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Comparison Between Zagreb Eccentricity Indices and the Eccentric Connectivity Index, the Second Geometric-arithmetic Index and the Graovac-Ghorbani Index

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Abstract: The concept of Zagreb eccentricity indices (E_1 and E_2) was introduced in the chemical graph theory very recently. The eccentric connectivity index (ξ^c) is a distance-based molecular structure descriptor that was used for mathematical modeling of biological activities of diverse nature. The second geometric-arithmetic index (GA_2) was introduced in 2010, is found to be useful tool in QSPR and QSAR studies. In 2010 Graovac and Ghorbani introduced a distance-based analog of the atom-bond connectivity index, the Graovac-Ghorbani index (ABC_{GG}), which yielded promising results when compared to analogous descriptors. In this note we prove that $E_1(T) > \xi^c(T)$ for chemical trees *T*. For connected graph *G* of order *n* with maximum degree Δ, it is proved that $\xi^c(G) > E_2(G)$ if $\Delta = n-1$ and $\xi^c(G) < E_2(G)$, otherwise. Moreover, we show that $GA_2 > ABC_{GG}$ for paths and some class of bipartite graphs.

Keywords: eccentric connectivity index, first Zagreb eccentricity index, second Zagreb eccentricity index, second geometric-arithmetic index, second atom-bond connectivity index.

1. INTRODUCTION

A topological index is a numerical descriptor of the molecular structure derived from the corresponding molecular graph. There are numerous topological descriptors that have found some applications in theoretical chemistry, especially in QSPR/QSAR research.[1,2] They can be classified based on the structural properties of graphs used for their calculation. The following topological indices are well-studied by the researchers: Wiener index,[3] Hosoya index,[4] the energy[5] and the Randić connectivity index.[6]

Let $G = (V,E)$ denote a simple graph with *n* vertices and *m* edges, where $V(G) = \{v_1, v_2, \ldots, v_n\}$ and $|E(G)| = m$ (The cardinality of a set *S*, denoted $|S|$, is the number of elements in *S*). The degree of a vertex $v_i \in V(G)$, $d_G(v_i)$ is the number of edges incident to v_i. The maximum degree of a graph *G* is denoted by Δ, that is, $\Delta = \max\{d_G(v_i) : v_i \in V(G)\}$. The distance between v_i and v_j in $V(G)$, $d_G(v_i, v_j)$, is the length of a shortest v_i to v_j path in *G*. The eccentricity, $\varepsilon_G(v_i)$ of a vertex $v_i \in V(G)$ is the maximum distance between v_i and any other vertex in *G*, that is, $\varepsilon_G(v_i) = \max\{d_G(v_i, v_j) : v_j \in V(G)\}$. The diameter of *G*, *d*, is defined as the maximum value of the eccentricities of the vertices of *G*, that is, $d = \max\{d_G(v_i, v_j) : v_i, v_j \in V(G)\}$.

Gutman and Trinajstić[7] derived a formula for estimating total π-electron energy of conjugated systems. Their formula contained two terms that later became known as the Zagreb indices M_1 and M_2. The first Zagreb index $M_1(G)$ and the second Zagreb index $M_2(G)$ of graph *G* (see Refs. [2], [7–11] and the references therein) are among the oldest and most studied topological indices. They are defined as:

$$M_1(G) = \sum_{v_i \in V(G)} d_G^2(v_i) \text{ and } M_2(G) = \sum_{v_i v_j \in E(G)} d_G(v_i) \cdot d_G(v_j),$$

where $d_G(v_i)$ is the degree of the vertex v_i of graph *G*.

The invariants based on vertex eccentricities attracted some attention in chemistry. In an analogy with the first and the second Zagreb indices, Ghorbani *et al.*[12] and Vukičević *et al.*[13] defined the first E_1, and the second E_2, Zagreb eccentricity indices by

$$E_1(G) = \sum_{v_i \in V(G)} \varepsilon_G^2(v_i) \quad (1)$$

and

$$E_2(G) = \sum_{v_i v_j \in E(G)} \varepsilon_G(v_i) \cdot \varepsilon_G(v_j), \quad (2)$$

where $\varepsilon_G(v_i)$ is the eccentricity of the vertex v_i in *G*. Upper and lower bounds for the Zagreb eccentricity indices of graphs have been reported in Refs. [12–15].

The eccentric connectivity index of a graph *G*, denoted by $\xi^c(G)$, is defined as[16]

$$\xi^c(G) = \sum_{v_i \in V(G)} d_G(v_i) \cdot \varepsilon_G(v_i) = \sum_{v_i v_j \in E(G)} (\varepsilon_G(v_i) + \varepsilon_G(v_j)),$$

where $d_G(v_i)$ and $\varepsilon_G(v_i)$ are the degree and the eccentricity of the vertex v_i in *G*, respectively. The eccentric connectivity index provides good correlations with regard to both physical and biological properties.[17] The simplicity amalgamated with high correlating ability of this index can be easily exploited in QSPR/QSAR studies. Such studies can easily provide valuable leads for the development of potential therapeutic agents. We encourage the reader to consult papers[18,19] for the mathematical properties of the eccentric connectivity index.

Let *e* be an edge of the graph *G* (which may contain cycles or be acyclic), connecting the vertices v_i and v_j. Here we define two sets $N_i(e \mid G)$ and $N_j(e \mid G)$ as follows:

$$N_i(e \mid G) = \{v_k \in V(G) | d_G(v_k, v_i) < d_G(v_k, v_j)\},$$

$$N_j(e \mid G) = \{v_k \in V(G) | d_G(v_k, v_j) < d_G(v_k, v_i)\}.$$

The number of elements of $N_i(e \mid G)$ and $N_j(e \mid G)$ are denoted by $n_i(e \mid G)$ $(=| N_i(e \mid G) |)$ and $n_j(e \mid G)$ $(=| N_j(e \mid G) |)$, respectively. Therefore $n_i(e \mid G)$ counts the number of vertices of *G* lying closer to the vertex v_i than to vertex v_j. The meaning of $n_j(e \mid G)$ is analogous. Vertices equidistant from both ends of the edge $v_i v_j$ belong neither to $N_i(e \mid G)$ nor to $N_j(e \mid G)$. Note that for any edge *e* of *G*, $n_i(e \mid G) \geq 1$ and $n_j(e \mid G) \geq 1$, because $v_i \in N_i(e \mid G)$ and $v_j \in N_j(e \mid G)$. For the sake of brevity, if there is no risk of confusion, we always simplify $n_i(e \mid G)$ and $n_j(e \mid G)$ as n_i and n_j, respectively.

Recently, Fath-Tabar, Furtula and Gutman[20] defined *second geometric–arithmetic index* by

$$GA_2(G) = \sum_{v_i v_j \in E(G)} \frac{\sqrt{n_i \cdot n_j}}{\frac{1}{2}[n_i + n_j]}. \quad (3)$$

For the mathematical properties of GA_2 index, the reader is referred to Refs. [20–22]

In Ref. [23], Graovac and Ghorbani proposed the following distance-based analog of the ABC index:

$$ABC_{GG}(G) = \sum_{v_i v_j \in E(G)} \sqrt{\frac{n_i + n_j - 2}{n_i \cdot n_j}}. \quad (4)$$

Some initial studies indicate that the Graovac-Ghorbani index could be an effective predictive tool in chemistry. For instance, it can be used to model both the boiling and the melting points of molecules.[24] Upper and lower bounds for the ABC_{GG} index of graphs have been given in Refs. [23], [25–28].

Let $\sum$ be the class of finite graphs. A topological index is a function *Top* from $\sum$ into real numbers, where for *G* and *H* being isomorphic: $Top(G) = Top(H)$. Suppose two topological indices Top_1 and Top_2. Since Top_1 and Top_2 are real numbers for any graph *G*, then it is interesting to compare these two topological indices Top_1 and Top_2 for *G*, that is, $Top_1(G) \geq Top_2(G)$ or $Top_1(G) < Top_2(G)$? Recently, Das and Trinajstić compared the first geometric-arithmetic index and the atom-bond connectivity index for trees and graphs. Moreover, they compared ξ^c with M_1 and M_2 for chemical trees, molecular graphs and some graph families. Several relations between the two *ABC*-indices are established. Geometric-arithmetic indices are compared for chemical trees, starlike trees and general trees in Ref. [30], and the Wiener index and the Zagreb indices and the eccentric connectivity index for trees in Ref. [31]. In this note we prove that $E_1(T) > \xi^c(T)$ for chemical trees *T*, and for connected graph *G*, $\xi^c(G) > E_2(G)$ if $\Delta = n-1$ and $\xi^c(G) < E_2(G)$, otherwise. Moreover, we show that $GA_2 > ABC_{GG}$ for paths and some class of bipartite graphs.

2. PRELIMINARIES

A connected graph with maximum vertex degree at most 4 is said to be a "*molecular graph*".[1] A tree in which the maximum vertex degree does not exceed 4 is said to be a "chemical tree". Denote, as usual, by $K_{1,n-1}$, P_n, C_n and K_n, the star, the path, the cycle and the complete graph on *n* vertices, respectively. A double star of order *n*, denoted by $DS(p,q)$ $(p \geq q, n = p + q + 2)$, is a tree, which is constructed by joining the central vertices of two stars $K_{1,p}$ and $K_{1,q}$. A vertex of a graph is said to be pendent if its neighborhood contains exactly one vertex. An edge of a graph is said to be pendent if one of its vertices is a pendent vertex.

3. COMPARISON BETWEEN E_1 AND ξ^c OF GRAPHS

In this section we compare the first Zagreb eccentricity index (E_1) and the eccentric connectivity index (ξ^c) for graphs. For $G \cong K_{1,n-1}$, $E_1(G) = 4(n-1)+1 > 3(n-1) = \xi^c(G)$ and for $G \cong K_n$ $(n > 2)$, $E_1(G) = n < n(n-1) = \xi^c(G)$. Therefore the first Zagreb eccentricity index and the eccentric connectivity index are incomparable on the class of general graphs. But we have the following theorem.

Theorem 3.1. *Let T be a chemical tree of order $n > 2$. Then*

$$E_1(T) > \xi^c(T).$$

Proof: Let d be the diameter of tree T. Then $d \geq 2$. For $d = 2$, $T \cong K_{1,n-1}$ and hence $E_1(T) > \xi^c(T)$. For $d = 3$, $T \cong DS(p,q)$ $(n = p+q+2)$. Then $E_1(T) = 9n-10 > 5n-6 = \xi^c(T)$. For $d = 4$, the number of non-pendent vertices in T is at most five and the number of pendent vertices in T is at least two (since T is a chemical tree). Exactly one non-pendent vertex, say v_i, has eccentricity 2 and all the other non-pendent vertices have eccentricity exactly 3. For each non-pendent vertex $v_j \in V(T)$ $(j \neq i)$,

$$\varepsilon_T^2(v_j) - d_T(v_j)\cdot\varepsilon_T(v_j) \geq 9 - 12 = -3.$$

For $v_i \in V(T)$,

$$\varepsilon_T^2(v_i) - d_T(v_i)\cdot\varepsilon_T(v_i) \geq 4 - 8 = -4.$$

For pendent vertex $v_k \in V(T)$ (vertex v_k is on a diametral path),

$$\varepsilon_T^2(v_k) - d_T(v_k)\cdot\varepsilon_T(v_k) = 16 - 4 = 12.$$

All the other pendent vertices $v_k \in V(T)$, we have

$$\varepsilon_T^2(v_k) - d_T(v_k)\cdot\varepsilon_T(v_k) > 0.$$

Since the number of non-pendent vertices is at most five and the number of pendent vertices is at least two in T, we have

$$\sum_{i=1}^{n}(\varepsilon_T^2(v_i) - d_T(v_i)\cdot\varepsilon_T(v_i)) > 0, \text{ that is, } E_1(T) > \xi^c(T).$$

For d = 5 or 6, there are at most two vertices of eccentricity 3 with $\varepsilon_T^2(v_i) - d_T(v_i)\varepsilon_T(v_i) \geq -3$ and there are at least two pendent vertices of eccentricity 5 with $\varepsilon_T^2(v_j) - d_T(v_j)\varepsilon_T(v_j) = 20$. For all other vertices $v_k \in V(G)$, the eccentricity is at least 4, therefore $\varepsilon_T^2(v_k) - d_T(v_k)\varepsilon_T(v_k) \geq 0$. Therefore again we get $E_1(T) > \xi^c(T)$.

For $d \geq 7$, we have $\varepsilon_T(v_i) \geq 4$ for any $v_i \in V(T)$. Since there is some vertex v_k such that $\varepsilon_T(v_k) > 4$ and $d_T(v_i) \leq 4$ for any $v_i \in V(T)$, again we get $\xi^c(T) < E_1(T)$. This completes the proof of the theorem. □

In Ref. [29], we compared M_1 and ξ^c for chemical tree and molecular graph. Moreover, we compare E_1 and ξ^c for chemical tree in Theorem 3.1. We now obtain the following result for any connected graph.

Theorem 3.2. *Let G be a connected graph. Then*

$$M_1(G) \geq \xi^c(G) \text{ or } E_1(G) \geq \xi^c(G).$$

Proof: We have

$$M_1(G) + E_1(G) - 2\xi^c(G) = \sum_{i=1}^{n}(d_G^2(v_i) + \varepsilon_G^2(v_i) - 2d_G(v_i)\varepsilon_G(v_i)) = \sum_{i=1}^{n}(d_G(v_i) - \varepsilon_G(v_i))^2 \geq 0.$$

Thus

$$M_1(G) + E_1(G) \geq 2\xi^c(G)$$

with equality holding if and only if $d_G(v_i) = \varepsilon_G(v_i)$, for every $i = 1,2,\ldots,n$. Moreover, we have

$$(M_1(G) - \xi^c(G)) + (E_1(G) - \xi^c(G)) \geq 0,$$

which implies that $M_1(G) \geq \xi^c(G)$ or $E_1(G) \geq \xi^c(G)$. This completes the proof. □

4. COMPARISON BETWEEN E_2 AND ξ^c OF GRAPHS

We now compare $E_2(G)$ and $\xi^c(G)$ for any connected graph *G*.

Theorem 4.1. *Let G be a connected graph of order $n > 1$ with maximum degree Δ. If $\Delta = n-1$, then $E_2(G) < \xi^c(G)$. Otherwise, $E_2(G) > \xi^c(G)$.*

Proof: First we assume that $\Delta = n-1$. Let $v_1, v_2, \ldots, v_k$ be the k (≥ 1) vertices of degree $n-1$ in *G*.Then $\varepsilon_G(v_i) = 1$ for $v_i \in V(G)$, $i = 1,2,\ldots,k$ and $\varepsilon_G(v_i) = 2$ for $v_i \in V(G)$, $i = k+1, k+2, \ldots, n$. Now,

$$\begin{aligned}
E_2(G) &= \sum_{v_iv_j \in (G)} \varepsilon_G(v_i)\cdot\varepsilon_G(v_j) \\
&= \sum_{\substack{v_iv_j \in E(G) \\ 1\leq i\neq j\leq k}} \varepsilon_G(v_i)\cdot\varepsilon_G(v_j) \\
&\quad + \sum_{\substack{v_iv_j \in E(G) \\ 1\leq i\leq k,\, k+1\leq j\leq n}} \varepsilon_G(v_i)\cdot\varepsilon_G(v_j) + \sum_{\substack{v_iv_j \in E(G) \\ k+1\leq i\neq j\leq n}} \varepsilon_G(v_i)\cdot\varepsilon_G(v_j) \\
&= \frac{k(k-1)}{2} + k(n-k)2 + \left[m - \frac{k(k-1)}{2} - k(n-k)\right]4 \\
&= 4m - \frac{3k(k-1)}{2} - 2k(n-k)
\end{aligned}$$

and

$$\begin{aligned}\xi^c(G) &= \sum_{i=1}^{n} d_G(v_i)\cdot\varepsilon_G(v_i)\\ &= \sum_{i=1}^{k} d_G(v_i)\cdot\varepsilon_G(v_i) + \sum_{i=k+1}^{n} d_G(v_i)\cdot\varepsilon_G(v_i)\\ &= k(n-1) + 2\sum_{i=k+1}^{n} d_G(v_i)\\ &= k(n-1) + 2\big[2m - k(n-1)\big] = 4m - k(n-1).\end{aligned}$$

Since $n \geq 2$, one can easily see that

$4m - k(n-1) > 4m - \frac{3k(k-1)}{2} - 2k(n-k)$, that is, $\xi^c(G) > E_2(G)$.

Next we assume that $\Delta \leq n-2$. Then $\varepsilon_G(v_i) \geq 2$ for every $v_i \in V(G)$. For any edge $v_iv_j \in E(G)$,

$$\varepsilon_G(v_i)(\varepsilon_G(v_j)-1) \geq 2(\varepsilon_G(v_j)-1) \geq \varepsilon_G(v_j),$$

that is,

$$\varepsilon_G(v_i)\cdot\varepsilon_G(v_j) \geq \varepsilon_G(v_i) + \varepsilon_G(v_j).$$

Thus we have

$$E_2(G) - \xi^c(G) = \sum_{v_iv_j\in E(G)} (\varepsilon_G(v_i)\cdot\varepsilon_G(v_j) - \varepsilon_G(v_i) - \varepsilon_G(v_j)) \geq 0.$$

This completes the proof of the theorem. □

5. COMPARISON BETWEEN ABC_{GG} AND GA_2 OF GRAPHS

Since each term of GA_2 and ABC_{GG} are function of n_i and n_j, it is interesting to compare of these two indices. We start with some examples:

Example 1.

$$GA_2(K_n) = \frac{n(n-1)}{2} > 0 = ABC_{GG}(K_n).$$

Example 2.

$$GA_2(C_n) = n \text{ and } ABC_{GG}(C_n) = \begin{cases} 2\sqrt{n-2} & \text{if } n \text{ is even} \\ \frac{2n}{n-1}\sqrt{n-3} & \text{if } n \text{ is odd}, \end{cases}$$

and therefore $GA_2(C_n) > ABC_{GG}(C_n)$.

Example 3. For $n \geq 5$,

$$GA_2(K_{1,n-1}) = \frac{2(n-1)\sqrt{n-1}}{n} < \sqrt{(n-1)(n-2)} = ABC_{GG}(K_{1,n-1}).$$

From these examples, we can conclude that GA_2 index and ABC_{GG} index are incomparable on the class of general graphs on *n* vertices. So now we compare these two indices for special class of graphs. For path $P_n: v_1v_2\ldots v_{n-1}v_n$ $(n \geq 5)$, one can easily see that

$$\frac{2\sqrt{n_in_{i+1}}}{n} > \sqrt{\frac{n-2}{n_in_{i+1}}},\ i = \left\lfloor\frac{n}{2}\right\rfloor, \text{ and } \frac{2\sqrt{n_1n_2}}{n} < \sqrt{\frac{n-2}{n_1n_2}}.$$

Therefore there is a term in $GA_2(P_n)$ is greater than the corresponding term in $ABC_{GG}(P_n)$ and also there exists a term in $GA_2(P_n)$ is less than the corresponding term in $ABC_{GG}(P_n)$. So it is interesting to compare these two indices (GA_2 and ABC_{GG}) for path P_n. For this we need the following result:

Lemma 5.1. For $1 \leq x \leq \frac{n-2}{4}$ $(n \geq 2)$,

$$4x^3 - 3nx^2 - \frac{n^2}{2}x + \frac{n^3}{4} > 0.$$

Proof: Let us consider a function

$$g(x) = 4x^3 - 3nx^2 - \frac{n^2}{2}x + \frac{n^3}{4},\ 1 \leq x \leq \frac{n-2}{4}.$$

Then $g'(x) = 12x^2 - 6nx - \frac{n^2}{2}$. Since $1 \leq x \leq \frac{n-2}{4}$ $(n \geq 2)$, one can easily see that $g'(x) < 0$ and hence $g(x)$ is a decreasing function on $1 \leq x \leq \frac{n-2}{4}$. Thus we have

$$g(x) \geq g\left(\frac{n-2}{4}\right) = \frac{1}{8}(5n^2-4) > 0.$$

□

We are now ready to give the proof of $GA_2(P_n) > ABC_{GG}(P_n)$

Theorem 5.2. *Let $n > 2$ be a positive integer. Then $GA_2(P_n) > ABC_{GG}(P_n)$.*

Proof: Let $n = 2p$ and $p = 2r+1$. We have

$$\begin{aligned}GA_2(P_n) &= \sum_{k=1}^{n-1}\frac{2\sqrt{k(n-k)}}{n}\\ &= \frac{4}{n}\Big[\sqrt{1\cdot(n-1)} + \sqrt{2\cdot(n-2)} + \cdots + \sqrt{(p-2)\cdot(p+2)}\\ &\quad + \sqrt{(p-1)\cdot(p+1)}\Big] + 1\\ &= \frac{4}{n}\sum_{x=1}^{r}\left[\sqrt{nx-x^2} + \sqrt{\frac{n^2}{4}-x^2}\right] + 1.\end{aligned} \tag{5}$$

$$\begin{aligned}ABC_2(P_n) &= \sum_{k=1}^{n-1}\frac{\sqrt{n-2}}{\sqrt{k(n-k)}}\\ &= 2\sqrt{n-2}\Big[\frac{1}{\sqrt{1\cdot(n-1)}} + \frac{1}{\sqrt{2\cdot(n-2)}} + \cdots\\ &\quad + \frac{1}{\sqrt{(p-2)\cdot(p+2)}} + \frac{1}{\sqrt{(p-1)\cdot(p+1)}}\Big] + \frac{2}{n}\sqrt{n-2}\\ &= 2\sqrt{n-2}\sum_{x=1}^{r}\left[\frac{1}{\sqrt{nx-x^2}} + \frac{1}{\sqrt{\frac{n^2}{4}-x^2}}\right] + \frac{2}{n}\sqrt{n-2}.\end{aligned} \tag{6}$$

Let us consider a function

$$f(x) = (nx - x^2)\left(\frac{n^2}{4} - x^2\right),\ 1 \le x \le \frac{n-2}{4}.$$

Then by Lemma 5.1, we have

$$f'(x) = 4x^3 - 3nx^2 - \frac{n^2}{2}x + \frac{n^3}{4} > 0.$$

Therefore $f(x)$ is an increasing function on $1 \le x \le \frac{n-2}{4}$ and hence

$$f(x) \ge f(1) = (n-1)\left(\frac{n^2}{4} - 1\right) \ge \frac{n^2}{4}(n-2).$$

From the above, one can easily see that

$$\sqrt{nx - x^2}\sqrt{\frac{n^2}{4} - x^2} \ge \frac{n\sqrt{n-2}}{2},\ 1 \le x \le \frac{n-2}{4}$$

that is,

$$\frac{4}{n}\left[\sqrt{nx - x^2} + \sqrt{\frac{n^2}{4} - x^2}\right] \ge 2\sqrt{n-2}\left[\frac{1}{\sqrt{nx - x^2}} + \frac{1}{\sqrt{\frac{n^2}{4} - x^2}}\right],$$

$$1 \le x \le \frac{n-2}{4}.$$

Since $n > 2\sqrt{n-2}$, from Eq. (5),Eq (6) and the above result, we have

$$GA_2(P_n) > ABC_2(P_n).$$

Similarly, one can easily prove the result for $n = 2p$, $p = 2r$; $n = 2p+1$, $p = 2r+1$; $n = 2p+1$, $p = 2r$. This completes the proof of the theorem. □

We now compare GA_2 and ABC_{GG} for bipartite graph.

Theorem 5.3. *Let G be a bipartite graph of order n. If*

$$\left\lceil \frac{n}{2} - \sqrt{\frac{n^2}{4} - \frac{n}{2}\sqrt{n-2}} \right\rceil \le n_i \le n_j \le \left\lfloor \frac{n}{2} + \sqrt{\frac{n^2}{4} - \frac{n}{2}\sqrt{n-2}} \right\rfloor$$

for any edge $v_iv_j \in E(G)$, then $GA_2(G) \ge ABC_{GG}(G)$.

Proof. Since G is bipartite, $n_i + n_j = n$ for any edge $v_iv_j \in E(G)$. Therefore we have

$$GA_2(G) - ABC_{GG}(G) = \sum_{v_iv_j \in E(G)} \left[\frac{2\sqrt{n_i \cdot n_j}}{n_i + n_j} - \frac{\sqrt{n_i + n_j - 2}}{\sqrt{n_i \cdot n_j}}\right]$$

$$= \sum_{v_iv_j \in E(G)} \left[\frac{2n_in_j - n\sqrt{n-2}}{n\sqrt{n_i \cdot n_j}}\right].$$

The expression (7) is certainly non-negative if for every $v_iv_j \in E(G)$ we have

$2n_in_j - n\sqrt{n-2} \ge 0$, that is, $2n_i(n - n_i) - n\sqrt{n-2} \ge 0$,

that is,

$$n_i^2 - nn_i + \frac{n}{2}\sqrt{n-2} \le 0,$$

that is,

$$\frac{n}{2} - \sqrt{\frac{n^2}{4} - \frac{n}{2}\sqrt{n-2}} \le n_i \le n_j \le \frac{n}{2} + \sqrt{\frac{n^2}{4} - \frac{n}{2}\sqrt{n-2}}.$$

This completes the proof of the theorem. □

Remark 5.4. *We can construct several graphs such that the condition in Theorem 5.3 is satisfied. For $n = 100$, we have*

$$\left\lceil \frac{n}{2} - \sqrt{\frac{n^2}{4} - \frac{n}{2}\sqrt{n-2}} \right\rceil = 6, \left\lfloor \frac{n}{2} + \sqrt{\frac{n^2}{4} - \frac{n}{2}\sqrt{n-2}} \right\rfloor = 94 .$$

In the above theorem the condition is $6 \le n_i \le n_j \le 94 = n - 6$ for any edge $v_iv_j \in E(G)$. So we can find several graphs of order $n(=100)$ with that condition, for example, C_n. Moreover, we can construct several graphs such that $GA_2(G) > ABC_{GG}(G)$ without satisfy the condition in Theorem 5.3. For example, we have $GA_2(P_n) > ABC_{GG}(P_n)$, by Theorem 5.2.

CONCLUSION

Topological indices are graph invariants and are used for quantitative structure - activity relationship (QSAR) and quantitative structure - property relationship (QSPR) studies. Many topological indices have been defined in the literature and several of them have found applications as means to model physical, chemical, pharmaceutical and other properties of molecules. The eccentric connectivity index provides good correlations with regard to both physical and biological properties. In this note we presented that the eccentric connectivity index (ξ^c) is less than the first Zagreb eccentricity index (E_1) for chemical trees. For connected graph G, we prove that $\xi^c(G) > E_2(G)$ if $\Delta = n - 1$ and $\xi^c(G) < E_2(G)$, otherwise. The Graovac-Ghorbani index is a distance-based analog of the atom-bond connectivity index, one of the most meaningful degree-based molecular structure descriptors. In this work, we show that the second geometric-arithmetic index (GA_2) is greater than the Graovac-Ghorbani index (ABC_{GG}) for paths and some class of bipartite graphs. There are many unsolved problems regarding the comparison between topological indices of graphs. The comparison between the first Zagreb eccentricity index (E_1) and the eccentric connectivity index (ξ^c) , in the case of trees and general graphs is left as an open problem. The comparison between the second geometric-arithmetic index (GA_2) and the Graovac-Ghorbani index (ABC_{GG}) for general graphs remains a task for future.

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