

# Performance Evaluation of Parallel Concatenated Chaos Coded Modulations

Francisco J. Escribano, Luis López and Miguel A. F. Sanjuán

Original scientific paper

**Abstract**—In this article we study the performance of a class of parallel concatenated encoders similar to a turbo trellis coded modulation, but where the constituent encoders are chaos coded modulators. We show that, even when the uniform error property does not hold for the kind of constituent chaos coded modulations employed, it is still possible to draw a reasonable bound for the bit error probability at the error floor region based on the interleaver structure. The simulations validate the bounds and show that the dynamics of the underlying chaotic maps, rather than the quantization level of the constituent encoders, is the most important factor to account for both the bit and frame error rate behavior at the error floor region. We also show that these chaos based parallel concatenated schemes yield performances comparable to binary turbocodes and are thus of potential interest in communications.

**Index Terms**—Channel coding, Chaos, Concatenated coding, Modulation coding, Error analysis.

## I. INTRODUCTION

The possibility of using chaotic signals to carry information was first considered in 1993 [1]. This aroused a big deal of work on chaotic communications, which became a hot topic in both nonlinear science and engineering. The interest in chaotic communications was due to the foreseen good properties of the chaotic signals in the fields of secure systems or broadband multiple access systems. In the case of secure systems, one can take advantage of the uncorrelation and unpredictability of the chaotic signals to build encryption algorithms. These are the same properties desirable for the spread sequences of a code division multiple access (CDMA) system. On the other hand, chaotic modulations and channel encoders derived from chaotic systems attracted much attention, but the interest on this kind of chaotic communications dropped somewhat due to the bad performance of the systems proposed so far, since they did not outperform other usual coded communication schemes, and they did not have even better performance than uncoded systems [2].

However, in later times and in some contexts we have witnessed the arising of some proposals with good performance

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as compared with classical communication systems [3], and this has reopened the way to look further into the possibilities of chaotic systems to act efficiently in coded modulation schemes. Chaos based modulation systems working at the waveform level have already shown to be of potential use in multipath fading channels [4], as well as chaos based systems working at the coding level [5]. Other recent works have stressed the fact that chaos coded modulation (CCM) systems working at a joint waveform and coding level can be efficient in AWGN [3], [6]. It has been also shown that communications based on high dimensional chaotic systems and belief propagation decoding can offer an excellent performance characterized with thresholds [7]. Moreover, chaos based coded modulation systems have shown to be of potential interest in flat fading channels [8]. This contrasts strongly with previous state of the art [9]. Such success has been achieved by building a comprehensive bridge linking the fields of chaos theory and digital communications.

In particular, the proposal of a kind of chaos based coded modulations which could be seen under a trellis encoding view [3] opened the road to evaluate such kind of encoders in the same schemes where convolutional codes or trellis coded modulation (TCM) systems are able to yield high coding gains. For example, one can build very efficient coded and modulated systems using concatenation: parallel concatenation, such as in the so-called turbocodes or turbo TCM systems; and serial concatenation, such as in serially concatenated convolutional codes, or in serially concatenated TCM systems [10].

The mentioned convolutional encoder view of such kind of chaos based coded modulations is much like Ungerboeck's TCM [10], and since the turbo TCM systems combining the bandwidth efficient TCM systems and the philosophy of turbocoding offer good performance in white noise and radio channels, it was expected that new systems built under the same principles could lead to comparable results. This was ascertained in [11], where parallel concatenated chaos coded modulations were shown to give coding gains as high as with standard binary turbocodes. Another advantage of designing systems under these similarities is that we can use well established tools to design the encoders and decoders and to evaluate their performance, thus avoiding the need to start from scratch. The use of the EXIT charts to evaluate the convergence of the decoding algorithm in [11] is an instance of this. Other examples of the success of this philosophy in chaotic communications are the proposals of [12] and [6], where the convolutional encoder view allowed the design of systems with serially concatenated channel and chaotic encoders leading to good bit error rate (BER) results.

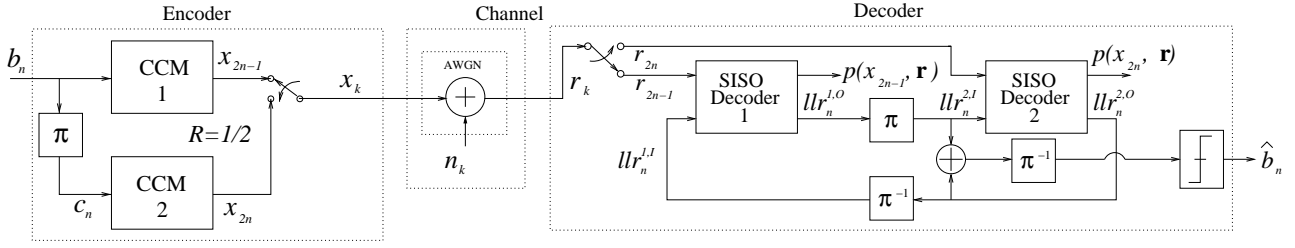


Fig. 1. Scheme of the whole communications system. Left side: parallel concatenated encoder structure for  $R = 1/2$ ; center: channel model for AWGN; right side: soft-input soft-output (SISO) iterative decoder.

The successful proposal of turbo-like structures based on chaos coded modulations in [11] for the AWGN channel is still lacking of a deep and comprehensive study which could allow us to address the design task. That is the reason why we look here into the possibility to draw tight bounds for the bit error rate performance in the error floor region based on the minimum distance concept [13], with the aim also to link the dynamical properties of the underlying chaotic systems and the final performance of the concatenated system. This work is an extension of a previous conference paper [14].

According to all this, the article is structured as follows. Section II is devoted to the description of the whole system, including the concatenated encoder for some constituent chaos coded modulations, the channel and the iterative decoder. In Section III, we show how the minimum distance analysis can be extended to this kind of nonlinear concatenated codes, and how to draw bounds for the error floor region. Section IV exhibits the simulation results and compares them with the proposed bounds. Finally, Section V is devoted to the conclusions.

## II. SYSTEM MODEL

### A. Encoder

The concatenated encoder consists in the parallel concatenation of two chaos coded modulations (CCM's) [3], and, accordingly, we will call these systems parallel concatenated chaos coded modulations (PCCCM's). They have been first proposed in [11], where it is shown that these systems can reach comparable performance to binary turbo codes and turbo TCM systems. The general structure of the system, including the concatenated encoder, the AWGN channel and the iterative decoder is shown in Fig. 1.

The first CCM receives an *i.i.d.* binary input word  $\mathbf{b} = (b_1, \dots, b_N)$  of size  $N$ . The second CCM receives as input a scrambled sequence  $\mathbf{c} = (c_1, \dots, c_N)$ , which is the word  $\mathbf{b}$  interleaved by an  $S$ -random interleaver [15] of size  $N$ . The factor  $S$  of this kind of interleaver is the minimum separation between bit positions at the output for any two contiguous bits at the input. The permutation function is chosen in a semi-random basis. This function will map an index  $j$  into an index  $\pi_j$ , which means that the bit in position  $j$ ,  $b_j$ , will be taken as bit in position  $\pi_j$ ,  $c_{\pi_j}$ , within the input word for the second encoder. The index permutation is chosen according to the following steps:

- Choose an integer  $S$ .

- For index  $\pi_j$  corresponding to position  $j$  draw a random integer  $t$  between 1 and  $N$ .
- If  $t$  has not been chosen before, verify if the  $S$  previously chosen indexes lie at least at a distance of  $S$  from  $t$ , i.e.  $|\pi_p - t| > S$  for  $p = j - S, \dots, j - 1$ .
- If  $t$  satisfies the previous conditions, keep it as  $\pi_j = t$  and proceed in the same way until all  $N$  indexes are chosen.

This algorithm converges in a reasonable time when  $S$  is chosen according to:

$$S < \sqrt{\frac{N}{2}}. \quad (1)$$

When  $S = 1$ , we have a purely random interleaver.

Each CCM belongs to the class of discrete chaotic switched maps driven by small perturbations [3]. They follow the general expression

$$z_n = f(z_{n-1}, b_n) + g(b_n, z_{n-1}) \cdot 2^{-Q}, \quad (2)$$

$$x_n = 2z_n - 1, \quad (3)$$

where  $f(\cdot, 0) = f_0(\cdot)$  and  $f(\cdot, 1) = f_1(\cdot)$  are chaotic maps that leave the interval  $[0, 1]$  invariant. They are piecewise linear maps with slope  $\pm 2$ . The natural number  $Q$  indicates the number of bits to represent  $z_n$  (thus  $x_n$ ), and  $g(b_n, z_{n-1}) \in \{0, 1\}$  is the small perturbations term [3], given by

$$g(b, z) = \begin{cases} b & z < \frac{1}{2} \\ \bar{b} & z \geq \frac{1}{2} \end{cases}, \quad (4)$$

where  $\bar{b} = b \oplus 1$  and  $\oplus$  is the binary XOR operation. If  $f_0(\cdot) = f_1(\cdot)$  is the Bernoulli shift map,  $g(b, z)$  is equivalent to a precoder defined by the polynomial  $1 + D^{Q-1}$  [16], where  $D$  stands for delay and the exponent  $Q - 1$  means that input is delayed  $Q - 1$  sample periods. In any case, the function  $g(b, z)$  ensures that two binary input words ( $\mathbf{b}$ ,  $\mathbf{b}'$ ) differing in only 1 bit do not lead to output sequences ( $\mathbf{x}$ ,  $\mathbf{x}'$ ) for any individual CCM with a low squared Euclidean distance  $d_E^2 = \sum_{n=1}^N (x_n - x'_n)^2$ . This recursive term is included to ensure a good interleaver gain, as it is mandatory for the inner encoders in any concatenated encoder [17].

With these definitions, it is easy to see that the recursion of (2) leaves the finite set  $S_Q = \{i \cdot 2^{-Q} | i = 0, \dots, 2^Q - 1\}$  invariant and, therefore, we can restrict (2) to  $S_Q$  by taking as initial condition  $z_0 = 0$ . We shall consider the following pairs of maps  $f_0(\cdot)$  and  $f_1(\cdot)$ :

- 1) Bernoulli shift map (BSM).

$$f_0(z) = f_1(z) = 2z \bmod 1. \quad (5)$$

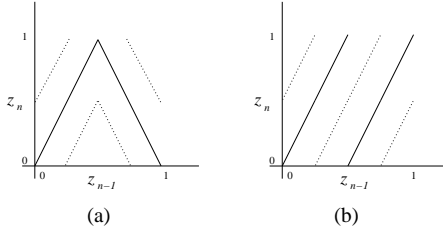


Fig. 2. Maps for the constituent CCM encoders. The continuous line corresponds to  $f_0(\cdot)$ ; the dotted line, to  $f_1(\cdot)$ . (a) mTM (continuous line: TM); (b) mBSM (continuous line: BSM).

- 2) Tent map (TM), corresponding to equations

$$f_0(z) = f_1(z) = \begin{cases} 2z & 0 \leq z < \frac{1}{2} \\ 2 - 2z & \frac{1}{2} \leq z \leq 1 \end{cases}. \quad (6)$$

- 3) The BSM and a shifted version of the same (multi-Bernoulli shift map, mBSM), following

$$f_0(z) = 2z \bmod 1, \quad (7)$$

$$f_1(z) = \begin{cases} 2z + \frac{1}{2} & 0 \leq z < \frac{1}{4} \\ 2z - \frac{1}{2} & \frac{1}{4} \leq z < \frac{1}{2} \\ 2z - \frac{3}{2} & \frac{3}{4} \leq z \leq 1 \end{cases}. \quad (8)$$

- 4) The tent map and a shifted version of the same (multi-tent map, mTM), following

$$f_0(z) = \begin{cases} 2z & 0 \leq z < \frac{1}{2} \\ 2 - 2z & \frac{1}{2} \leq z \leq 1 \end{cases}, \quad (9)$$

$$f_1(z) = \begin{cases} 2z + \frac{1}{2} & 0 \leq z < \frac{1}{4} \\ 2z - \frac{1}{2} & \frac{1}{4} \leq z < \frac{3}{4} \\ 2z - \frac{3}{2} & \frac{3}{4} \leq z \leq 1 \end{cases}. \quad (10)$$

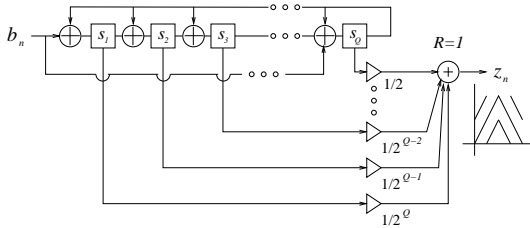


Fig. 3. Encoding structure for the mTM CCM.  $s_i$  stands for a memory position holding one bit,  $\oplus$  is the binary XOR operation and the feedback path before  $s_1$  implements the recursive precoding of (4). The rest of feedback paths account for the structure of the equivalent finite-state machine of the discretized map chosen.

In Fig. 2 we have depicted the corresponding maps. These CCM systems, when restricted to  $S_Q$ , allow an equivalent representation in terms of a *trellis encoder*, closely related to a trellis coded modulation (TCM) system [3]. In Fig. 3, for example, we can see the equivalent trellis encoder structure for the mTM CCM.

The constituent encoders seen work at a rate of one chaotic sample per bit, and therefore, the parallel concatenation of two CCM's will have a coding rate of  $R = 1/2$ . The output words have thus size  $2N$ ,  $\mathbf{x} = (x_1, \dots, x_{2N})$ , with  $x_k$ ,  $k = 1, \dots, 2N$ , taking values alternatively from each of the CCM's: when  $k$  is odd ( $k = 2n - 1$ ,  $n = 1, \dots, N$ ), the

sample corresponds to the first CCM; when  $k$  is even ( $k = 2n$ ,  $n = 1, \dots, N$ ), the sample corresponds to the second CCM (see Fig. 1). Note that the overall system is similar to a turbo TCM system, but now the sequence  $x_n$  is a chaos-like broadband signal not intended for spectral efficiency.

### B. Channel

The channel corresponds to the well known AWGN model. The sequence arriving at the decoder side,  $\mathbf{r} = (r_1, \dots, r_{2N})$ , will then be

$$r_k = x_k + n_k, \quad (11)$$

where  $n_k$  are *i.i.d.* samples of a Gaussian RV with zero mean and power  $\sigma^2$ . Since the joint PCCCM system proposed has rate  $R = 1/2$ , the relationship between the power of the noise and the signal to noise ratio in terms of bit energy to noise spectral density will be

$$\sigma^{-2} = 2 \frac{R E_b}{P N_0} = \frac{1 E_b}{P N_0}, \quad (12)$$

where  $P$  is the power of the chaos coded modulated signal<sup>1</sup>.

### C. Decoder

A chaos based signal coded with the mentioned finite state machine scheme (see Fig. 3) can be decoded with any standard method intended for general state machine coding, in particular *maximum likelihood* (ML) [3] or *maximum a posteriori* (MAP) algorithms [18]. They are based on the set of  $2^Q$  possible states defined by the memory positions  $s_i$  (see Fig. 3), and the transitions between those states. As seen in Fig. 4, each transition happens for time  $n$  over an edge  $e_n$  between a starting state  $S^S(e_n)$  and an ending state  $S^E(e_n)$ . The transition is driven by an input bit  $b_n$  and produces as output a chaotic sample  $x_n = 2z_n - 1$  as a function of  $b_n$  and of the previous chaotic sample  $x_{n-1}$  (equivalent to the previous state). This is indicated by function  $h(\cdot, \cdot)$ . The state  $S^E(e_n)$  is the starting state for the subsequent transition,  $S^S(e_{n+1})$ .

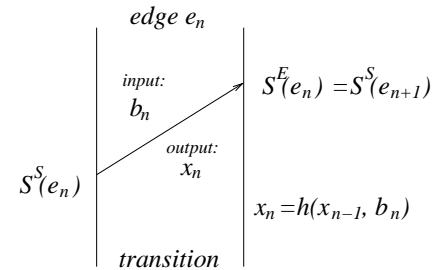


Fig. 4. Transition diagram of the equivalent finite state machine for a chaos based coded modulated system.

In particular, the iterative decoder for a concatenated system can use standard soft-input soft-output (SISO) decoding modules [19]. As seen in Fig. 1, the decoder chosen consists on two SISO decoding blocks working iteratively over each set

<sup>1</sup>For the kind of maps considered, the invariant density of  $x_k$  is uniform in  $[-1, 1]$  and, therefore, we can assume  $P \approx 1/3$  when  $Q > 4$ , since  $P$  is equivalent to the square standard deviation  $\sigma_x^2$  of the invariant density.

of received samples (the ones coming from the first encoder,  $r_{2n-1}$ ,  $n = 1, \dots, N$ , and the ones coming from the second one,  $r_{2n}$ ,  $n = 1, \dots, N$ ). This decoder reproduces the well known structure of the SISO iterative decoders for parallel concatenated binary channel codes, but with channel metrics adapted to the CCM setup

$$p(r_n|x_n) = \frac{1}{\sqrt{2\pi\sigma}} \exp\left(-\frac{(r_n - x_n)^2}{2\sigma^2}\right), \quad (13)$$

where  $x_n = 2z_n - 1$ ,  $z_n \in S_Q$ , is the candidate quantized chaotic sample for the corresponding branch metric [12]. We do not review here further details of the adapted SISO decoders for brevity. The calculation of the output binary log likelihood ratios ( $llr$ 's, see Fig. 1) as a function of the input  $llr$ 's and the channel metrics is straightforward [2] using the already known log-MAP algorithm [19].

### III. MINIMUM DISTANCE ANALYSIS

The CCM's of the kind described are not linear and do not comply in general with the uniform error property [10], and so a ML bound for the bit error probability based on the PCCCM transfer function is almost unfeasible. Nevertheless, we will see that we can give a reasonable bound in the error floor region. In binary turbocodes or turbo TCM systems, the minimum or free distance of the resulting parallel concatenated code or coded modulation is usually employed to provide this error floor bound [10].

Each individual CCM has a characteristic binary input error event  $\mathbf{e} = \mathbf{b} \oplus \mathbf{b}'$  associated to its output minimum squared Euclidean distance. For example, in the case of the TM CCM, the minimum squared Euclidean distance is given by an input error event with length  $L = Q + 1$  and Hamming weight  $w(\mathbf{e}) = Q + 1$ . This event produces  $Q$  different output values for the resulting encoded sequences  $x_n$  and  $x'_n$ , and a minimum squared Euclidean distance  $d_E^2 = \sum_{n=m}^{m+L-2} (x_n - x'_n)^2$  tending to 0 for  $Q \rightarrow \infty$  [3].

Nevertheless, due to the presence of the parallel concatenated scheme with the S-random interleaver, the input binary error events leading to the output minimum squared Euclidean distance for the joint PCCCM system are normally to be found among the Hamming weight 2 input binary error events (or concatenations of the same). Such error events  $\mathbf{e}^*$  consist in two 1's separated by  $L^* - 2$  0's, and have length  $L^* = Q + 1$  for the BSM and mBSM CCM's, and  $L^* = Q + 2$  for the TM and the mTM CCM's. The Hamming weight 2 binary error events with length  $p(L^* - 1) + 1$ ,  $p \in \mathbb{N}$  will lead to increasing values in the associated output squared Euclidean distances by a factor of  $p$ . In the case of the BSM, each pair of sequences produced by a pair of input binary words differing only on a Hamming weight 2 error event of length  $L^*$  will determine a squared Euclidean distance given by [3]

$$d_{min}^2 = 4 \sum_{n=m}^{L^*+m-1} (z_n - z'_n)^2 = 4 \sum_{i=1}^Q \frac{1}{4^i} = \frac{4}{3} \left(1 - \frac{1}{4^Q}\right). \quad (14)$$

For the rest of CCM's, the squared Euclidean distances associated to such error events will depend generally on the input words  $\mathbf{b}$  and  $\mathbf{b}'$ . In Figs. 5(a), 5(b) and 6(a) we can see the

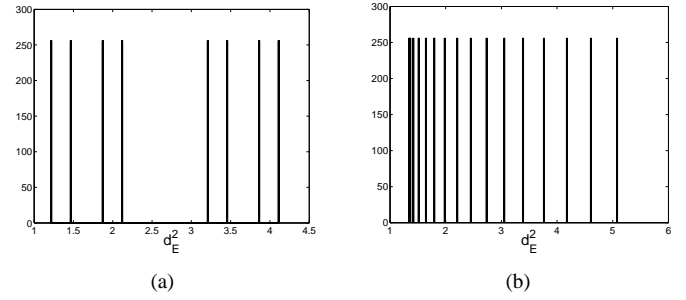


Fig. 5. Histogram of the  $d_{E_i}^2$ , corresponding to the Hamming weight 2 binary input error events in the case of a CCM system with  $Q = 5$  for all the possible input sequences. (a) mBSM CCM; (b) TM CCM.

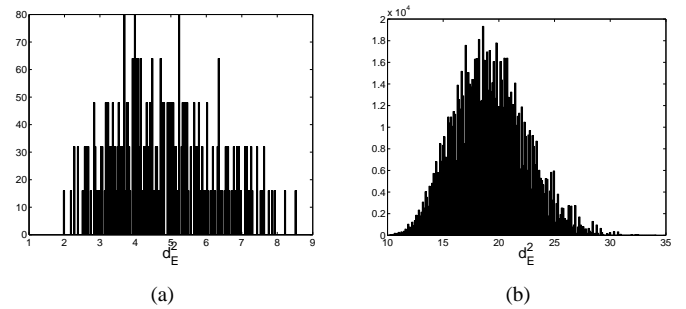


Fig. 6. (a) Histogram of  $d_{E_i}^2$  for the Hamming weight 2 binary input error events in the case of an mTM CCM with  $Q = 5$  for all the possible input sequences; (b) Histogram of  $d_E^2$  when the binary error events consist on the concatenation of 2 binary input error events with Hamming weight 2 and length  $L^*$  each, in the case of a PCCCM with two mTM CCM's with  $Q = 5$ .

distance spectrum for the corresponding maps, produced for all possible  $\mathbf{b}$  and  $\mathbf{b}'$  sequences differing on a Hamming weight 2 error event with length  $L^*$ .

Let us denote as  $D_2^{L^*}$  the set of all the possible squared Euclidean distances given by such binary error events in a CCM, that is to say

$$D_2^{L^*} = \{d^2(\mathbf{x}, \mathbf{x}') \mid \mathbf{x} \leftrightarrow \mathbf{b}, \mathbf{x}' \leftrightarrow \mathbf{b}', \mathbf{b} \oplus \mathbf{b}' = \mathbf{e}^*\} \\ d^2(\mathbf{x}, \mathbf{x}') = \sum_{n=1}^N (x_n - x'_n)^2. \quad (15)$$

If the S-random interleaver is designed with  $S$  taking a value at least  $3L^*$ , the input binary error events with Hamming weight 2 will not be the dominant ones in the error floor region, since they would lead to squared Euclidean distances consisting on the sum of the distances associated with an error event of length  $L^*$  for one CCM, and of length  $p(L^* - 1) + 1 > S$  for the other CCM ( $p > 3$ ). That is to say, the squared Euclidean distances in the PCCCM system will be around  $d_{E_1}^2 + pd_{E_2}^2$ , with  $d_{E_i}^2 \in D_2^{L^*}$ .

Since the S-random interleaver does not put any restriction about the concatenation of error events, in the case of  $S > 3L^*$ , the dominant error events will have Hamming weight 4 and will consist in two Hamming weight 2 binary error events differing in  $\mathbf{b}$  at indexes  $i, i + L^* - 1$  and  $j, j + L^* - 1$ , and leading to an analogous combination in  $\mathbf{c}$ , so that, for example,  $|\pi(i) - \pi(j)| = L^* - 1$  and  $|\pi(i + L^* - 1) - \pi(j + L^* - 1)| = L^* - 1$ . These Hamming weight 4 compound binary error events will lead to squared Euclidean distances in the PCCCM system,  $d_E^2 = \sum_{i=1}^4 d_{E_i}^2$ ,  $d_{E_i}^2 \in$

$D_2^{L^*}$ , lower than the squared Euclidean distances associated to the Hamming weight 2 binary error events allowed by the interleaver structure.

In the case of two concatenated BSM CCM's, all Hamming weight 2 binary error events of length  $L^*$  have the same associated squared Euclidean distance ( $D_2^{L^*}$  has only one element), so that, as with turbocodes, the error floor determined by the concatenation of two of such events will be given by

$$P_{b_{\text{floor}}} \approx \frac{w_4 N_4}{2N} \operatorname{erfc} \left( \sqrt{\frac{4d_{\min}^2}{4P} R \frac{E_b}{N_0}} \right), \quad (16)$$

where  $d_{\min}^2$  is given in (14),  $w_4 = 4$  is the number of error bits, and  $N_4$  is the number of compound binary error events with Hamming weight 4 consisting in the mentioned concatenation of Hamming weight 2 binary error events with length  $L^*$  allowed by the interleaver structure. In the rest of cases, such compound binary error events will lead to squared Euclidean distances  $d_E^2 = d_{E_1}^2 + d_{E_2}^2 + d_{E_3}^2 + d_{E_4}^2$ , where  $d_{E_i}^2 \in D_2^{L^*}$  is the individual squared Euclidean distance induced by each individual Hamming weight 2 binary error event in each of the CCM's, which depends generally on  $\mathbf{b}$  and  $\mathbf{b}'$ . For the mBSM CCM and the TM CCM, these distances  $d_{E_i}^2$  have the same frequency (see Fig. 5), and so it is enough to consider the combinations with repetition of 4 elements in  $D_2^{L^*}$  to account for all possible values of  $d_E^2$  between two PCCCM words. Therefore, the bound is given as

$$P_{b_{\text{floor}}} \approx \frac{w_4 N_4}{2N} \binom{t+3}{4}^{-1} \sum_{d_E^2} \operatorname{erfc} \left( \sqrt{\frac{d_E^2}{4P} R \frac{E_b}{N_0}} \right), \quad (17)$$

where  $t$  is the number of elements in  $D_2^{L^*}$  and  $d_E^2 = \sum_{i=1}^4 d_{E_i}^2$ ,  $d_{E_i}^2 \in D_2^{L^*}$ . The term  $\binom{t+3}{4}$  is the number of said combinations of the elements of  $D_2^{L^*}$ .

When we have two mTM CCM's, the number of distances in  $D_2^{L^*}$  is remarkably larger and their frequency is not uniform (see Fig. 6(a)), so that the overall squared Euclidean distance spectrum for the PCCCM outputs,  $d_E^2 = \sum_{i=1}^4 d_{E_i}^2$ ,  $d_{E_i}^2 \in D_2^{L^*}$ , follows the almost Gaussian distribution of Fig. 6(b). To provide a bound based on the related binary error events, we can resort to computing the probability density function (pdf) of  $d_E^2$  with the help of the related histogram and calculate numerically

$$P_{b_{\text{floor}}} \approx \frac{w_4 N_4}{2N} \int_{d_{E_{\min}}^2}^{d_{E_{\max}}^2} p(v) \cdot \operatorname{erfc} \left( \sqrt{\frac{v}{4P} R \frac{E_b}{N_0}} \right) dv, \quad (18)$$

where  $p(v)$  is the estimated pdf for  $d_E^2$ .

With respect to the frame error probability ( $P_e$ ), i.e., the probability for a frame of  $N$  bits to contain bit errors, it is easy to calculate for the error floor region once we have calculated the mentioned bound for  $P_{b_{\text{floor}}}$ . Assuming that the dominant errors will be those with Hamming weight  $w_4 = 4$ , then

$$P_{e_{\text{floor}}} \approx \frac{N}{w_4} P_{b_{\text{floor}}}. \quad (19)$$

The next section will show the accurateness of this result with the help of the frame error rate (FER) results.

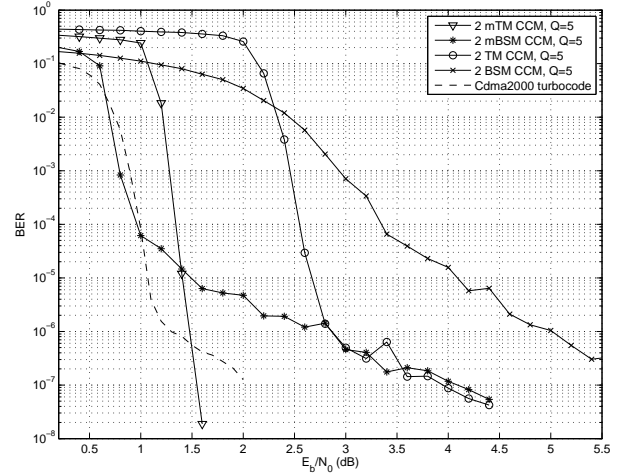


Fig. 7. BER for different parallel concatenation of two equal CCM's in AWGN,  $Q = 5$ ,  $N = 10000$  and  $S = 23$ , compared with a Cdma2000 binary turbocode.

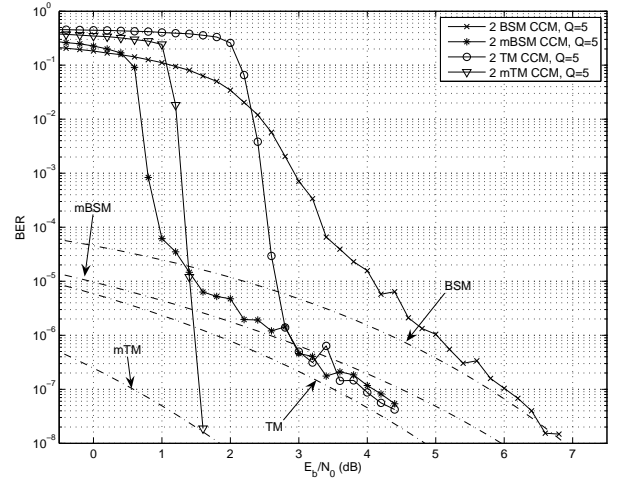


Fig. 8. BER and error floor bounds for different parallel concatenation of two equal CCM's in AWGN,  $Q = 5$ ,  $N = 10000$  and  $S = 23$ . Bounds are depicted with dash-dotted lines.

#### IV. SIMULATION RESULTS

For all the simulations, 20 decoding iterations were performed and the BER and FER results were recorded after finding 20 frames on error. In Figs. 7 and 8 we have depicted the BER results for the parallel concatenation of two equal CCM's with an S-random interleaver of length  $N = 10000$  and  $S = 23$ . All the CCM's have a quantization factor of  $Q = 5$ . In Fig. 7 we plot the simulation results for a Cdma2000 binary turbocode [10] of similar complexity, with an interleaver of size  $N = 6138$  specially designed for its constituent codes. We see that we can get comparable results in the waterfall region in the case of the concatenation of 2 mBSM CCM's, though the error floor of the latter is higher. On the other side, the concatenation of 2 mTM CCM's allows us to achieve a much lower error floor than with the Cdma2000 binary turbocode. It has been shown that a good combination of CCM's can yield chaos based concatenated systems competing with standard alternatives in both the waterfall and error floor

regions [11]. In Fig. 8, we also show the bounds for the BER at the error floor region calculated according to the principles and expressions seen in the previous section. Though they only take into account the kind of error events mentioned there, they are reasonably tight. In the case of BSM and mBSM, the interleavers employed allowed a total of  $N_4 = 5$  possible concatenations of pairs of Hamming weight 2 binary error events with  $L = L^* = Q + 1$ , and the interleavers of the TM and mTM cases allowed a total of  $N_4 = 4$  of such compound events (this time with  $L = L^* = Q + 2$ ). It is remarkable the fact that the theoretical error floor, together with the error floor shown by the simulations, decreases as the regularity in the spectrum of the squared Euclidean distances associated to the individual Hamming weight 2 binary error events decreases. In fact, the mTM case, which has the most complex squared Euclidean distance spectrum as seen in Fig. 6(a), exhibits a very good behavior respecting the error floor. The fact that this system reaches the waterfall region some tenths of dB after the system concatenating of 2 mBSM's is related to the fact that the decoding algorithm starts converging for a higher  $E_b/N_0$  [11], but, once decoding convergence is reached, the better squared Euclidean distance spectrum manifests itself in a lower error floor. This means that the properties of  $E_b/N_0$  convergence threshold (that determines the waterfall region) and the error floor level are to be dealt with separately when choosing a possible concatenation of CCM systems. The rest of cases, excepting the very bad performing concatenation of BSM CCM's, agree well with the principles of parallel concatenation: a general good behavior for low-mid  $E_b/N_0$  once reached the turbo cliff threshold, but a relatively high BER error floor for  $E_b/N_0 \rightarrow \infty$ . Note that the results shown here are comparable to the ones attained by binary turbocodes or turbo TCM systems of similar complexity [10], [11].

In Fig. 9 we show the BER and the error floor bounds for the mBSM PCCCM with different values of  $N$ ,  $S$  and  $Q$ . The difference between  $Q = 5$  and  $Q = 6$  is small, and seems to affect mainly to the turbo cliff, determining a slightly steeper slope for the  $Q = 6$  case. Nevertheless, the error floor is very similar: though for  $Q = 6$  there are some important differences (the minimal loop length is now  $L^* = 7$ ), the values of the squared Euclidean distances are practically the same, since the changes in such values are small as  $Q \rightarrow \infty$  when  $Q > 4$ . Therefore, the main change with a change in  $Q$  is in the multiplicity  $N_4$  of the compound error events allowed by the interleaver permutation. This makes it clear that the dynamics of the map, which determines the spectrum of the squared Euclidean distances between pairs of chaotic sequences, is a more influential factor at the error floor region than the ad-hoc quantization level  $Q$ .

On the other hand, when we change the value of  $N$ , we can appreciate in Fig. 9 that the interleaver gain in the error floor region changes as  $N^{-1}$ . This is the expected behavior for any parallel concatenated system with interleavers [10]. There is not even a noticeable improvement in the  $E_b/N_0$  threshold for the waterfall region with a growing value of  $N$  with respect to the case with  $N = 10000$ . Only when  $N < 10000$  (see the  $N = 1000$  case), the situation is clearly worse and the behavior starts resembling the pure BSM case. In fact, with  $N = 1000$

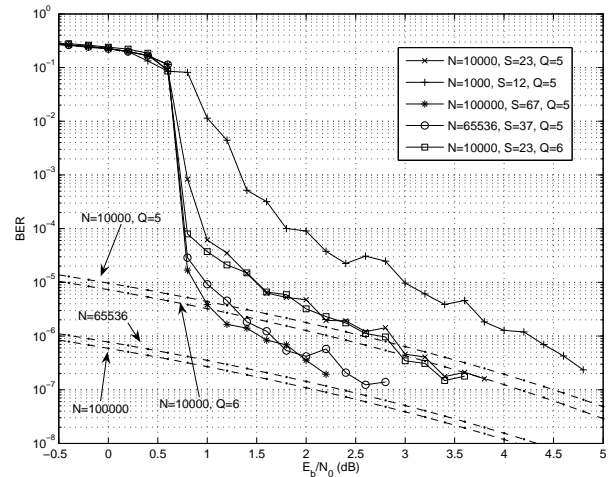


Fig. 9. BER and error floor bounds for the parallel concatenation of two equal mBSM CCM's in AWGN. Bounds are depicted with dash-dotted lines.

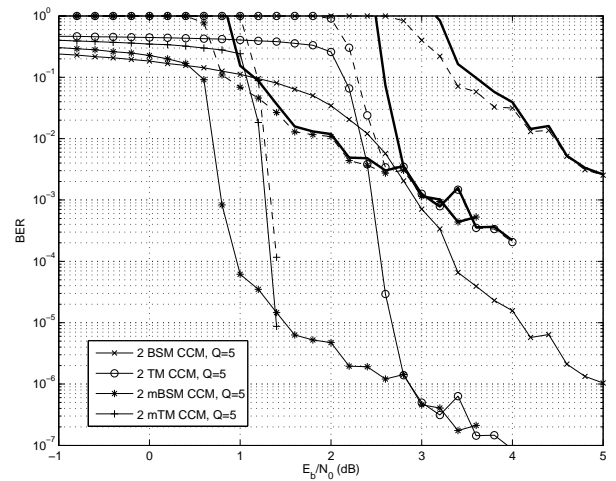


Fig. 10. BER and FER for different parallel concatenation of two equal CCM's in the AWGN channel,  $Q = 5$ ,  $N = 10000$  and  $S = 23$ . BER: continuous line. FER: dashed line. Thick continuous lines merging for high  $E_b/N_0$  with the dashed lines depict the relation  $FER = (N/w_4)BER$  for each corresponding PCCCM kind (except for the mTM).

the interleaver requires a lower value for  $S$ , and, since it takes now the value  $S = 12$ , the dominant error events in the error floor region are those with overall Hamming weight 2 (instead of 4) and given by one individual binary error loop with  $L = L^*$  in one encoder and  $L = 3(L^* - 1) + 1 = 16 > S$  in the other. These error events have associated squared Euclidean distances lower than the squared Euclidean distances for the compound Hamming weight 4 error events of the cases with  $S > 3L^*$ . Note that the maximum possible value for  $S$  is limited by the size of the interleaver  $N$ , since  $S < \sqrt{N/2}$ . This also puts a limit on the value of  $Q$  once we have a pair  $S$  and  $N$ , because  $L^* = Q + a$ ,  $a \in \mathbb{N}$ , and we require that  $L^* < S$  to avoid low squared Euclidean distance weight 2 error events in the concatenated code.

Finally, we have plotted in Fig. 10 the BER and FER results for several concatenation of CCM's. We have also included the plots of  $(N/w_4)BER$  (see (18)). We can see how the FER curves follow the expected behavior for a parallel concatenated

scheme, with the waterfall and the error floor regions located where is hinted by the BER curves (with the commented exception of the pathological BSM system). With respect to the error floor region, we verify that the relationship (18) holds for the FER, and we can be sure that the dominant error events present in each frame with errors are those of Hamming weight 4 and structure seen. We have not depicted  $(N/w_4)BER$  for the concatenation of two mTM's since the results obtained are still far from reaching the error floor steady state.

### V. CONCLUSIONS

In this article we have reviewed a class of chaos coded modulations with switched discrete chaotic maps and small perturbations control, and we have analyzed the behavior at the error floor region of the parallel concatenation of such modulations. This scheme has shown to be of potential interest since the attainable performance is quite similar to that of binary turbocodes or turbo TCM. The analysis of the error events leading to low output square Euclidean distances has allowed us to draw bounds for the bit error probability at said error floor region. The same analysis has provided a valuable insight into the properties of the concatenated system and has stressed the fact that the dynamics of the underlying map is a major factor in the system design, since this dynamics controls the squared Euclidean distance spectrum. In fact, the best results were obtained when the uniform error property was furthest from being met.

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