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# Comparison Between Two Eccentricity-based Topological Indices of Graphs

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THIS PAPER IS DEDICATED TO PROFESSOR NENAD TRINAJSTIĆ ON THE OCCASION OF HIS 80TH BIRTHDAY

**Abstract:** For a connected graph *G*, the eccentric connectivity index (ECI) and the first Zagreb eccentricity index of *G* are defined as $\xi^c(G)=\sum_{v_i\in V(G)}\deg_G(v_i)\varepsilon_G(v_i)$ and $E_1(G)=\sum_{v_i\in V(G)}\varepsilon_G(v_i)^2$, respectively, where $\deg_G(v_i)$ is the degree of $v_i$ in *G* and $\varepsilon_G(v_i)$ denotes the eccentricity of vertex $v_i$ in *G*. In this paper we compare the eccentric connectivity index and the first Zagreb eccentricity index of graphs. It is proved that $E_1(T)>\xi^c(T)$ for any tree *T*. This improves a result by Das[25] for the chemical trees. Moreover, we also show that there are infinite number of chemical graphs *G* with $E_1(G)>\xi^c(G)$. We also present an example in which infinite graphs *G* are constructed with $E_1(G)=\xi^c(G)$ and give some results on the graphs *G* with $E_1(G)<\xi^c(G)$. Finally, an effective construction is proposed for generating infinite graphs with each comparative inequality possibility between these two topological indices.

**Keywords**: Graph, First Zagreb eccentricity index, Eccentric connectivity index.

## 1. INTRODUCTION

We only consider finite, undirected and simple graphs throughout this paper. Let *G* be a graph with vertex set $V(G)=\{v_1,v_2,\ldots\ldots,v_n\}$ and edge set $E(G)$. The *degree* of $v_i\in V(G)$, denoted by $\deg_G(v_i)$, is the number of vertices in *G* adjacent to $v$. For any two vertices $v_i,v_j$ in a graph *G*, the distance between them, denoted by $d_G(v_i,v_j)$, is the length of a shortest path connecting them in *G*. Other undefined notations and terminology on the graph theory can be found in.[1]

For any vertex of graph *G*, the eccentricity $\varepsilon_G(v_i)$ is the maximum distance from $v_i$ to other vertices of *G*, i.e., $\varepsilon_G(v_i)=\max_{v_j\neq v_i} d_G(v_i,v_j)$. If $\varepsilon_G(v_i)=d_G(v_i,v_j)$, then $v_j$ is an *eccentric vertex* of vertex $v_i$. For any graph *G*, we denote by $\bar{G}$ the complement of *G*. As usual, let $S_n$, $P_n$, $C_n$, $K_n$ be the star graph, path graph, cycle graph and complete graph, respectively, on $n$ vertices. We denote by $K_{n_1,n_2}$ the complete bipartite graph with bipartition of sizes $n_1$ and $n_2$. The *Cartesian product* $G\square H$ of graphs *G* and *H* is the graph with $V(G\square H)=V(G)\times V(H)$ and $(g,h)$ is adjacent to $(g',h')$ if and only if $gg'\in E(G)$ and $h=h'$, or $g=g'$ and $hh'\in E(H)$. If $G=H$, then $G\square H$ is denoted by $G^{(2)}$ for short. Moreover, $G^{(k)}$ can be similarly defined.

A graphical invariant is a number related to a graph which is a structural invariant, in other words, it is a fixed number under graph automorphisms. In chemical graph theory, these invariants are also known as the topological indices. Two of the oldest graph invariants are the well-known Zagreb indices first introduced in Ref. [2] where Gutman and Trinajstić examined the dependence of total $\pi$-electron energy on molecular structure and elaborated in Ref. [3] For a (molecular) graph *G*, the first Zagreb index $M_1(G)$ and the second Zagreb index $M_2(G)$ are, respectively, defined as follows:

$$M_1=M_1(G)=\sum_{v_i\in V(G)}\deg_G(v_i)^2,$$

$$M_2=M_2(G)=\sum_{v_iv_j\in E(G)}\deg_G(v_i)\deg_G(v_j).$$

These two classical topological indices reflect the extent of branching of the molecular carbon-atom skeleton.[4] The main properties of $M_1$ and $M_2$ were summarized in Refs. [5,6]. Other recent results on Zagreb indices can be found in Ref. [7] and the references cited therein.

In analogy with the first and second Zagreb indices of graphs, some variants of them are invented, such as multiplicative Zagreb indices,[8,9] multiplicative sum Zagreb index,[10,11] Zagreb coindex[12,13] and so on. In particular, Vukičević and Graovac[14] defined the first and second Zagreb eccentricity indices as follows:

$$E_1(G)=\sum_{v_i\in V(G)}\varepsilon_G(v_i)^2, E_2(G)=\sum_{v_iv_j\in E(G)}\varepsilon_G(v_i)\varepsilon_G(v_j).$$

Some mathematical properties of $E_1$ and $E_2$ can be found in Refs. [15,16].

In 1997, Sharma, Goswami and Madan[17] introduced a distance-based molecular structure descriptor, which is named as "eccentric connectivity index" and defined as

$$\xi^c(G)=\sum_{v_i\in V(G)}\deg_G(v_i)\varepsilon_G(v_i).$$

The eccentric connectivity index (ECI) has been employed successfully for the development of numerous mathematical models for the prediction of biological activities of diverse nature.[18–20] The ECI also can be written as follows:

$$\xi^c(G)=\sum_{v_iv_j\in E(G)}\left(\varepsilon(v_i)+\varepsilon(v_j)\right).$$

Some properties of ECI have been reported in Refs. [21–23].

A tree with maximum degree at most 4 is called *chemical tree*, which provides the graph representation of alkanes.[24] In particular, a graph with maximum degree at most 4 is called *chemical graph*. Denote by $\mathcal{T}_n(d)$ the set of trees of order *n* and with diameter *d*.

From definition, we have $E_1(S_n)>\xi^c(S_n)$ and $E_1(K_n)<\xi^c(K_n)$ for $n\geq 3$. Therefore these two topological indices $E_1$ and $\xi^c$ are incomparable. In Ref. [25], Das proved that $E_1(T)>\xi^c(T)$ for any chemical tree *T*. The paper is organized as follows. In Section 2, we show that $E_1(T)>\xi^c(T)$ for any tree *T*. And we present that there are infinite number of chemical graphs *G* with $E_1(G)>\xi^c(G)$. In Section 3, we give an example in which infinite number of graphs *G* are constructed with $E_1(G)=\xi^c(G)$. Also several sufficient conditions are proved for graphs *G* with $E_1(G)<\xi^c(G)$. In Section 4, an effective construction is presented for generating infinite graphs with each comparison possibility between these two topological indices.

## 2. THE GRAPHS WITH $E_1(G)>\xi^c(G)$

In this section we characterize some graphs *G* with $E_1(G)>\xi^c(G)$. Clearly, $E_1(P_2)=\xi^c(P_2)$. Then we deal with the case when *T* is a tree of order $n\geq 3$. Next we will prove that there are infinite number of chemical graphs *G* with $E_1(G)>\xi^c(G)$.

Before presenting the main results, we need to introduce some notations. A *caterpillar*,[26] denoted by $P^n_{k+1}(a_2,a_3,\ldots,a_k)$ with $\sum_{i=2}^k a_i=n-k-1$, is a tree of order *n* with diameter *k* obtained from a path $P_{k+1}=v_1v_2\cdots v_{k+1}$ by attaching $a_i\geq 0$ pendant vertices to the vertex $v_i$ for $i=2,3,\ldots,k$. If *k* is even, then $P_{k+1}$ has a unique central vertex $v_{\frac{k}{2}+1}$. Otherwise, $P_{k+1}$ has two adjacent central vertices $v_{\frac{k-1}{2}}$ and $v_{\frac{k+1}{2}}$. If $k=3$ in $P^n_{k+1}(a_2,a_3,\ldots,a_k)$ with $a_2=p-1\geq 0$, $a_k=q-1\geq 0$ and $p+q=n-2$, then $P^n_{k+1}(a_2,a_3,\ldots,a_k)$ is a *double star* and denoted by $DS_n(p,q)$ for short. And $P^n_{k+1}(a_2,a_3,\ldots,a_k)$ is a *dumbbell* and denoted by $DB_n(a_2,a_k)$ if $a_2\geq 0$, $a_k\geq 0$ and $a_t=0$ for $3\leq t\leq k-1$. Moreover, $P^n_{k+1}(a_2,a_3,\ldots,a_k)$ is called a *volcano tree* and denoted by $V_n(n-k-1)$ if $a_t=n-k-1$ when $v_t$ is a central vertex of $P_{k+1}$ and $a_j=0$ for any $j\neq t$ for even *k*, and in the set $\mathcal{V}_n(n-k-1)$ if $a_t+a_{t+1}=n-k-1$ when $v_t,v_{t+1}$ are two central vertices of $P_{k+1}$ and $a_j=0$ for any $j\neq t, t+1$ for odd *k*. If $d=2$, $\mathcal{T}_n(d)$ contains a single tree $S_n$. The case is same when $n=3$. For $n=4$, there are exactly two trees $S_n$ and $P_n$ with $E_1(P_n)>\xi^c(P_n)$ and $E_1(S_n)>\xi^c(S_n)$. So in the following we always assume that $d\geq 3$ and $n\geq 5$. For convenience, here we set $\xi^A(G)=E_1(G)-\xi^c(G)$ for any connected graph *G*.

**Lemma 2.1.** *Suppose that* $T\in\mathcal{T}_n(d)$ *with* $n\geq 5$ *and* $3\leq d<n-1$ *minimizes the value of* $\xi^A$. *Then T must be a caterpillar.*

**Proof.** We choose an arbitrary tree $T\in\mathcal{T}_n(d)$ with $\xi^A(T)$ as small as possible. If $T\cong P^n_{d+1}(a_2,a_3,\ldots,a_d)$, then our result holds immediately. Otherwise, we can assume that $P_{d+1}=v_1v_2\cdots v_dv_{d+1}$ is a diametral path in *T*. Then *T* can be viewed as a tree obtained by attaching a subtree $T_i$ to each of vertex $v_i$ with $i\in\{2,3,\ldots\ldots,d\}$ such that $diam(T_i)\leq d-\varepsilon_T(v_i)$. There must be a pendant vertex, say $v_k$, from $V(T_m)$ with $d_T(v_k,v_m)\geq 2$ where $m\in\{3,4,\ldots,d-1\}$ and $v_kv_j\in E(T)$ $(j\notin\{1,2,\ldots,d,d+1\})$. Without loss of generality, we assume that $\varepsilon_T(v_m)=d_T(v_m,v_{d+1})=d+1-m$ and $\varepsilon_T(v_k)=t$ with $t>d+1-m+1$.

Now we construct a new tree $T'$ obtained from *T* by deleting the edge $v_kv_j$ and adding a new edge $v_kv_m$. Then $T'$ still belongs to $\mathcal{T}_n(d)$ with $\varepsilon_{T'}(v_k)=d+1-m+1$ with $\varepsilon_{T'}(v_m)=d+1-m$ and $\varepsilon_{T'}(v_x)=\varepsilon_T(v_x)$ where $x\neq k$. Note that $t>d+1-m+1$. Therefore, only considering the contribution of the vertices $v_j,v_k$ and $v_m$ to $\xi^A$, we have

$$\begin{aligned}\xi^A(T)-\xi^A(T')=&\left[t-1-\deg_T(v_j)\right](t-1)+t(t-1)\\&+(d+1-m)\left[d+1-m-\deg_T(v_m)\right]\\&-\left[t-1-(\deg_T(v_j)-1)\right](t-1)\\&-(d+2-m-1)(d+2-m)\\&-(d+1-m)\left[d+1-m-(\deg_T(v_m)+1)\right]\end{aligned}$$

$$= -(t-1)+(d+1-m)+t(t-1)$$
$$-(d+1-m)(d+2-m)$$
$$=(t-1)^2-(d+1-m)^2>0,$$

that is, $\xi^A(T^{'})<\xi^A(T)$. If $T^{'}$ is a caterpillar, our result follows. If not, we can continue the above construction process until we obtain a caterpillar $P^n_{d+1}(a_2,a_3,\ldots,a_d)\in\mathcal{T}_n(d)$. Then our result holds from the fact that $\xi^A$ strictly decreases in the above construction process. □

**Lemma 2.2.** *Let $T\cong P^n_{d+1}(a_2,a_3,\ldots,a_d)$ be a non-volcano tree. Then there is another caterpillar $T^{'}\in\mathcal{T}_n(d)$ with $\xi^A(T^{'})<\xi^A(T)$.*

**Proof.** By assumption, there is a non-central vertex $v_i$ in the diametral path of $T\cong P^n_{d+1}(a_2,a_3,\ldots,a_d)$ such that $a_i>0$. First, let *d* be even. When *d* is odd, our proof can be similarly completed and so omitted here. Without loss of generality, assume that $i<\frac{d}{2}$. Now we construct a new tree $T^{'}$ from *T* by deleting all the pendant edges incident with $v_i$ and joining them with the vertex $v_{i+1}$. Note that $\deg_T(v_i)=a_i+2$ and $T^{'}$ is still a caterpillar with diameter *d*. Moreover, we have

$$\xi^A(T)-\xi^A(T^{'})$$
$$=a_i\left[(d-i+2)(d-i+2-1)-(d-i+1)(d-i+1-1)\right]$$
$$+(d-i+1)\left[(d-i+1-a_i-2)-(d-i+1-2)\right]$$
$$+(d-i)\left[\left(d-i-\deg_T(v_{i+1})\right)-\left(d-i-\deg_T(v_{i+1})-a_i\right)\right]$$
$$=a_i\left[(d-i+2)^2-(d-i+1)^2-1\right]-a_i(d-i+1)+a_i(d-i)$$
$$=a_i\left[2(d-i)+1\right]$$
$$>0,$$

which finishes the proof of this lemma. □

Now we define the a function as follows:

$$h(n,d)=\begin{cases} 2\sum\limits_{i=\frac{d+1}{2}}^{d}(i^2-2i)+\dfrac{(d+1)}{4}\left[(n-d-2)d+n\right]+\dfrac{7d-1}{4} & \text{if } d \text{ is odd;}\\[2ex] 2\sum\limits_{i=\frac{d}{2}}^{d}(i^2-2i)+\dfrac{d^2(n-d-2)}{4}+3d & \text{if } d \text{ is even.}\end{cases}$$

By some calculations, we have $\xi^A(V_n(n-d-1))=h(n,d)$ for even *d* and $\xi^A(T)=h(n,d)$ for any $T\in\mathcal{V}_n(n-d-1)$ when *d* is odd.

**Theorem 2.3.** *Let T be a tree of order $n\geq 4$. Then*

$$\xi^A(T)\geq n$$

*with equality holding if and only if $T\cong S_n$.*

**Proof.** Note that the set of all trees of order *n* can be partitioned into the union of the sets $\mathcal{T}_n(d)$ with $d\in\{2,3,\ldots,n-1\}$. Moreover, it can be easily checked that $h(n,d)>h(n,d-2)$ for any polarity of *d*.

Assume that *T* is a tree of order $n\geq 3$ with $\xi^A$ as small as possible. By the above argument, we conclude that *T* is a tree in $\mathcal{T}_n(2)$ or in $\mathcal{T}_n(3)$, that is, $T\cong S_n$ or $T\cong DS_n(n_1,n_2)$ with $n_1\geq 1$, $n_2\geq 1$ and $n_1+n_2=n-2$. By definition, we have $\xi^A(S_n)=n<4n-4=\xi^A(DS_n(n_1,n_2))$. Thus $T\cong S_n$ from the choice of *T*.

Conversely, if $T\cong S_n$, we have $\xi^A(T)=n$, finishing the proof of this theorem. □

From Theorem 2.3, the following corollary can be easily obtained.

**Corollary 2.4.** *Let T be a tree of order $n\geq 3$. Then $E_1(T)>\xi^c(T)$.*

From Corollary 2.4, any chemical tree *T* fulfills the property that $E_1(T)>\xi^c(T)$, which is also recently proved by Das.[25]

Note that any vertex  in the cycle $C_n$ has the same eccentricity $\lfloor\frac{n}{2}\rfloor$. Therefore we have $E_1(C_n)>\xi^c(C_n)$ for $n>4$. In the following theorem we prove the existence of chemical graphs *G* with $E_1(G)>\xi^c(G)$.

**Theorem 2.5.** *There are infinite number of chemical graphs G such that $E_1(G)>\xi^c(G)$.*

**Proof.** Now we consider the graph $G=C_n\square K_2$ with $n\geq 6$. Note that *G* is a 3-regular graph of order 2*n*. Assume that $V(G)=\{v_1,v_2,\ldots,v_n,u_1,u_2,\ldots,u_n\}$ where $v_1v_2\ldots v_n$ and $u_1u_2\ldots u_n$ with their natural adjacency relation form two induced cycle $C_n$ in *G*.

For any positive integer *k* with $1\leq k\leq n$, we set $|k|_n=\min\{k,n-k\}$. It can be verified that $\varepsilon_G(v_i)=\lfloor\frac{n}{2}\rfloor+1>3$ for any vertex $v_i$ with $u_t$ as its eccentric vertex in *G* where $|t|_n=\lfloor\frac{n}{2}\rfloor$. By symmetry, we have $\varepsilon_G(u_j)=\lfloor\frac{n}{2}\rfloor+1$ for any vertex $u_j$. Thus $\xi^A(G)>0$, that is, $E_1(G)>\xi^c(G)$ as desired. □

## 3. THE GRAPHS WITH $E_1(G)\leq\xi^c(G)$

In this section we prove several results on the graphs *G* with $E_1(G)\leq\xi^c(G)$.

Recall that $E_1(K_2)=\xi^c(K_2)$. Also $E_1(C_4)=\xi^c(C_4)$ with $C_4=K_2^{(2)}$. Note that, for $k\geq 2$, $K_2^{(k)}$ is just the *k*-cube which is a *k*-regular graph with each vertex with eccentricity k. Now we give a more general result.

**Example 3.1.** $E_1(K_2^{(k)})=\xi^c(K_2^{(k)})$ *for any* $k\geq 1$.

Next we turn to the results for the graphs *G* with $E_1(G)<\xi^c(G)$. Although in Section 2 we prove that $E_1(T)>\xi^c(T)$ for any tree of order $n>2$, we have the opposite result for the complements of all trees of order $n>2$. Below we first list an essential lemma for the complement of a graph.

**Lemma 3.2.**[27] *Let G be a connected graph with the connected complement.*

(*i*) *If* $d > 3$*, then* $\overline{G}$ *has diameter* $\overline{d} = 2$*.*

(*ii*) *If* $d = 3$*, then* $\overline{G}$ *has a spanning subgraph which is a double star.*

**Lemma 3.3.** *Let G be a self-centered graph of order* $n > 2$ *with* $v_i \in V(G)$*. Then* $\deg_G(v_i) \geq 2$*.*

**Proof.** To the contrary, we assume that $\deg_G(v_i) = 1$ with $v_i v_j \in E(G)$. Note that $n > 2$. Then $\varepsilon_G(v_i) = \varepsilon_G(v_j) + 1$. This is a contradiction since *G* is self-centered. □

Note that $\overline{P_3}$ is disconnected and $\overline{P_4} = P_4$. Moreover, it can be verified that $E_1(\overline{T}) < \xi^c(\overline{T})$ for the trees of order 5 except $DS_5(1,2)$ with $E_1(\overline{DS_5(1,2)}) > \xi^c(\overline{DS_5(1,2)})$. So we assume that $n > 5$ in the following theorem.

**Theorem 3.4.** Let *T* be a tree of order $n > 5$ with diameter $d > 2$. Then $E_1(\overline{T}) < \xi^c(\overline{T})$.

**Proof:** If $d = 3$, then $T \cong DS_n(n_1, n_2)$ with $n_1 + n_2 = n - 2 > 2$, $n_1 \geq 1$ and $n_2 \geq 1$. Assume that the only two vertices with eccentricity 2 in T are $v_1$ and $v_2$ with $\deg_T(v_1) = n_1 + 1 \leq n_2 + 1 = \deg_T(v_2)$. Then $\varepsilon_{\overline{T}}(v_1) = \varepsilon_{\overline{T}}(v_2) = 3$, and all other vertices have the same eccentricity 2 in $\overline{T}$. Moreover, all vertices other than $v_1, v_2$ in $\overline{T}$ have degrees $n - 2$. If $n_1 = 1$, then $n_2 = n - 3$ and $\deg_{\overline{T}}(v_1) = n - 3$, $\deg_{\overline{T}}(v_2) = 1$. Note that $n > 5$. Then we have

$$\xi^c(\overline{T}) - E_1(\overline{T}) = 3(n-3-3) + 3(1-3) + 2(n-2)(n-2-2)$$
$$= 2n^2 - 9n - 8 > 0.$$

For $n_1 \geq 2$, similarly as above, we get

$$\xi^c(\overline{T}) - E_1(\overline{T}) = 2n^2 - 9n - 8 > 0.$$

If $d > 3$, then, by Lemma 3.2 (*i*), $\overline{T}$ has diameter $\overline{d} = 2$. Now we claim that $\overline{T}$ is a 2-self-centered graph. If not, $\overline{T}$ has a vertex $v_i$ with $\varepsilon_{\overline{T}}(v_i) = 1$, *i.e.*, $\deg_{\overline{T}}(v_i) = n - 1$. Thus $v_i$ is an isolated vertex in T, which is a contradiction since T is a tree. Observe that a connected 2-regular graph is just a cycle $C_n$ whose complement is not a tree for $n > 5$, in view of Lemma 3.3, we find that $\overline{T}$ is a 2-self-centered graph with $\deg_{\overline{T}}(v_i) \geq 2$ for any vertex $v_i$ and there is at least one vertex $v_j$ with $\deg_{\overline{T}}(v_j) > 2$. Thus $\xi^c(\overline{T}) - E_1(\overline{T}) > 0$, finishing the proof of the theorem. □

In the following theorem we give a sufficient condition for the graphs *G* of order *n* and with $E_1(G) < \xi^c(G)$.

**Theorem 3.5.** *Let G be a connected graph of order* $n \geq 5$ *with* $K_{2,n-2}$ *as its subgraph. Then* $E_1(G) < \xi^c(G)$*.*

**Proof.** If $G \cong K_{2,n-2}$, then we have $\xi^c(G) = 8(n-2)$ and $E_1(G) = 4n$. It follows that $\xi^c(G) > E_1(G)$ from $n \geq 5$. Therefore in the following we assume that *G* contains $K_{2,n-2}$ as a proper spanning subgraph.

Since *G* contains $K_{2,n-2}$ as a proper spanning subgraph, the eccentricity of any vertex in *G* is 1 or 2. Moreover, $\deg_G(v_k) \geq 2$ for any vertex $v_k \in V(G)$ with $\varepsilon_G(v_k) = 2$, and $d_G(v_i) = n - 1$ for any vertex $v_i \in V(G)$ with $\varepsilon_G(v_i) = 1$. If there is a vertex $v_k \in V(G)$ with degree $n - 1$, then $\xi^c(G) > E_1(G)$ immediately. Otherwise, *G* is 2-self-centered graph. Considering that $K_{2,n-2}$ is a proper spanning subgraph of *G*, there are at least two vertices $v_i, v_j$ with $\deg_G(v_i) > \varepsilon_G(v_i)$ and $\deg_G(v_j) > \varepsilon_G(v_j)$. So $\xi^c(G) > E_1(G)$, finishing the proof of the theorem. □

We can easily observe that the graph *G* in Theorem 3.5 has minimum degree at least 2 and $\deg_G(v_i) \geq \varepsilon_G(v_i)$ for any vertex $v_i \in V(G)$ with at least one strict inequality. But these conditions are not necessary for $\xi^c(G) > E_1(G)$. Next we will give an example with $\xi^c(G) > E_1(G)$ but not satisfying these above conditions. Denote by $G^*$ the graph obtained by attaching a pendant vertex to each of the vertices in a connected graph *G*. Clearly, for $n \geq 3$, any pendant vertex in $K_n^*$ has eccentricity 3, and any other vertex has a same eccentricity 2 in it. Thus $\xi^c(K_n^*) - E_1(K_n^*) = 2n(n-2) + 3n(1-3) = 2n(n-5) > 0$ if $n > 5$.

**Example 3.6.** $E_1(K_n^*) < \xi^c(K_n^*)$ *for* $n > 5$*.*

Note that all the graphs described in Theorems 3.4 and 3.5 have diameter at most 3. In Ref. [28], a class of graphs are constructed with exactly two distinct eccentricities (see Figure 1). Here $G_0$ is an arbitrary graph, each of whose vertices is adjacent to any vertex from $\{x_1, y_1, z_1, w_1\}$. The black vertices have eccentricity $r - 1$ and the white vertices have eccentricity *r* in G. The graph schematically shown in Figure 1 is denoted by $G = S(G_0; r-1, r)$ with diameter *r*. In the following theorem we prove the existence of graphs *G* with diameter more than 2 fulfilling $E_1(G) < \xi^c(G)$.

**Theorem 3.7.** *For any integer* $r \geq 2$*, there is a graph G with diameter r and fulfilling* $E_1(G) < \xi^c(G)$*.*

**Proof.** For any integer $r \geq 2$, we choose a graph $G = S(K_n; r-1, r)$ with $n \geq 3r$. From the structure of $S(K_n; r-1, r)$, we find that there are n vertices in $K_n$ of $S(K_n; r-1, r)$ with eccentricity $r - 1$ and degree $n + 3$, other vertices have a

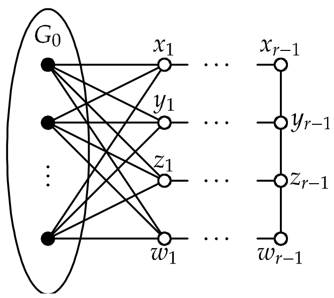


Figure 1. Graph *G* with only two eccentricities $r - 1$ and *r*.

same eccentricity $r$ in $S(K_n;r-1,r)$. Note that $n \geq 3r$ and $r \geq 2$. Then

$$\begin{aligned}\xi^c(G)-E_1(G) &= \left[n+3-(r-1)\right]n(r-1)+(3-r)2r \\ &\quad +(2-r)\left[4(r-3)+2\right]r+(n+1-r)4r \\ &= (n+4-r)n(r-1)+(n+1-r)4r \\ &\quad -2r(2r^2-8r+7) \\ &\geq 3r(2r+4)(r-1)+(2r+1)4r \\ &\quad -2r(2r^2-8r+7) \\ &= 2r(r^2+15r-11)>0.\end{aligned}$$

This completes the proof of the theorem. □

## 4. AN EFFECTIVE CONSTRUCTION

From Example 3.1, we would like to give a more general result for generating infinite graphs with different comparison relations. To do it, we first prove a useful lemma as follows.

**Lemma 4.1.** *Let G be a connected graph of order* $n>2$ *and* $H=G\square K_2$. *Then* $\varepsilon_H(w)=\varepsilon_G(w)+1$ *for any vertex* $w\in V(H)$.

**Proof.** Assume that $V(G)=\{v_1,v_2,\ldots,v_n\}$ and $V(H)=\{v_1,v_2,\ldots,v_n,v_1',v_2',\ldots,v_n'\}$ with $H[\{v_1,v_2,\ldots,v_n\}]=H[\{v_1',v_2',\ldots,v_n'\}]\cong G$ where $v_i'$ is the copy of vertex $v_i$ in $H$ for $i=1,2,\ldots,n$. By symmetry, it suffices to prove that $\varepsilon_H(v_i)=\varepsilon_G(v_i)+1$ for $i=1,2,\ldots,n$.

Assume that $\varepsilon_G(v_i)=k$ for an arbitrary vertex $v_i\in V(G)\subseteq V(H)$. Then there exists a vertex $v_j$ as an eccentric vertex of $v_i$ in $G$. Therefore $\varepsilon_H(v_i)\geq d_H(v_i,v_j')=k+1$ from the structure of $H$. Next we prove that $\varepsilon_H(v_i)\leq k+1$. Otherwise, we have $\varepsilon_H(v_i)\geq k+2$. Then there is a vertex $v_m'\in V(H)\setminus V(G)$ with $d_H(v_i,v_m')\geq k+2$. By the structure of $H$, again, we have $d_G(v_i,v_m)\geq k+1$, contradicting the fact that $\varepsilon_G(v_i)=k$. So $\varepsilon_H(v_i)=k+1=\varepsilon_G(v_i)+1$, finishing the proof of the lemma. □

**Theorem 4.2.** Assume that $G$ is a connected graph of order $n\geq 2$ with $\varepsilon_G(v_i)\geq \deg_G(v_i)$ for any vertex $v_i\in V(G)$. Then $E_1(G\square K_2)\geq \xi^c(G\square K_2)$.

**Proof.** Let $H=G\square K_2$. From the structure of $H$, we have $\deg_H(w)=\deg_G(w)+1$ for any vertex $w\in V(G)$. Thus $E_1(G)\geq \xi^c(G)$. By Lemma 4.1, we have

$$\begin{aligned}&E_1(G\square K_2)-\xi^c(G\square K_2)\\ &=2\sum_{v_i\in V(G)}(\varepsilon_G(v_i)+1)\left[(\varepsilon_G(v_i)+1)-(\deg_G(v_i)+1)\right]\\ &\geq 4(E_1(G)-\xi^c(G))\geq 0.\end{aligned}$$

Therefore, our result holds immediately. □

Similarly as above, we can easily obtain the corollary below.

**Corollary 4.3.** *Assume that G is a connected graph of order* $n>2$ *with* $\varepsilon_G(v_i)<\deg_G(v_i)$ *for any vertex* $v_i\in V(G)$. *Then* $E_1(G\square K_2)<\xi^c(G\square K_2)$.

By Theorem 4.2 and Corollary 4.3, we can get infinite graphs with each possibility for comparison between the first Zagreb eccentricity index and eccentric connectivity index.

Recall that the graph $G^*$ is defined in Section 3. In addition to Example 3.1 for $E_1=\xi^c$, by Theorem 4.2 and Corollary 4.3, we can give the following examples for other comparative inequalities between $E_1$ and $\xi^c$.

**Example 4.4.** $E_1(C_n^*\square K_2)>\xi^c(C_n^*\square K_2)$ for any $n\geq 4$, $E_1(K_{n_1,n_2}\square K_2)<\xi^c(K_{n_1,n_2}\square K_2)$ for $n_2\geq n_1>2$.

## 5. CONCLUSION

In this paper we present some results on the comparison between $E_1(G)$ and $\xi^c(G)$ on the graphs $G$ including some chemical graphs. In particular, we show that $E_1(T)>\xi^c(T)$ for any trees $T$ including chemical trees. Some sufficient conditions are obtained on the graphs $G$ with $E_1(G)\leq \xi^c(G)$. Moreover, we also give a construction method for generating infinite graphs $G$ for each comparative inequality between $E_1(G)$ and $\xi^c(G)$. Now it seems to be an open and attractive problem to characterize completely chemical graphs with some comparative inequality between $E_1$ and $\xi^c$.

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