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The Column-Convex Polyominoes Perimeter Generating Function for Everybody

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The function mentioned in the title will be leisurely derived in two different ways. The apparatus used in the proofs consists of easy bijections and some generating functionology.

DEFINITIONS, CONVENTIONS AND NOTATIONS

All the geometric objects studied in this paper are assumed to lie in the x-y plane. When counting such objects, we make no difference between the two of them that can be transformed one into another by a translation.

A *cell* is a closed unit square whose vertices have integer coordinates.

Let P be a finite union of cells. For $i \in \mathbb{Z}$, let P_i be the union of those cells of P whose left side has abscissa i. If nonempty, the set P_i is said to be a *column* of P. P is a *column-convex* set if each column of P is an unbroken line of cells. P is a *polyomino* (or *animal*) if the interior of P is connected. Needless to say, P is a *column-convex polyomino* if it is both a column-convex set and a polyomino. See Figures 1 and 2.

Notation 1. If a polyomino P has k_1 horizontal edges and k_2 vertical edges, we write $\mathrm{He}(P) = k_1$ and $\mathrm{Ve}(P) = k_2$.

Definition 1. Let Ω be some family of polyominoes. By the *perimeter generating function* (perimeter *gf*, *pgf*) of Ω we mean the formal sum

$$pgf(\Omega) = \sum_{P \in \Omega} x^{\mathrm{He}(P)} y^{\mathrm{Ve}(P)} .$$

The *perimeter+ area gf* of Ω is the *gf* that has the same variables as the perimeter *gf*, plus a third variable q = area.

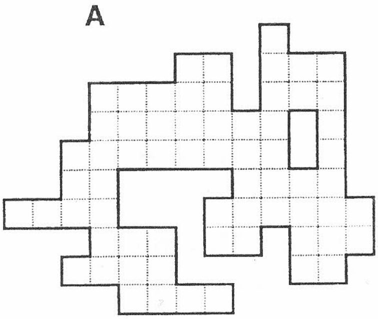


Figure 1. A polyomino.

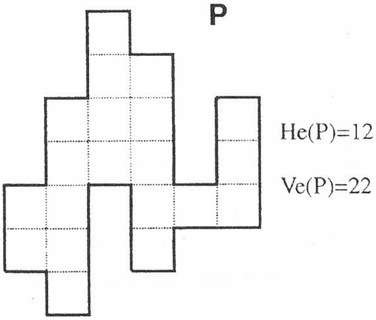


Figure 2. A column-convex polyomino.

The set of all column-convex (*c-c*) polyominoes will be denoted $\mathscr{P}$, and the perimeter *gf* of $\mathscr{P}$ will be denoted G. Further, we put

$$H = \frac{G}{1 - y^2} \,. \tag{1}$$

INTRODUCTION

Polyomino enumeration and/or generation problems arise in several chemical and physical contexts. Some examples are the study of molecular aggregates on catalyst surfaces, and of the thermodynamic properties of a polymer in dilute solution (Müller *et al.*[1]); modelling of a fluid which percolates through a medium having random properties (Stauffer and Aharony,[2] Bousquet-Mélou[3]); the solution of Baxter's lattice-gas model (Dhar[4]).

In its full force, the enumeration of polyominoes is a very difficult open problem. Nevertheless, a few asymptotic bounds are known. As an example, let us quote a result of Klarner and Rivest:[5]

Let a_n denote the number of polyominoes whose area is n. The sequence $<a_1, a_2^{1/2}, a_3^{1/3}, \ldots>$ converges, and its limit M satisfies $3.87 < M < 4.65$.

On the other hand, various special kinds of polyominoes have been counted, with respect to perimeter, area and/or other properties. For a survey of these exact enumerations, see Viennot.[6] For some of the current progress in this field, see Bousquet-Mélou,[7,8] as well as Svrtan and Feretić.[9]

The *c-c* polyominoes are undoubtedly quite a remarkable special class of polyominoes. Their importance is due to two factors. First, the *c-c* polyominoes are relatively numerous, so that their perimeter + area *gf* exactly reproduces many of the early terms of the perimeter + area *gf* for all polyominoes. Second, the area *gf* (Pólya,[10] Temperley[11]), perimeter *gf* and perimeter

+ area *gf* (Bousquet-Mélou[7]) of the *c-c* polyominoes are all given by reasonably simple expressions. (The area *gf* just mentioned is a function of two variables: x = horizontal edges and q = area.)

The first person who tackled the *c-c* polyominoes perimeter enumeration was H. N. V. Temperley[11] in 1956. Let g_r be the perimeter *gf* for the *c-c* polyominoes whose left-hand column contains exactly r cells. Taking account of the various possible overlaps between adjacent columns, Temperley found the following set of equations:

$$g_r = x^2 \left\{ y^{2r} + \sum_{s=1}^{r-1} \left[2 \cdot \frac{y^{2(r-s)} - y^{2r}}{1-y^2} + (r-s-1) \cdot y^{2(r-s)} \right] \cdot g_s + \right.$$

$$\left. + \sum_{s=r}^{\infty} \left[2 \cdot \frac{y^2 - y^{2r}}{1-y^2} + s - r + 1 \right] \cdot g_s \right\} \qquad (r \in \mathbb{N}) \,. \tag{2}$$

By doing a few things of the type »subtract the equation for $y^2 g_r$ from that for g_{r+1}«, hence he got the difference equation

$$y^4 g_{r-2} - 2(1+y^2)y^2 g_{r-1} + (1+4y^2+y^4)g_r - 2(1+y^2)g_{r+1} + g_{r+2} = x^2(1-y^2)^2 g_r \,. \tag{3}$$

In Eqs. (2)–(3) the initial values g_1, g_2, ... are not known, so at first glance it is not clear how to get the solution. However, Temperley found a method to do the job and solved a similar, but simpler system for the *c-c* polyominoes area *gf*. Applying the same method to Eq. (2) involves an extensive calculation which Temperley, who certainly had no computer algebra at that time, left undone. Only many years later (in 1990) Brak *et al.*[12] employed *Mathematica* to solve Eq. (2) in the case $x = y$. The general case $x \neq y$ was solved shortly thereafter by Lin,[13] who used *Reduce*.

In the meantime (1984), M. P. Delest[14] derived the function $G(x,x)$ by a different approach, namely by means of the algebraic language (or DSV, *i.e.* Dyck-Schützenberger-Viennot) methodology.[15] Delest's proof also resorts to computer algebra.

The function $G(x,y)$ was definitively unmasked in 1993, when Svrtan and Feretić[16] found the formula given in Theorem 1 below. (The previously known formulas for G were more complicated.) Svrtan and Feretić derived the function $G(x,y)$ both by the algebraic language and by the Temperley approach. It is interesting to mention that in these two derivations all the algebra was done by hand. In the algebraic language approach, which involves establishing and then solving an algebraic equation for the function to be found, much work was saved by noticing that the function $L = (1 - 3H) / (1 - H)$ satisfies the simple biquadratic equation

$$(1 - L^2)(L^2 - x^2) = \frac{4x^2y^2}{(1 - y^2)^2} . \qquad (4)$$

The Temperley equations were tamed by prof. Svrtan, who devised an efficient new method for their solution. Incidentally, this new method can handle the q-enumeration (the perimeter + area enumeration) as well. Some aspects of Svrtan's method were streamlined and further developed by Bousquet-Mélou.[7]

Very recently Feretić[17] has, in a sense, reduced the algebraic-language derivation of $G(x,y)$ by a factor of four.

The present paper explores the possibilities of deriving the function G in still more popular ways. So first we shall see what Feretić's[17] proof looks like when the algebraic languages are replaced by the so-called wall polyominoes. Then we move on to another derivation of G, which is new, and is, in a sense, complementary to the derivation presented before it.

OTHER DEFINITIONS, CONVENTIONS AND NOTATIONS

Let P be an arbitrary column-convex (*c-c*) polyomino. The top left corner of the first column of P is called the *north-west pole* of P and is denoted by NW(P). The bottom right corner of the last column of P is the *south-east pole* of P (notation: SE(P)).

Imagine a plane figure T obtained by appending a vertical segment of $d \in \mathbb{N}_0$ lattice units to the south-east pole of a *c-c* polyomino P. We say that T is a *tailed polyomino* (a *tapo*, for short). Naturally, the appended segment is termed the *tail* of T. By the columns of a tapo T we mean the columns of the underlying *c-c* polyomino P. The north-west pole of T is defined by NW(T) = NW(P), while the south-east pole SE(T) is defined as the lower endpoint of the tail of T. See Figure 3.

Now suppose that, for some $n \in \mathbb{N}$, $n - 1$ arbitrary tapoes $T_1, ..., T_{n-1}$ and a tapo with a null tail T_n are given. Let $T_1, ..., T_n$ be disposed in a way that, for $2 \le i \le n$, the north-west pole of T_i coincides with the south-east pole of T_{i-1}.

In a situation like this we say that the union $S = \bigcup_{1 \le i \le n} T_i$ is a *stapo* (*sequence of tailed polyominoes*). The tapoes $T_1, ..., T_n$ are termed the *parts* of S. By the columns of a stapo we mean the columns of its parts. See Figure 4. Observe that the one-part stapoes are *c-c* polyominoes.

It is useful to adopt the following convention:

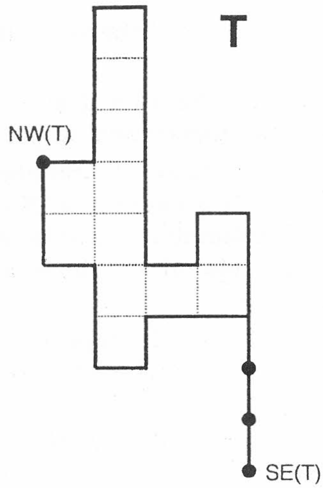
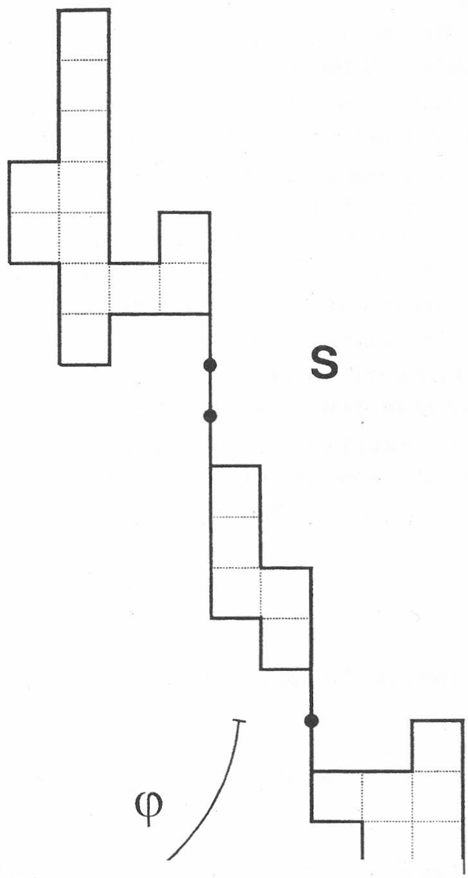


Figure 3. A tapo.
He(T) = 8, Ve(T) = 22

Figure 4. A stapo.
He(S) = 18, Ve(S) = 42



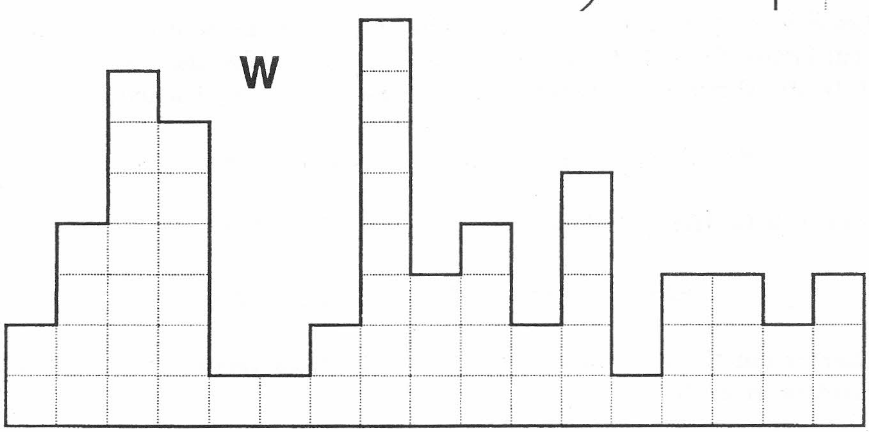


Figure 5. A wall polyomino. Co(W) = 17, Ve(W) = 42

Convention 1. Let a tapo T be obtained by appending a tail of length d to a c-c polyomino whose vertical perimeter is $2v$ (*i.e.* which has $2v$ vertical edges). Then T has $2v + 2d$ vertical edges.

By the vertical perimeter of a stapo we mean the sum of the vertical perimeters of its parts.

With this convention, in the sequel we shall apply Notation 1 and Definition 1 not only to the polyominoes, but also to the tapoes and stapoes.

Notation 2. Let P be a c-c polyomino, a tapo or a stapo. The number of columns of P will be denoted $\mathrm{Co}(P)$. (Of course, $\mathrm{Co}(P)$ is one half of $\mathrm{He}(P)$.) The minimal and the maximal ordinates of the i^{th} column of P will be written $y_i(P)$ and $Y_i(P)$, respectively. The height of that column, *i.e.* the number of cells contained in it, will be denoted $h_i(P)$.

We denote the set of all tapoes by $\mathscr{T}$, and the set of all stapoes by $\mathscr{S}$. Next, we put $I = pgf(\mathscr{S})$. As to the function $pgf(\mathscr{T})$, it is equal to H of Eq. (1) (this is proved in section »Some easy remarks«).

A column-convex polyomino whose bottoms of columns all have the same ordinate is called a *wall polyomino*. See Figure 5.

Definition 2. For Ω a family of wall polyominoes, we put

$$wgf(\Omega) = \sum_{W \in \Omega} x^{\mathrm{Co}(W)}\, y^{\mathrm{Ve}(W)} \ .$$

We denote the set of all wall polyominoes by $\mathscr{W}$ and we put $K = wgf(\mathscr{W})$.

A ONE-TO-ONE MAPPING OF THE STAPOES INTO WALL POLYOMINOES

Let S be a stapo with c columns. The stapo S is incident on $c + 1$ of the vertical lattice lines. Let these lines be $l_o, \ldots, l_c$, and let the contribution of $S \cap l_i$ to the vertical perimeter of S be denoted by $v_i(S)$. Clearly,

$$v_o(S) = Y_1(S) - y_1(S) \qquad \text{and} \qquad v_c(S) = Y_c(S) - y_c(S) \ .$$

Further, a little thought shows that for $i \in \underline{c-1}$ we have*

$$v_i(S) = \left| Y_{i+1}(S) - Y_i(S) \right| + \left| y_{i+1}(S) - y_i(S) \right|$$

whether or not $S \cap l_i$ contains a tail of S. The total vertical perimeter of S is therefore given by

* The symbol $\underline{k}$ denotes the set of integers $\{1, 2, \ldots, k\}$.

$$\mathrm{Ve}(S) = \Big(Y_1(S) - y_1(S)\Big) + \Big(Y_c(S) - y_c(S)\Big) + \\ + \sum_{i=1}^{c-1} \Big(\big|Y_{i+1}(S) - Y_i(S)\big| + \big|y_{i+1}(S) - y_i(S)\big|\Big) . \tag{5}$$

Owing to the geometry of our stapo S, the numbers $Y_i(S) - y_i(S)$ ($i \in \underline{c}$) and $Y_i(S) - y_{i+1}(S)$ ($i \in \underline{c-1}$) are all positive. Hence there exists a wall polyomino $\varphi(S)$, with $2c - 1$ columns, such that the 1^{st}, 2^{nd}, 3^{rd}, 4^{th}, ..., $2c - 1^{\mathrm{th}}$ column of $\varphi(S)$ contain

$$Y_1(S) - y_1(S), \quad Y_1(S) - y_2(S), \quad Y_2(S) - y_2(S), \quad Y_2(S) - y_3(S), \quad \ldots, \quad Y_c(S) - y_c(S)$$

cells, respectively. We readily find

$$v_o(\varphi(S)) = Y_1(S) - y_1(S), \quad v_{2c-1}(\varphi(S)) = Y_c(S) - y_c(S),$$

$$v_{2i-1}(\varphi(S)) = \big|y_{i+1}(S) - y_i(S)\big| \qquad (i \in \underline{c-1}) \qquad \text{and}$$

$$v_{2i}(\varphi(S)) = \big|Y_{i+1}(S) - Y_i(S)\big| \qquad (i \in \underline{c-1}) \; .$$

The numbers $v_i(\varphi(S))$ ($i = 0, \ldots, 2c - 1$) add up to the expression on the right side of Eq. (5). This means that S and $\varphi(S)$ have the same number of vertical edges.

For $c, v \in \mathbb{N}$, let

$$\mathscr{S}_{cv} = \big\{S \in \mathscr{S}\colon \; \mathrm{He}(S) = 2c, \quad \mathrm{Ve}(S) = 2v\big\}$$

and

$$\mathscr{W}_{cv} = \big\{W \in \mathscr{W}\colon \; \mathrm{Co}(W) = 2c - 1, \quad \mathrm{Ve}(W) = 2v\big\} \; .$$

So far we have shown that φ maps $\mathscr{S}_{cv}$ into $\mathscr{W}_{cv}$. But it is quite easy to prove a stronger result:

Proposition 1. φ is a bijection between $\mathscr{S}_{cv}$ and $\mathscr{W}_{cv}$.

As an example, the reader may verify that φ takes the stapo of Figure 4 into the wall polyomino of Figure 5.

Let $\mathscr{P}_{cv}$ be the set formed by those elements of $\mathscr{S}_{cv}$ which are c-c polyominoes. A little thought shows that φ maps $\mathscr{P}_{cv}$ onto the set, say $\mathscr{W}^{+}_{cv}$, of those W $\in \mathscr{W}_{cv}$ which satisfy the additional requirement

$$h_{2i}(W) < h_{2i-1}(W) + h_{2i+1}(W) \quad (i \in \underline{c-1}) \; . \tag{6}$$

We have to count the sets $\mathscr{W}^{+}_{cv}$ (that is, $\mathscr{P}_{cv}$), but the requirement Eq.(6) does not look very handy. However, we need not worry, because the stapoes will do the hard work for us.

SOME EASY REMARKS

Notation 3. Let f be a series in powers of z. By the symbol $<z^n> f$ we mean the coefficient of z^n in the series f.

For $d \in \mathbb{N}_o$, let $\mathscr{T}_d$ be the set of tapoes whose tail is exactly d units long. Cutting the tails is a 1-1 mapping of $\{T \in \mathscr{T}_d\text{: He}(T) = 2c,\ \text{Ve}(T) = 2v\}$ onto $\{P \in \mathscr{P}\text{: He}(P) = 2c,\ \text{Ve}(P) = 2v - 2d\}$. Hence,

$$<x^{2c}y^{2v}> pgf(\mathscr{T}_d) = <x^{2c}y^{2v-2d}> G = <x^{2c}y^{2v}> y^{2d}G \; ,$$

for all $c, v \in \mathbb{N}$. In other words, $pgf(\mathscr{T}_d) = y^{2d}\ G$. Recalling Eq.(1), we find

$$pgf(\mathscr{T}) = \sum_{d \geq o} pgf(\mathscr{T}_d) = \sum_{d \geq o} y^{2d}G = \frac{G}{1-y^2} = H \; . \tag{7}$$

Let $\mathscr{S}_n$ be the set of n-part stapoes. An n-part stapo is, in substance, a sequence of $n - 1$ tapoes and one c-c polyomino. Hence, pgf$(\mathscr{S}_n) = H^{n-1}G$, and we find

$$I = pgf\,(\bigcup_{n \geq 1} \mathscr{S}_n) = \sum_{n \geq 1} H^{n-1}G = \frac{G}{1-H} \; . \tag{8}$$

Now it is convenient to put

$$J = \frac{I}{1-y^2} \; . \tag{9}$$

The function J can be interpreted as the perimeter gf for the generalized stapoes whose last part, too, is allowed to have a tail. From Eqs. (1), (8), and (9) it follows that

$$J = \frac{H}{1-H} \; , \tag{10}$$

so that
$$H = 1 - \frac{1}{1+J} \,. \tag{11}$$

Let
$$\bar{I} = \frac{x}{2} \cdot \left[K(x,y) - K(-x,y) \right] . \tag{12}$$

By Proposition 1, for c, $v \in \mathbb{N}$ we have $|\mathscr{W}_{cv}| = |\mathscr{S}_{cv}|$, so that

$$<x^{2c}y^{2v}> \bar{I} = <x^{2c-1}y^{2v}> K = |\mathscr{W}_{cv}| = |\mathscr{S}_{cv}| = <x^{2c}y^{2v}> I \,. \tag{13}$$

Both $\bar{I}$ and I have only that kind of terms where both x and y are raised to even powers. Eq.(13) therefore implies

$$\bar{I} = I \,. \tag{14}$$

Now, instead of attacking the *pgf* G directly, we shall rather take the easier route of first deriving the function K. With K in our hands, we shall successively obtain $\bar{I}$, I, J, H and G from Eqs. (12), (14), (9), (11) and (1).

DERIVATION OF K

In order to derive an algebraic equation for K, we shall partition the set $\mathscr{W}$ of all wall polyominoes into some classes and study how the *wgf's* of these classes relate to $K = wgf(\mathscr{W})$.

i) Let $\mathscr{W}_\alpha = \{W \in \mathscr{W}$: W has no one-cell columns$\}$.

There is a bijection between $\mathscr{W}$ and $\mathscr{W}_\alpha$: with $U \in \mathscr{W}$ we associate the polyomino $W \in \mathscr{W}_\alpha$, produced by putting one additional cell on the top of each column of U. (See Figure 6.a). We have Ve(W) = Ve(U) + 2, which implies

$$wgf(\mathscr{W}_\alpha) = y^2 K \,. \tag{15}$$

Each $W \in \mathscr{W} \setminus \mathscr{W}_\alpha$ possesses at least one one-cell column. The leftmost one-cell column of such a W will be denoted $\mathscr{C}_1(W)$.

ii) Let $\mathscr{W}_\beta = \{W \in \mathscr{W} \setminus \mathscr{W}_\alpha$: $\mathscr{C}_1(W)$ is both the leftmost and the rightmost column of $W\}$.

The set $\mathscr{W}_\beta$ contains just one element: the one-cell polyomino. Hence

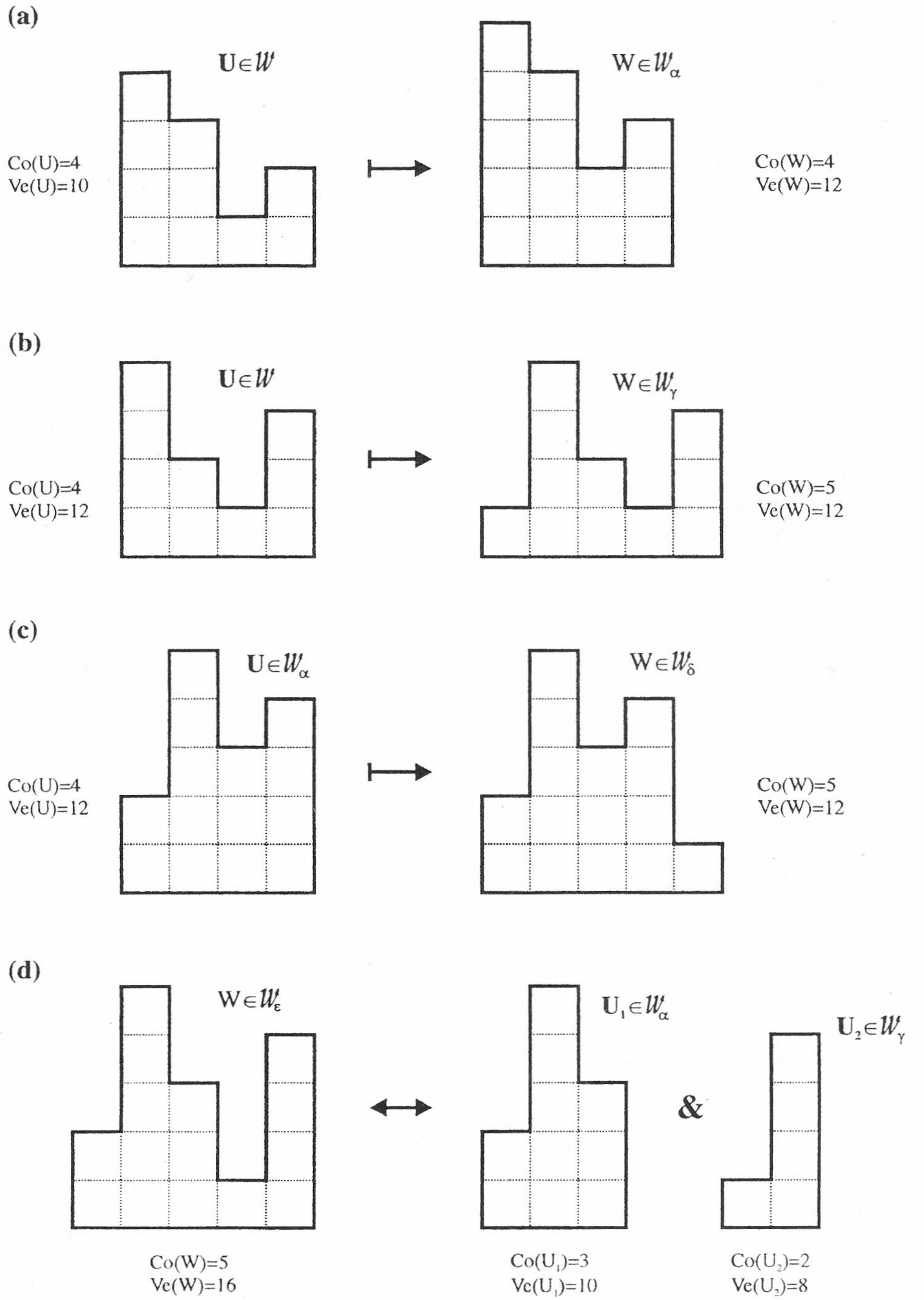


Figure 6. The bijections used in establishing Eq. (20).

iii) Let $\mathscr{W}_\gamma = \{W \in \mathscr{W} \setminus \mathscr{W}_\alpha$: $\ell_1(W)$ is the leftmost, but is not the rightmost column of $W\}$.

Suppose we take a polyomino $U \in \mathscr{W}$ and paste an additional one-cell column on its left side. We then get a polyomino $W \in \mathscr{W}_\gamma$ such that Co(W) = Co(U) + 1 and Ve(W) = Ve(U). (See Figure 6.b).

In fact, this adjoining-a-column operation is a bijection between $\mathscr{W}$ and $\mathscr{W}_\gamma$. Thus,

$$wgf(\mathscr{W}_\gamma) = x \cdot wgf(\mathscr{W}) = xK \ . \tag{17}$$

iv) Let $\mathscr{W}_\delta = \{W \in \mathscr{W} \setminus \mathscr{W}_\alpha$: $\ell_1(W)$ is the rightmost, but is not the leftmost column of $W\}$.

With $U \in \mathscr{W}_\alpha$ we associate the polyomino W produced by adjoining one one-cell column on the right side of U. (See Figure 6.c). We have $W \in \mathscr{W}_\delta$, Co(W) = Co(U) + 1 and Ve(W) = Ve(U). Again, the pasting operation just defined is actually a bijection between $\mathscr{W}_\alpha$ and $\mathscr{W}_\delta$. Thus

$$wgf(\mathscr{W}_\delta) = x \cdot wgf(\mathscr{W}_\alpha) = xy^2K \ . \tag{18}$$

v) Let $\mathscr{W}_\varepsilon = \{W \in \mathscr{W} \setminus \mathscr{W}_\alpha$: $\ell_1(W)$ is neither the first nor the last column of $W\}$.

The columns of a given $W \in \mathscr{W}_\varepsilon$ that lie strictly to the left of $\ell_1(W)$ form a polyomino, say U_1, which belongs to $\mathscr{W}_\alpha$. On the other hand, $\ell_1(W)$ and the columns to the right of it form a polyomino U_2 which is an element of $\mathscr{W}_\gamma$. (See Figure 6.d). We have Co(W) = Co(U_1) + Co(U_2) and Ve(W) = Ve(U_1) + Ve(U_2) − 2. Further, the decomposition just described is a bijection between $\mathscr{W}_\varepsilon$ and the cartesian product $\mathscr{W}_\alpha \times \mathscr{W}_\gamma$. From these remarks we conclude that

$$wgf(\mathscr{W}_\varepsilon) = y^{-2}\, wgf(\mathscr{W}_\alpha) \cdot wgf(\mathscr{W}_\gamma) = xK^2 \ . \tag{19}$$

Clearly, $\{\mathscr{W}_\alpha, \mathscr{W}_\beta, \mathscr{W}_\gamma, \mathscr{W}_\delta, \mathscr{W}_\varepsilon\}$ is a partition of $\mathscr{W}$. Using Eqs. (15)–(19), we now find

$$K = wgf(\mathscr{W}) = wgf(\mathscr{W}_\alpha) + \ldots + wgf(\mathscr{W}_\varepsilon) = y^2K + xy^2 + xK + xy^2K + xK^2 \ . \tag{20}$$

The above equation for K can be rewritten as

$$x \cdot K^2 + \left[x(1+y^2) - (1-y^2)\right] \cdot K + xy^2 = 0 \ . \tag{21}$$

Proposition 2. The *wgf* of the wall polyominoes is given by

$$K(x,y) = \frac{1 - y^2 - x(1 + y^2) - \sqrt{(1-x)^2(1-y^2)^2 - 4xy^2(1-y^{2})}}{2x} \,. \tag{22}$$

Proof. Eq. (22) was obtained from the solution of quadratic Eq. (21) by rearranging the discriminant.

Remark. I know of no reference for the result of Proposition 2. The wall polyominoes themselves appear in the papers of Privman and Švrakić[18] and Bertoli *et al.*,[19] but these papers deal with some other aspects of polyominology, namely with the area enumeration[18] and random generation.[19]

FUNCTION G

As announced before, now we rapidly advance through Eqs. (12), (14), (9), (11), and (1) finding the functions $\bar{I}$, I, J, H and G one after the other. Thus we get:

Theorem 1. The perimeter *gf* for the column-convex polyominoes is given by

$$G(x,y) = (1 - y^2)\left[1 - \frac{4}{6 - \sqrt{(1-x)^2 - \frac{4xy^2}{1-y^2}} - \sqrt{(1+x)^2 + \frac{4xy^2}{1-y^2}}}\right] . \tag{23}$$

ANOTHER DERIVATION OF G

As we shall now see, the formula for G can as well be derived without recourse to the tapoes and stapoes. This simplification is, however, purchased at a price: the wall polyominoes need to be studied in more depth.

Let us first recapitulate some facts from section »A one-to-one mapping of the stapoes into wall polyominoes«. For $c, v \in \mathbb{N}$, we defined the sets

$$\mathscr{P}_{cv} = \{P \in \mathscr{P}\colon \mathrm{He}(P) = 2c, \mathrm{Ve}(P) = 2v\} \quad \text{and}$$

$$\mathscr{W}_{cv}^{+} = \{W \in \mathscr{W}\colon \mathrm{Co}(W) = 2c - 1, \mathrm{Ve}(W) = 2v,$$
$$h_{2i}(W) < h_{2i-1}(W) + h_{2i+1}(W) \quad (\forall_i \in \underline{c-1})\} \,.$$

With $P \in \mathscr{P}_{cv}$ we associated the wall polyomino $\varphi(P)$ which has $2c - 1$ columns, and whose 1^{st}, 2^{nd}, 3^{rd}, 4^{th}, ... , $2c - 1^{\text{th}}$ column contain

$$Y_1(P) - y_1(P),\ \ Y_1(P) - y_2(P),\ \ Y_2(P) - y_2(P),\ \ Y_2(P) - y_3(P),\ \ \ldots,\ \ Y_c(P) - y_c(P)$$

cells, respectively. The correspondence φ proved to be a bijection between $\mathscr{P}_{cv}$ and $\mathscr{W}^+_{cv}$.

Now it is time for some new definitions and notations.

Definition 3. Let W be a wall polyomino.

i) By the even columns of W we mean the 2^{nd}, 4^{th}, 6^{th}, ... of the columns of W.

ii) Let $1 < j < \text{Co}(W)$. When $h_j(W) \geq h_{j-1}(W) + h_{j+1}(W)$, the j^{th} column of W is an *extra high* (*xhigh*) one, and it has $h_j(W) - h_{j-1}(W) - h_{j+1}(W)$ *extra cells* (*xcells*).

We shall write $\mathscr{W}'$ for the set of those wall polyominoes which have an odd number of columns.

For $r \in \mathbb{N}_o$, let $\mathscr{W}'(r) = \{W \in \mathscr{W}': W$ has precisely r xhigh even columns$\}$.

For $r \in \mathbb{N}$ and $d \in \mathbb{N}_o$, let $\mathscr{W}'(r,d) = \{W \in \mathscr{W}'(r)$: W has d xcells in its r^{th} xhigh even column$\}$.

And now we start crossing the wall polyomino waters. For c, $v \in \mathbb{N}$ we have

$$<x^{2c}y^{2v}>G = |\mathscr{P}_{cv}| = |\mathscr{W}^+_{cv}| = <x^{2c-1}y^{2v}>\ wgf(\mathscr{W}'(0)) = <x^{2c}y^{2v}>\ x \cdot wgf(\mathscr{W}'(0))\ ,$$

which implies
$$G = x \cdot wgf(\mathscr{W}'(0))\ . \tag{24}$$

Let $r \in \mathbb{N}$ and $d \in \mathbb{N}_o$. With $U_1 \in \mathscr{W}'(r-1)$ and $U_2 \in \mathscr{W}'(0)$ we associate the polyomino $W = \psi(U_1,U_2)$, generated by the following two-step procedure:

i) Create the one-column polyomino X consisting of $i + j + d$ cells, where i and j are the heights of the last column of U_1 and of the first column of U_2, respectively.

ii) Put U_1, U_2, and the new creature X side by side, in the order $U_1 - X - U_2$. Level the bottoms, then glue these three polyominoes into one polyomino. This latter polyomino is the output of the procedure, *i.e.* it is W.

We have $W \in \mathscr{W}'(r,d)$, $\text{Co}(W) = \text{Co}(U_1) + \text{Co}(U_2) + 1$ and $\text{Ve}(W) = \text{Ve}(U_1) + \text{Ve}(X) + \text{Ve}(U_2) - 2i - 2j = \text{Ve}(U_1) + \text{Ve}(U_2) + 2d$. The polyomino U_1 (resp. U_2) can now be recognized as that part of W which lies to the left (resp. right) of W's rightmost xhigh even column. See Figure 7.

The correspondence ψ is, in fact, a bijection between the cartesian product $\mathscr{W}'(r-1) \times \mathscr{W}'(0)$ and the set $\mathscr{W}'(r,d)$. So we have

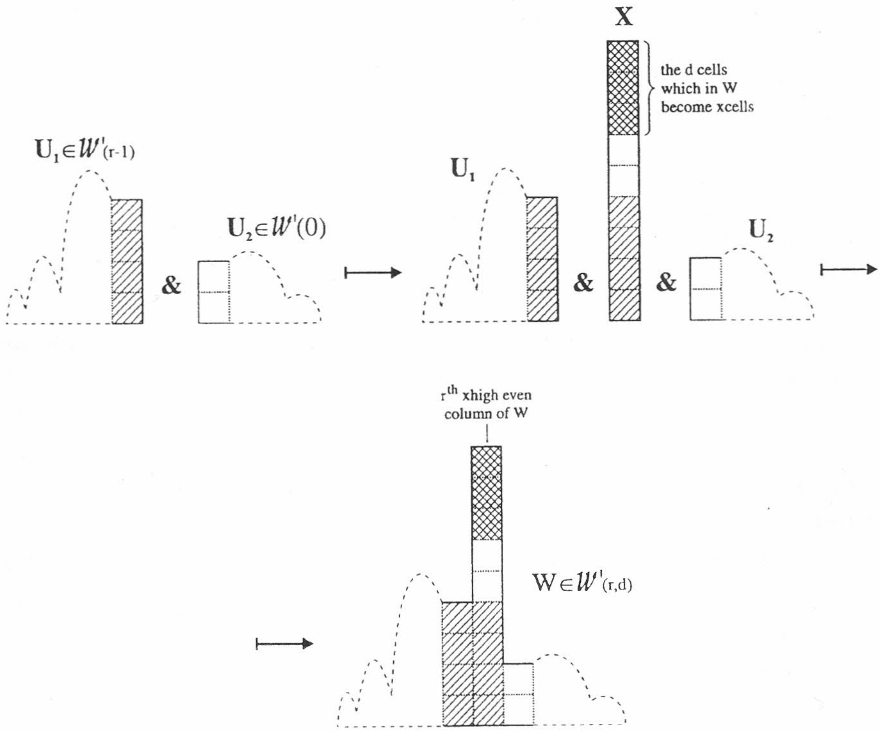


Figure 7. The correspondence between $\mathscr{W}'(r-1) \times \mathscr{W}'(0)$ and $\mathscr{W}'(r,d)$.

$$wgf(\mathscr{W}'(r,d)) = xy^{2d} \cdot wgf(\mathscr{W}'(r-1)) \cdot wgf(\mathscr{W}'(0)) \,, \tag{25}$$

from which follows that

$$wgf(\mathscr{W}'(r)) = \sum_{d \geq o} wgf(\mathscr{W}'(r,d)) = wgf(\mathscr{W}'(r-1)) \cdot \frac{x}{1-y^2}\, wgf(\mathscr{W}'(0)) \,. \tag{26}$$

Iteration of Eq.(26) gives

$$\begin{aligned} wgf(\mathscr{W}'(r)) &= wgf(\mathscr{W}'(r-2)) \cdot \left[\frac{x}{1-y^2}\, wgf(\mathscr{W}'(0))\right]^2 = \ldots \\ \ldots &= wgf(\mathscr{W}'(0)) \cdot \left[\frac{x}{1-y^2}\, wgf(\mathscr{W}'(0))\right]^r . \end{aligned} \tag{27}$$

Observe that Eq. (27) holds for $r = 0$ as well. Next we sum over $r \geq 0$ to find

$$wgf(\mathscr{W}') = \sum_{r \geq o} wgf(\mathscr{W}'(r)) = \frac{wgf(\mathscr{W}'(0))}{1 - \frac{x}{1-y^2} wgf(\mathscr{W}'(0))} . \tag{28}$$

Hence
$$wgf(\mathscr{W}'(0)) = \frac{wgf(\mathscr{W}')}{1 + \frac{x}{1-y^2} wgf(\mathscr{W}')} . \tag{29}$$

The formula for $wgf(\mathscr{W}')$ follows immediately from Proposition 2:

$$wgf(\mathscr{W}') = \frac{1}{2} \cdot \left[K(x,y) - K(-x,y)\right] = \tag{30}$$

$$= \frac{2(1-y^2) - \sqrt{(1-x)^2(1-y^2)^2 - 4xy^2(1-y^2)} - \sqrt{(1+x)^2(1-y^2)^2 + 4xy^2(1-y^2)}}{4x}$$

Theorem 1 can now be re-established with ease: we just need to combine Eqs. (24), (29) and (30).

Acknowledgements. – My original intention was to put the stapoes in a bijection with certain lattice paths. Then Goran Igaly made a remark that the paths could be replaced by the wall polyominoes. To my mind, that replacement was a good idea.

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SAŽETAK

Prebrojavanje vertikalno konveksnih poliominoa prema opsegu na svima pristupačan način

Svjetlan Feretić

Nedavno (1993.) su o vertikalno konveksnim (vk-) poliominoima sa zadanim opsegom dobiveni novi rezultati, znatno jednostavniji od dotadašnjih. Do tih se novih rezultata došlo na dva načina: preko algebarskih jezika i preko Temperleyevih jednadžbi. Nešto kasnije (1995.) se ustanovilo da prebrojavanje pomoću algebarskih jezika postaje lakše ako se vk-poliominoe shvati kao poseban slučaj takozvanih stapoa. U ovom se radu nastojalo taj olakšani dokaz učiniti još pristupačnijim, i zbog toga su algebarski jezici zamijenjeni zid-poliominoima. Na kraju rada pokazano je i to da se, uz malo pažljivije proučavanje zid-poliominoa, do rezultata o vk-poliominoima može doći i bez korištenja stapoa.