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A ZERO-SUM GAME APPROACH FOR H_∞ ROBUST CONTROL OF SINGULARLY PERTURBED BILINEAR QUADRATIC SYSTEMS

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Original scientific paper
A zero-sum game approach for H_∞ robust control of continuous-time singularly perturbed bilinear quadratic systems with an additive disturbance input is presented. By regarding the stochastic disturbance (or the uncertainty) as "the nature player", the H_∞ robust control problem is transformed into a two-person zero-sum dynamic game model. By utilizing the singular perturbation decomposition method to solve the composite saddle-point equilibrium strategy of the system, the H_∞ robust control strategy of the original singularly perturbed bilinear quadratic systems is obtained. A numerical example of a chemical reactor model is considered to verify the efficiency of the proposed algorithm.

Keywords: *bilinear quadratic system; H_∞ robust control; singularly perturbed; zero-sum game theory*

Pristup igre nulte-sume za H_∞ robusno reguliranje jedinstveno perturbiranih bilinearnih kvadratnih sustava

Izvorni znanstveni članak
U radu se opisuje pristup igre nulte sume za H_∞ robusno reguliranje trajnih jedinstveno perturbiranih bilinearnih kvadratnih sustava s dodatnim unosom smetnji. Smatrajući stohastičke smetnje (ili nesigurnost) kao "igrača prirode", problem H_∞ robusnog reguliranja pretvara se u model dinamičke igre nulte-sume za dvije osobe. Primjenom metode dekompozicije singularne perturbacije za rješavanje složene strategije ravnoteže točke opterećenja toga sustava, dobiva se strategija H_∞ robusnog reguliranja originalnih jedinstveno uznemirenih bilinearnih kvadratnih sustava. Provjera učinkovitosti predloženog algoritma daje se na numeričkom primjeru modela kemijskog reaktora.

Ključne riječi: *bilinearni kvadratni sustav; H_∞ robusno reguliranje; jedinstveno perturbiran; teorija igre nulte sume*

1 Introduction

Robust control is a branch of control theory that explicitly deals with uncertainty in its approach to controller design. The established game theory can be used to solve a robust control problem. The idea is to regard the control designer as one player, and the stochastic disturbance (or the uncertainty) as "the nature player". Thus a robust control problem is converted into a two-player game problem, that is when anticipating the nature player's various disturbance, how the controller will design his strategy to optimize his goal, and at the same time to realize the equilibrium with the nature player. Then by solving the saddle-point equilibrium strategy or the Nash equilibrium strategy, the robust control strategy of various performance indexes can be further obtained.

The approach of game theory has achieved great success in the robust control of linear systems. David J. N. Limebeer et al. studied a H_∞ control problem for linear time-varying systems using a game theoretic approach [1], Ihnseok Rhee and Jason L. Speyer considered a finite-time interval disturbance attenuation problem for a time-varying system with uncertainty in the initial conditions of state based on a LQ game theoretic formulation where the control plays against adversaries composed of the process and measurement disturbances and initial conditions [2]. T. Basar showed that the discrete-time disturbance rejection problem, formulated in finite and infinite horizons, and under perfect state measurements, can be solved by making direct use of some results on linear-quadratic zero-sum dynamic games [3]. Dan Shen and Jose B. Cruz converted H_∞ optimal control problems with linear quadratic objective functions to a regular optimal regulator problem by improving a game theory based approach [4]. Huai-nian Zhu, Chengke-Zhang et al. presented a Nash game approach to obtain a class of stochastic H_2/H_∞ control for continuous-time Markov jump linear systems [5]. Hiroaki Mukaidani presented that the H_2/H_∞ robust control problem for linear stochastic system governed by *Itô* differential equation could be formulated as a Stackelberg differential game where the leader minimizes an H_2 criterion while the follower deals with the H_∞ constraint [6]. Hai-ying Zhou, Huai-nian Zhu et al. discussed linear quadratic stochastic zero-sum differential games for discrete-time Markov jump systems, and constructed the explicit expressions of the optimal strategies [7]. Tian-liang Zhang, Yu-hong Wang et al. reviewed newly development in H_2/H_∞ control of stochastic linear systems with multiplicative noise based on Nash game approach [8].

However, game theories for singularly perturbed bilinear systems are seldom discussed, while singularly perturbed bilinear systems are a quite proper and essential description tool in describing many practical systems such as neutron level control problem in a fission reactor, dc-motor, induction motor drives [9], and in financial engineering problems, Black-Scholes Option Pricing Model, M. Aoki's two sector macroeconomic growth model, P. Chander and F. Tokao's non-linear input-output model can all be extended to singularly perturbed bilinear models in [10÷12].

H_∞ robust control of singularly perturbed bilinear quadratic systems is studied in this paper. By regarding the stochastic disturbance (or the uncertainty) as "the nature player", the H_∞ robust control problem is transformed into a two-person zero-sum dynamic game model. Utilizing the singular perturbation decomposition method to solve the composite saddle-point equilibrium strategy of the system, we obtain the H_∞ robust control strategy of the original singularly perturbed bilinear quadratic systems.

2 Preliminaries

We introduce some necessary notations. Let $R^{n\times 1}$ denote the n-dimensional vector space and let the norm of a vector $\boldsymbol{x} = [x_1 x_2 \dots x_n]^{\mathrm{T}}$ be denoted by

$$\|\boldsymbol{x}\| = (x * x)^{1/2} = \left(\sum_{i=1}^{n} |x_i|^2\right)^{1/2}$$

The norm of matrix $\boldsymbol{A} \in R^{n\times m}$ is defined by

$$\|\boldsymbol{A}\| = \left(\sum_{i=1}^{n}\sum_{j=1}^{m} a_{ij}^2\right)^{1/2}$$

and the weighted 2-norm $\|\boldsymbol{x}\|_Q^2$ is defined by $x^{\mathrm{T}}Qx$.

Consider the following nonlinear system

$$\begin{aligned} \dot{x} &= f(x) + g(x)u \\ y &= h(x) \end{aligned} \tag{1}$$

with $f(0) = 0$, $h(0) = 0$ and $\boldsymbol{x} \in R^n$ is a state vector, $\boldsymbol{u} \in R^m$ is an input vector, and $\boldsymbol{y} \in R^p$ is a measurable vector. Then the definition of finite $L_{2\text{-gain}}$ is as follows:

Definition 1 [13]: Let $\gamma \geq 0$. System (1) is said to have $L_{2\text{-gain}}$ less than or equal to g if

$$\int_0^T \|y(t)\|^2 \mathrm{d}t \leq \int_0^T \gamma^2 \|u(t)\|^2 \mathrm{d}t \tag{2}$$

or to have weighted $L_{2\text{-gain}}$ not larger than g if

$$\int_0^T \|y(t)\|_Q^2 \mathrm{d}t \leq \int_0^T \gamma^2 \|u(t)\|_R^2 \mathrm{d}t \tag{3}$$

for all $T \geq 0$ and all $\boldsymbol{u} \in L_2(0,T)$, with positive definite matrices Q, R and $y(t) = h(\varphi(t,0,x_0,u))$ denoting the output of (1) resulting from u for initial state $x(0) = 0$. The system has $L_{2\text{-gain}} < \gamma$ if there exists some $0 \leq \tilde{\gamma} < \gamma$ such that (3) holds for $\tilde{\gamma}$.

3 Problem statement

Consider the H_∞ robust control strategy for the following time-invariant singularly perturbed bilinear system:

$$\begin{bmatrix} \dot{x}_1(t) \\ \varepsilon\dot{x}_2(t) \end{bmatrix} = A\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \left(B + \left\{\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}\begin{bmatrix} M_s \\ M_f \end{bmatrix}\right\}\right)u(t) + Ew(t) \tag{4a}$$

$$z(t) = C\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + Du(t) \tag{4b}$$

with initial condition

$$\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} x_{10} \\ x_{20} \end{bmatrix}$$

where $A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$, $B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}$, $E = \begin{bmatrix} E_1 \\ E_2 \end{bmatrix}$, $x_1(t) \in R^{n_1}$, $x_2(t) \in R^{n_2}$ are respectively slow and fast state variables, $x(t) = [x_1(t), x_2(t)]^{\mathrm{T}} \in R^n$ are state vectors with $n_1 + n_2 = n$, $\boldsymbol{u} \in R^m$ is a control vector, $\boldsymbol{w} \in R^l$ denotes the disturbance input, $z(t) \in R^q$ is the penalty function to be used in the cost function, the small singular perturbation parameter $\varepsilon > 0$ represents small time constants, inertias, masses, etc. , and A_{ij}, B_i, E_i, M_i, C, D (i, j = 1, 2) are constant matrices of appropriate dimensions, with $\left\{\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}\begin{bmatrix} M_s \\ M_f \end{bmatrix}\right\} = \sum_{j=1}^{n_1} x_{1j}\begin{bmatrix} M_{sj} \\ M_{fj} \end{bmatrix} + \sum_{j=n_1+1}^{n_1+n_2} x_{2j}\begin{bmatrix} M_{sj} \\ M_{fj} \end{bmatrix}$ let $\tilde{B}(x) = \begin{bmatrix} \tilde{B}_1(x) \\ \tilde{B}_2(x) \end{bmatrix} = B + \left\{\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}\begin{bmatrix} M_s \\ M_f \end{bmatrix}\right\}$, then the state Eq. (4a) can be written as:

$$\begin{bmatrix} \dot{x}_1(t) \\ \varepsilon\dot{x}_2(t) \end{bmatrix} = A\begin{bmatrix} x_1(t) \\ x_2(\mathrm{t}) \end{bmatrix} + \tilde{B}u(t) + Ew(t) \tag{5}$$

Letting

$$x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}, A_\varepsilon = \begin{bmatrix} A_{11} & A_{12} \\ \frac{A_{21}}{\varepsilon} & \frac{A_{22}}{\varepsilon} \end{bmatrix}, \tilde{B}_\varepsilon = \begin{bmatrix} \frac{\tilde{B}_1}{\varepsilon} \\ \frac{\tilde{B}_2}{\varepsilon} \end{bmatrix}, E_\varepsilon = \begin{bmatrix} \frac{E_1}{\varepsilon} \\ \frac{E_2}{\varepsilon} \end{bmatrix},$$

(4a) can be further written as follows:

$$\dot{x}(t) = A_\varepsilon x(t) + \tilde{B}_\varepsilon u(t) + E_\varepsilon w(t). \tag{6}$$

Thus, the nonlinear robust H_∞ control guarantees that the performance index (7) remains within an upper bound for a given positive number γ.

$$J(u,w) = \frac{1}{2}\int_0^\infty \left\{\|z\|^2 - \gamma^2\|w\|^2\right\}\mathrm{d}t \tag{7}$$

The basic game theory idea of robust control design is to regard the control designer as one player P1, and the stochastic disturbance (or the uncertainty) as "the nature player" P2. Thus a robust control problem is converted into a two-player game problem, that is when anticipating the nature player P2's various disturbance, how the controller P1 will design his strategy to optimize his goal, and at the same time to realize the equilibrium with the nature player. Accordingly, the design method of H_∞ robust control for system (4) is that: the $w^*(t, x)$ tries to maximize the energy, while the controller or $u^*(t, x)$ simultaneously seeks to minimize it.

Then the problem is converted to solve the equilibrium strategy $u^*(t, x)$ for the player P1, and the equilibrium strategy $w^*(t, x)$ for the player P2, which satisfy the following condition:

$$J(u^*, w) \le J(u^*, w^*) \le J(u, w^*).$$

Thus a two-player zero-sum dynamic game for players P1 and P2 is constructed.

4 Decomposition of slow and fast systems

Assumption 1 [13]: The pair (A, B) is completely controllable and x stays in the controllability domain defined by

$$X_c \overset{\Delta}{=} \{x \in R^n \mid (A, B + \{xM\})\, controllable\}$$

Assumption 2 [13]: The differential Eq. (4) has a solution defined on $[0,+\infty)$ for each admissible input function and $x(t) \to 0$ as $t \to \infty$.

The Hamiltonian $H(t)$ corresponding to the system (4) and performance (7) is:

$$H = \frac{1}{2}\left\{\|z\|^2 - \gamma^2\|w\|^2\right\} + \lambda^{\mathrm{T}}\left(A_\varepsilon x + \widetilde{B}_\varepsilon u + E_\varepsilon w\right) \tag{8}$$

where $\lambda \in R^{n\times 1}$ is the Lagrangian multiplier.

Then the optimal control is given by:

$$u = -R_1^{-1}\widetilde{B}_\varepsilon^{\mathrm{T}}\frac{\partial J(x)}{\partial x} \tag{9}$$

And worst disturbance input is:

$$w = \frac{1}{\gamma^2}E_\varepsilon^{\mathrm{T}}\frac{\partial J(x)}{\partial x} \tag{10}$$

And $J(x)$ satisfies the following HJI equation

$$\begin{aligned} 0 = &\frac{1}{2}x^{\mathrm{T}}C^{\mathrm{T}}Cx - \frac{1}{2}\left(\frac{\partial J(x)}{\partial x}\right)^{\mathrm{T}}\widetilde{B}_\varepsilon R_1^{-1}\widetilde{B}_\varepsilon^{\mathrm{T}}\frac{\partial J(x)}{\partial x} + \\ &+ \frac{1}{2\gamma^2}\left\|E_\varepsilon^{\mathrm{T}}\frac{\partial J(x)}{\partial x}\right\|^2 + \left(\frac{\partial J(x)}{\partial x}\right)^{\mathrm{T}}A_\varepsilon x \end{aligned} \tag{11}$$

under the assumption of $D^{\mathrm{T}} = 0$ and $R_1 = D^{\mathrm{T}}D > 0$.
Neglecting the fast modes is equivalent to assuming that they are infinitely fast, that is letting $\varepsilon = 0$. Without the fast modes the system (5) reduces to:

$$\dot{x}_1 = A_{11}x_1 + A_{12}x_2 + \widetilde{B}_1 u + E_1 w \tag{12a}$$

$$0 = A_{21}x_1 + A_{22}x_2 + \widetilde{B}_2 u + E_2 w \tag{12b}$$

Assuming that A_{22} is non-singular, we have

$$\dot{x}_{1s} = A_0 x_{1s} + \widetilde{B}_{01}u_s + E_0 w_s,\ x_{1s} = x_{10} \tag{13a}$$

$$x_{2s} = -A_{22}^{-1}(A_{21}x_{1s} + \widetilde{B}_2 u_s + E_2 w_s) \tag{13b}$$

where $A_0 = A_{11} - A_{12}A_{22}^{-1}A_{21}$, $\widetilde{B}_{01} = \widetilde{B}_1 - A_{12}A_{22}^{-1}\widetilde{B}_2$, $E_0 = E_1 - A_{12}A_{22}^{-1}E_2$ let $C^{\mathrm{T}}C = Q_1 = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{12}^{\mathrm{T}} & Q_{22}\end{bmatrix}$, and then $x^{\mathrm{T}}C^{\mathrm{T}}Cx = \begin{bmatrix} x_1 \\ x_2\end{bmatrix}^{\mathrm{T}} Q_1 \begin{bmatrix} x_1 \\ x_2\end{bmatrix}$.

Substituting the above into (7), we can obtain the quadratic cost function for the slow subsystem

$$\begin{aligned} J_s(u, w) &= \frac{1}{2}\int_0^\infty \left\{\|z\|^2 - \gamma^2\|w\|^2\right\}\mathrm{d}t = \\ &= \frac{1}{2}\int_0^\infty \left(x^{\mathrm{T}}Q_1 x + u^{\mathrm{T}}R_1 u - \gamma^2 w^{\mathrm{T}}w\right)\mathrm{d}t = \\ &= \frac{1}{2}\int_0^\infty \left(x_{1s}^{\mathrm{T}}Q_0 x_{1s} + 2x_{1s}^{\mathrm{T}}D_1 u_s + 2x_{1s}^{\mathrm{T}}D_2 w_s + \right. \\ &\quad \left. + 2u_{1s}^{\mathrm{T}}D_3 w_s + u_{1s}^{\mathrm{T}}R_{1s}u_s - w_s^{\mathrm{T}}R_{2s}w_s\right)\mathrm{d}t \end{aligned} \tag{14}$$

where

$$Q_0 = Q_{11} + A_{21}^{\mathrm{T}}A_{22}^{-\mathrm{T}}Q_{22}A_{22}^{-1}A_{21},\ D_1 = A_{21}^{\mathrm{T}}A_{22}^{-\mathrm{T}}Q_{22}A_{22}^{-1}\widetilde{B}_2,$$
$$D_2 = A_{21}^{\mathrm{T}}A_{22}^{-\mathrm{T}}Q_{22}A_{22}^{-1}E_2,\ D_3 = \widetilde{B}_2^{\mathrm{T}}A_{22}^{-\mathrm{T}}Q_{22}A_{22}^{-1}E_2,$$
$$R_{1s} = R_1 + \widetilde{B}_2^{\mathrm{T}}A_{22}^{-\mathrm{T}}Q_{22}A_{22}^{-1}\widetilde{B}_2,\ R_{2s} = \gamma^2 - E_2^{\mathrm{T}}A_{22}^{-\mathrm{T}}Q_{22}A_{22}^{-1}E_2.$$

Assumption 3: The triplet $(A, B, \sqrt{Q})$ is stabilizable and detectable.

Theorem 1: Under Assumption 3, suppose that the following algebraic Riccati equation has solution p_s

$$A^{\mathrm{T}}p_s + p_s A - p_s B p_s + Q = 0 \tag{15}$$

Then the equilibrium solution of the slow subsystem can be given by

$$u_s^* = -(T_1 + T_2 p_s)x_{1s} \tag{16a}$$

$$w_s^* = (T_3 + T_4 p_s)x_{1s} \tag{16b}$$

Proof: Substituting (14) into (9), we can obtain the optimal control u_s of the slow subsystem:

$$u_s = -R_1^{-1}\widetilde{B}_{01}^{\mathrm{T}}\frac{\partial J_s(x)}{\partial x} = -R_{1s}^{-1}(D_1^{\mathrm{T}}x_{1s} + D_3 w_s + \widetilde{B}_{01}^{\mathrm{T}}\lambda) \tag{17a}$$

Substituting (14) into (10), we can obtain the worst disturbance input w_s of the slow subsystem:

$$w_s = \frac{1}{\gamma^2}E_0^{\mathrm{T}}\frac{\partial J_s(x)}{\partial x} = R_{2s}^{-1}(D_2^{\mathrm{T}}x_{1s} + D_3^{\mathrm{T}}u_s + E_0^{\mathrm{T}}\lambda) \tag{17b}$$

We get

$$u_s = -T_1 x_{1s} - T_2\lambda,\ w_s = T_3 x_{1s} + T_4\lambda$$

where

$$T_1=(R_{1s}^{-1}D_3R_{2s}^{-1}D_3^{\mathrm T}+1)^{-1}(R_{1s}^{-1}D_1^{\mathrm T}+R_{1s}^{-1}D_3R_{2s}^{-1}D_2^{\mathrm T})$$
$$T_2=(R_{1s}^{-1}D_3R_{2s}^{-1}D_3^{\mathrm T}+1)^{-1}(R_{1s}^{-1}D_3R_{2s}^{-1}E_0^{\mathrm T}+R_{1s}^{-1}\widetilde{B}_{01}^{\mathrm T})$$
$$T_3=R_{2s}^{-1}(D_2^{\mathrm T}-D_3^{\mathrm T}T_1)$$
$$T_4=R_{2s}^{-1}(E_0^{\mathrm T}-D_3^{\mathrm T}T_2)$$

For $-\dot{\lambda}=Q_0x_{1s}+D_1u_s+D_2w_s+A_0^{\mathrm T}\lambda$, letting $\lambda=p_sx_{1s}$, we have

$$p_s(A_0-\widetilde{B}_{01}T_1+E_0T_3)+(A_0^{\mathrm T}-D_1T_2+D_2T_4)p_s+ + p_s(E_0T_4-\widetilde{B}_{01}T_2)p_s+(Q_0-D_1T_1+D_2T_3)=0 \tag{18}$$

Because $(A_0-\widetilde{B}_{01}T_1+E_0T_3)^{\mathrm T}=A_0^{\mathrm T}-D_1T_2+D_2T_4$, let $A=A_0-\widetilde{B}_{01}T_1+E_0T_3$, $B=\widetilde{B}_{01}T_2-E_0T_4$, $Q=Q_0-D_1T_1+D_2T_3$.

Then (18) can be written as the following algebraic Riccati equation:

$$A^{\mathrm T}p_s+p_sA-p_sBp_s+Q=0$$

where p_s is the solution of the above Riccati equation.

For convenience, let $G_0=-\widetilde{B}_{21}(T_1+T_2p_s)-\widetilde{B}_{22}(T_3+T_4p_s)$, then $x_{2s}=-A_{22}^{-1}(A_{21}+G_0)x_{1s}$.

In the fast subsystem, we assume that the slow variables are constant in the boundary layer. Redefining the fast variables $x_{2f}=x_2-x_{2s}$, and the fast controls $u_f=u-u_s$, $w_f=w-w_s$, the fast subsystem is formulated as:

$$\dot{x}_{2f}=\frac{1}{\varepsilon}A_{22}x_{2f}+\frac{1}{\varepsilon}\widetilde{B}_2u_f+\frac{1}{\varepsilon}E_2w_f,\quad x_{2f}(0)=x_{20}-x_{2s}(0). \tag{19}$$

Then we can obtain the quadratic cost function for the fast subsystem

$$J_f(u,w)=\frac{1}{2}\int_0^{\infty}\left\{\|z\|^2-\gamma^2\|w\|^2\right\}\mathrm{d}t= =\frac{1}{2}\int_0^{\infty}\left(x_{2f}^{\mathrm T}Q_{22}x_{2f}+u_f^{\mathrm T}R_1u_f-\gamma^2w_f^{\mathrm T}w_f\right)\mathrm{d}t \tag{20}$$

Assumption 4: the triplet ($A_{22},\widetilde{B}_{21},\sqrt{Q_{22}}$) and ($A_{22}$, $\widetilde{B}_{22},\sqrt{Q_{22}}$) are stabilizable and detectable.

Theorem 2: Under Assumption 4, suppose that the following algebraic Riccati equation has solution p_f

$$p_fA_{22}+A_{22}^{\mathrm T}p_f- -p_f\left(B_2R_1^{-1}\widetilde{B}_2^{\mathrm T}-E_2\frac{E_2^{\mathrm T}}{\gamma^2}\right)p_f+Q_{22}=0 \tag{21}$$

Then the equilibrium strategies of the fast subsystem are given by

$$u_f^*=-R_1^{-1}\widetilde{B}_2^{T}p_fx_{2f} \tag{22a}$$

$$w_f^*=\frac{E_2^{\mathrm T}}{\gamma^2}p_fx_{2f} \tag{22b}$$

The proof is similar to that of the slow subsystem.

5 Composite strategy

The composite strategy pair of the full-order singularly perturbed system (4) is constructed as follows [14]:

$$u_c=u_s^*+u_f^*=-(T_1+T_2p_s)x_{1s}-R_1^{-1}\widetilde{B}_2^{\mathrm T}p_fx_{2f} \tag{23a}$$

$$w_c=w_s^*+w_f^*=(T_3+T_4p_s)x_{1s}+\frac{E_2^{\mathrm T}}{\gamma^2}p_fx_{2f} \tag{23b}$$

with x_1 replacing x_{1s}, x_2 replacing $x_{2s}+x_{2f}$, for $x_{2s}=-A_{22}^{-1}(A_{21}x_{1s}+\widetilde{B}_2u_s+E_2w_s)$, we obtain

$$u_c=G_1x_1+G_2x_2 \tag{24a}$$
$$w_c=G_3x_1+G_4x_2 \tag{24b}$$

where

$$G_1=-(T_1+T_2p_s)+R_1^{-1}\widetilde{B}_2^{\mathrm T}p_fA_{22}^{-1}[-A_{21}+\widetilde{B}_2(T_1+T_2p_s)- -E_2(T_3+T_4p_s)]$$
$$G_2=-R_1^{-1}\widetilde{B}_2^{\mathrm T}p_f$$
$$G_3=(T_3+T_4p_s)-\frac{E_2^{\mathrm T}}{\gamma^2}p_fA_{22}^{-1}[-A_{21}+\widetilde{B}_2(T_1+T_2p_s)- -E_2(T_3+T_4p_s)]$$
$$G_4=\frac{E_2^{\mathrm T}}{\gamma^2}p_f$$

The composite strategy pair constitutes an $o(\varepsilon)$ (near) saddle-point equilibrium of the full-order game. The proof can be found in [15].

6 Numerical example

The bilinear model of a chemical reactor [16] is given by

$$\begin{bmatrix}\dot{x}_1(t)\\ \varepsilon\dot{x}_2(t)\end{bmatrix}=A\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix}+(B+x_1M_1+x_2M_2)u(t)+Ew(t)$$
$$z(t)=C\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix}+Du(t)$$

Where

$$A=\begin{bmatrix}3/16 & 5/12\\ -50/3 & -8/3\end{bmatrix},B=\begin{bmatrix}-1/8\\ 0\end{bmatrix},$$

$$M_1 = \begin{bmatrix} -1 \\ 0 \end{bmatrix},\ M_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix},\ E = \begin{bmatrix} 0 \\ 1 \end{bmatrix},\ C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},\ D = \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

$$Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},\ x_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

and x_1 and x_2 represent the temperature and concentration of a chemical reaction while u represents the coolant flow rate around the reactor. We choose $\gamma = 0,5$ and $\varepsilon = 0,001$, and obtain the simulation curves for the optimal control strategy and the state as follows:

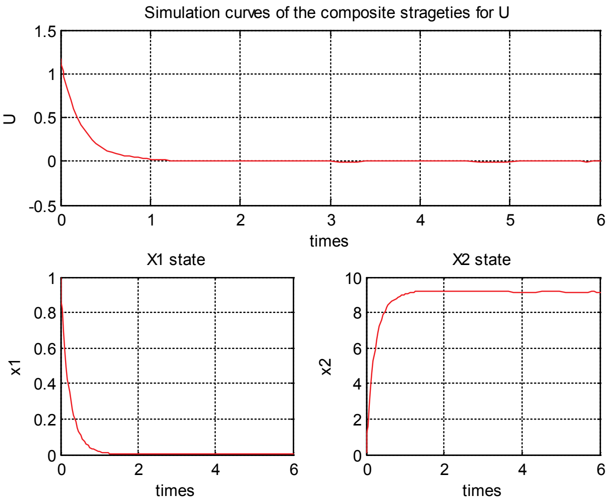

Figure 1 Simulation curves of the control strategy and the state trajectories

7 Conclusions

Many real systems possess the structure of the singularly perturbed bilinear control systems such as motor drives, robust control, multi-sector input-output analysis and option pricing. A game approach for H_∞ robust control of continuous-time singularly perturbed bilinear quadratic systems with an additive disturbance input is presented in this paper. By regarding the stochastic disturbance (or the uncertainty) as "the nature player", the H_∞ robust control problem is transformed into a two-person zero-sum dynamic game model. By utilizing the singular perturbation decomposition method to solve the composite saddle-point equilibrium strategy of the system, the H_∞ robust control strategy of the original singularly perturbed bilinear quadratic systems is obtained. A numerical example of a chemical reactor model is considered to verify the efficiency of the proposed algorithm. The conclusion obtained in this paper could be applied to deal with many industry engineering and financial engineering problems.

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