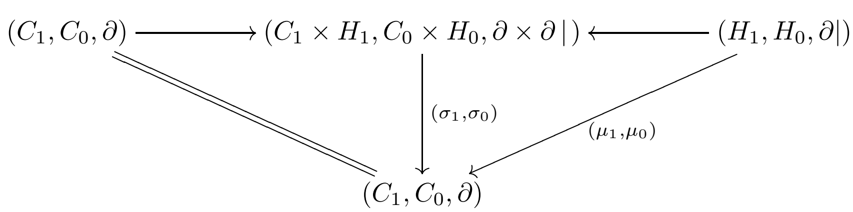
Proposition 5. *Let* (C_1, C_0, ∂) *be a crossed module. Then* $Z(C_1, C_0, \partial|)$ *is the maximal central subobject of* (C_1, C_0, ∂)*.*

Proof. Consider the diagram



Define $\alpha_1 : C_1 \times Z_1 \longrightarrow C_1$, $\alpha_0 : C_0 \times Z_0 \longrightarrow C_0$ by $\alpha_1(c_1, z_1) = c_1 + z_1$, $\alpha_0(c_0, z_0) = c_0 + z_0$, respectively, (β_1, β_0) as an inclusion and the others in a usual way. Then the diagram is commutative from which we get that $Z(C_1, C_0, \partial|)$ is a central subobject.

For any central object $(H_1, H_0, \partial|)$ of (C_1, C_0, ∂). Then there exist a monomorphism $(\mu_1, \mu_0) : (H_1, H_0, \partial|) \longrightarrow (C_1, C_0, \partial)$ and a homomorphism $(\sigma_1, \sigma_0) : (C_1 \times H_1, C_0 \times H_0, \partial \times \partial\,|) \longrightarrow (C_1, C_0, \partial)$, which makes a diagram



commutative. By direct checking we have $(\mu_1, \mu_0)(H_1, H_0, \partial|) \subseteq Z(C_1, C_0, \partial|)$, which means that $Z(C_1, C_0, \partial|)$ is the maximal central subobject of (C_1, C_0, ∂), as required. □

Corollary 2. *Definition 10 is equivalent to the definition in terms of [26].*

Proof. Follows from Definitions 12 and Proposition 5. □

Definition 13. *A singular (pre)crossed module in* $\mathbb{C}$ *is the crossed module coinciding with its center.*

4.2. The commutator of a (pre)crossed module in MCI

In this subsection, we introduce the notion of commutator of a precrossed module in $\mathbb{C}$ modules which recovers Huq's commutator [26] and relative commutator [24] as well.
Let (C_1, C_0, ∂) be a precrossed module. The commutator of the corresponding precat^1-object $(C_1 \rtimes C_0, \omega_0, \omega_1)$ is the ideal $[(C_1 \rtimes C_0, \omega_0, \omega_1), (C_1 \rtimes C_0, \omega_0, \omega_1)]$ generated by the set

$$\{(x_1, x_0) + (y_1, y_0) - (x_1, x_0) - (y_1, y_0),\ (x_1, x_0) + (0, y_0) - (x_1, x_0) - (0, y_0),$$
$$(x_1, x_0) + (0, \partial(y_1) + y_0) - (x_1, x_0) - (0, \partial(y_1) + y_0),\ (x_1, x_0) * (y_1, y_0),$$
$$(x_1, x_0) * (0, y_0), (x_1, x_0) * (0, \partial(y_1) + y_0) \mid\ (x_1, x_0), (y_1, y_0) \in C_1 \rtimes C_0 \text{ and } * \in \Omega_2{}'\}.$$

The image $\mathfrak{X}([(C_1 \rtimes C_0, \omega_0, \omega_1), (C_1 \rtimes C_0, \omega_0, \omega_1)])$ is the object $(K_1, K_0, \partial|)$, where K_1 and K_0 are the ideals generated by the sets

$$\{x_0 \cdot x_1 - x_1,\ x_1 + y_1 - x_1 - y_1,\ x_1 * y_1, x_0 * x_1 \mid x_0 \in C_0,\ x_1, y_1 \in C_1\}$$

and

$$\{x_0 + y_0 - x_0 - y_0,\ x_0 * y_0 \mid x_0, y_0 \in C_0\},$$

respectively.

Definition 14. *Let (C_1, C_0, ∂) be a precrossed module. Then $(K_1, K_0, \partial|)$ is called the commutator subcrossed module of (C_1, C_0, ∂).*

If (C_1, C_0, ∂) is a crossed module, then K_1 is the set generated by the set

$$\{x_0 \cdot x_1 - x_1, x_0 * x_1 \mid x_0 \in C_0,\ x_1 \in C_1\}.$$

4.3. Central extensions of (pre)crossed modules in MCI

Now, we introduce the central extensions of (pre)crossed modules in $\mathbb{C}$. Similarly to Proposition 4, the definition coincides with the notion of centrality in the terms of [27].

Definition 15. *Let (C_1, C_0, ∂_C) be a (pre)crossed module and (A_1, A_0, ∂_A) a singular object in* **(P)Xmod**$(\mathbb{C})$*. A central extension of (C_1, C_0, ∂_C) by (A_1, A_0, ∂_A) is an extension*

$$(A_1, A_0, \partial_A) \rightarrowtail (B_1, B_0, \partial_B) \twoheadrightarrow (C_1, C_0, \partial_C)$$

such that (A_1, A_0, ∂_A) is a crossed ideal of $Z(B_1, B_0, \partial_B)$.

As a consequence, one can construct the classification of central extensions of (pre)crossed modules. See [1, 6, 8, 9, 20, 37, 38] for various cases.

5. Conclusion

Internal category objects are nowadays (for example, in the context of application of homotopical methods) much more widely known objects and intuitively easier received by practical mathematicians. By using the correspondence between cat^1-objects and internal category objects, given in subsection 3.2, one can obtain the notions of center, singularity commutator and central extensions of internal category objects. On the other hand, the crossed modules are just the lowest case of crossed complexes; it has been considered to extend the main constructions in this paper to crossed complexes, as a generalization. For this, one needs an equivalence of the category of crossed complexes with a (modified) category of interest.

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