GLASNIK MATEMATIČKI
Vol. 52(72)(2017), 1 – 10

GEOMETRIC PROGRESSIONS ON ELLIPTIC CURVES

Abdoul Aziz Ciss and Dustin Moody
École Polytechnique de Thiès, Sénégal and National Institute of Standards and Technology, United States of America

Abstract. In this paper, we look at long geometric progressions on different models of elliptic curves, namely Weierstrass curves, Edwards and twisted Edwards curves, Huff curves and general quartics curves. By a geometric progression on an elliptic curve, we mean the existence of rational points on the curve whose x-coordinates (or y-coordinates) are in geometric progression. We find infinite families of twisted Edwards curves and Huff curves with geometric progressions of length 5, an infinite family of Weierstrass curves with 8-term progressions, as well as infinite families of quartic curves containing 10-term geometric progressions.

1. Introduction

Recently, several researchers have explored arithmetic and geometric progressions on various families of plane curves. By a progression on a curve, we mean there is a sequence of rational points on the curve whose x-coordinates (or y-coordinates) form an arithmetic or geometric progression. The historical motivation for this problem on elliptic curves seems to be an apparent connection between long progressions and high ranks for the corresponding Mordell-Weil groups (see [10, 18] for a lengthier discussion). Perhaps for this reason, much of the work in this area has pertained to elliptic (or hyperelliptic) curves.

Bremner ([4]), Campbell ([7]), Garcia-Selfa and Tornero ([10]), have looked at arithmetic progressions on elliptic curves defined by Weierstrass equations, while Campbell ([7]), MacLeod ([15]) and Ulas ([21]) have investigated progressions on curves represented by quartic models. Alvarado ([1]) and Ulas ([22]) extended similar results to genus 2 curves. In addition, Moody considered some alternate models of elliptic curves, Edwards curves ([16]) and

2010 *Mathematics Subject Classification.* 11B25, 11D41, 11G05.
Key words and phrases. Arithmetic progression, geometric progression, elliptic curves.

Huff curves ([17]), finding infinite families with long arithmetic progressions. Choudry ([8]), Bremner ([5]), and Gonzalez-Jiménez ([12]) also studied arithmetic progressions on Edwards and Huff curves.

While not having been explored quite as intensively, some researchers have also looked at geometric progressions on certain curves. Bérczes and Ziegler ([2]) examined geometric progressions on the set of solutions of Pell equations. They found infinitely many Pell equations containing geometric progressions of length 5. Bremner and Ulas ([6]) showed that for integers $n \geq 3$, there exist polynomials $a, b \in \mathbb{Z}[t]$ such that the curve defined by $y^2 = a(t)x^n + b(t)$ contains five points in geometric progression. They also show that there exist infinitely many parabolas $y^2 = ax + b$ which contain five points in geometric progression. González-Jiménez ([11]) studied geometric progressions on the set of solutions of the Markoff-Rosenberger equation $ax^2 + by^2 + cz^2 = dxyz$.

In this work, we look at finding long geometric progressions on different models of elliptic curves, including Weierstrass curves, Edwards curves, Huff curves and quartic curves. The main result of this paper is to show infinite families of (twisted) Edwards curves, and Huff curves containing 5 points in geometric progression, an infinite family of Weierstrass curves containing eight points in progression, as well as infinite families of quartic curves with a 10-term progression.

2. Edwards curves

An Edwards curve ([9]) is given by $E_d := x^2 + y^2 = 1 + dx^2y^2$, with $d \neq 0, 1$. The main result of this section is to show infinitely many Edwards curves with geometric progression of length 4. It is notable that with only one parameter, we obtain a progression almost as long as the Weierstrass case (which had four parameters).

Theorem 2.1. *There are infinitely many choices for d such that the Edwards curve E_d contains 4 points in geometric progression (of the x-coordinates).*

Proof. If we set the point (g, y_1) to be on E_d, then we can solve for d,

$$d = \frac{g^2 + y_1^2 - 1}{g^2 y_1^2}.$$

Using this value of d, we seek to force (g^2, y_2) to be on the Edwards curve E_d. This leads to the constraint of

$$-g^2y_2^2 + g^2y_1^2 - y_1^2y_2^2 + y_1^2 = 0.$$

This is a quadratic equation in y_1 and y_2, with parametric solutions given by

$$y_1 = -\frac{2mg^2 - g^2 - 1 + 2m - m^2}{1 - m^2 + g^2},$$

$$y_2 = -\frac{2mg^2 - g^2 - 1 + 2m - m^2}{g^2 + 1 - 2m + m^2}.$$

We now seek to force (g^3, y_3) to be on the curve E_d. Some simple arithmetic leads to a quadratic in y_3, whose discriminant modulo squares is

$$(g^4 + g^2 + 1)\left((g^2 - g + 1)m^2 + (-2g^2 - 2)m + 1 + g + 2g^2 + g^3 + g^4\right) \cdot \left((g^2 + g + 1)m^2 + (-2g^2 - 2)m + 1 - g + 2g^2 - g^3 + g^4\right).$$

The point (g^3, y_3) will be a rational point on E_d if and only if the discriminant is a square. A simple computation shows that we can take $m = 2(g^2 + g + 1)(g^2 - g + 1)(g^2 + 1)/(g^6 + 2g^4 + g^2 + 1)$. With this value of m, then we get points with x-coordinates $1, g, g^2$, and g^3 all on the Edwards curve, leading to infinitely many 4-term geometric progressions. We were not able to extend this to a 5-term progression. □

EXAMPLE 2.2. Let $g = 2$, so then $m = 210/101$. Then the points $(1, 0)$, $(2, 23399/1381)$, $(4, 23399/10537)$, and $(8, 23399/11481)$ are all on the Edwards curve E_d, with $d = 138308671/547513201$.

The curve E_d is birationally equivalent to the Weierstrass curve

$$E : y^2 = x^3 + 1371643744/547513201x^2 + 167448347372520900/299770705269266401x.$$

The point $(1, 0)$ corresponds to a point of order 4 on E, while the other three points with x-coordinates 2, 4, and 8 map to three linearly independent points with x-coordinates $-363594243/547513201$, $-620366415/2190052804$, and $-2237109903/8760211216$. The independence was checked by computing (using SAGE [19]) the determinant of the height pairing matrix, which is $145.76009242 \neq 0$.

Using Silverman's specialization theorem [20, III.11.4], Example 2.2 yields a lower bound on the rank of the free part of E_d over $\mathbb{Q}(g)$, where $d = d(g)$ is as given in Theorem 2.1. As we had three linearly independent points when $g = 2$, the rank is at least 3.

3. Twisted Edwards curves

A twisted Edwards curve ([3]) is given by $E_{a,d} : ax^2 + y^2 = 1 + dx^2y^2$, with $ad(a - d) \neq 0$. Edwards curves are thus a special case of twisted Edwards curves where a is set to 1. We can get a 5-term geometric progression by considering twisted Edwards curves, although we do so working with the y-coordinates.

THEOREM 3.1. *There are infinitely many twisted Edwards curves containing 5 points in geometric progression (of the y-coordinates).*

PROOF. Note that the point $(0,1)$ is always on the curve. To build a longer progression, we require the points (x_1, g) and (x_2, g^2) to be on the curve $E_{a,d}$. This leads to a system of two equations in the curve parameters a and d, which we solve:

$$a = \frac{x_1^2 + g^2 x_1^2 - g^2 x_2^2}{x_1^2 x_2^2},$$
$$d = \frac{x_1^2 + g^2 x_1^2 - x_2^2}{g^2 x_1^2 x_2^2}.$$

To extend the progression, we seek to have the point $(x_3, 1/g)$ on the curve. This is equivalent to the expression

$$-g^4 x_3^2 x_2^2 + x_3^2 g^4 x_1^2 - x_2^2 x_1^2 g^2 + 2x_3^2 x_1^2 g^2 - g^2 x_3^2 x_2^2 - x_3^2 x_2^2 + x_3^2 x_1^2 = 0.$$

We obtain a solution by the parameterization

$$x_1 = -\frac{g x_3 (4m^2 g^4 + 4g^4 m + 4g^2 m^2 + 4g^2 m - g^2 + 4m^2 + 4m)}{4m(g^2+1)(2mg^4 + 2mg^2 - g^2 + 2m)},$$

$$x_2 = -\frac{g(g^2+1) x_3 (4m^2 g^4 + 4g^4 m + 4g^2 m^2 + 4g^2 m - g^2 + 4m^2 + 4m)}{4(2g^8 + 5g^6 + 7g^4 + 5g^2 + 2)m^2 - 4g^2(g^4 + g^2 + 1)m + g^4}.$$

Finally, if the point $(x_4, 1/g^2)$ is on the curve, then we will have a 5-term progression. This leads to a quadratic equation in the variable x_3. The discriminant of this quadratic will be square, provided that

$$\begin{aligned} &16(5g^4+9g^2+5)(g^4+g^2+1)^2 m^4 + 32(g^4+g^2+1)(g^8-g^4+1)m^3 \\ &\quad + (16g^{12} + 40g^{10} + 96g^8 + 120g^6 + 96g^4 + 40g^2 + 16)m^2 \\ &\quad - 8g^2(g^4+g^2+1)^2 m + g^4(g^4+g^2+1) \end{aligned} \tag{3.1}$$

is square. We now show that if $5g^4 + 9g^2 + 5$ is square, then we can make the quartic (3.1) square. To do so, set

$$m = -\frac{16g^8 + 44g^6 + 59g^4 + 44g^2 + 16}{8(5g^4 + 9g^2 + 5)(2g^4 + 3g^2 + 2)}.$$

A simple calculation verifies that (3.1) yields a square, provided $5g^4 + 9g^2 + 5$ is square. By Lemma 3.2 (which will be proved below), we see that we have an infinite number of values of g such that $5g^4 + 9g^2 + 5$ is square. We therefore have an infinite number of twisted Edwards curves with y-coordinates $y = 1/g^2, 1/g, 1, g$, and g^2. □

LEMMA 3.2. *There are infinitely many values of g such that $5g^4 + 9g^2 + 5$ is square.*

PROOF. Note the point $(g, v) = (2, 11)$ is on the curve $5g^4 + 9g^2 + 5 = v^2$. Using standard transformations (see, for example, [23]), we have this curve is birationally equivalent to the elliptic curve $E : y^2 = x^3 - 127x - 546$. By

SAGE, it can be verified that the curve E has rank one. Given a point (x, y) on E, set

$$g_0 = 2\frac{23x - 11y + 176}{98x + 11y + 626}.$$

From the birationality, we have that $5g_0^4+9g_0^2+5$ will be square. For example, from the generator $P = (97/4, 825/8)$ of the free part of $E(\mathbb{Q})$ we obtain $g = -1282/6619$ and $5g^4 + 9g^2 + 5 = (101284931/43811161)^2$. As the rank of E is positive, we have an infinite number of rational points and thus infinitely many values of g as desired. □

Example 3.3. We are free to select g and x_4, provided $5g^4 + 9g^2 + 5$ is a square. So we set $g = 2$ and $x_4 = 1$. We then compute $m = -503/2783$, which leads to the points $(1, 1/4)$, $(166902131/159143645, 1/2)$, $(0, 1)$, $(166902131/202822175, 2)$, and $(166902131/196487183, 4)$ on the twisted Edwards curve $E_{a,d}$, with

$$a = 28488726729454945/27856321332341161,$$
$$d = 37974807686161705/27856321332341161.$$

It is straightforward to check that the four points with y-coordinates $1/4, 1/2, 2$, and 4 are linearly independent on the curve $E_{a,d}$. To do so, we mapped the curve to the Weierstrass curve and used SAGE to compute the determinant of the height pairing matrix, which was $8394.4755918 \neq 0$.

By the specialization in Example 3.3, we see that there is an infinite family of twisted Edwards curves with five points in geometric progression which has rank at least 4.

4. Huff curves

An elliptic curve in the Huff model ([13, 14]) can be written in the form

$$H_{a,b} : ax(y^2 - 1) = by(x^2 - 1),$$

with $ab(a^2 - b^2) \neq 0$. In this section, we construct a geometric progression on such curves. In fact, we show a construction which yields an infinite family of Huff curves containing five points in geometric progression.

Theorem 4.1. *There exists infinitely many Huff curves which contain a geometric progression of length five (of the x-coordinates).*

Proof. Let $H_{a,b}$ be a Huff curve. Without loss of generality, we may assume that $b = 1$. We begin by observing that if the point (x, y) is on $H_{a,1}$, then so is $(1/x, -y)$. Since $(1, 1)$ is trivially on the curve, to produce a geometric progression of length 5 it thus suffices to ensure that $x = g, g^2$ are both valid x-coordinates of rational points on $H_{a,1}$.

The requirement that (g, y_1) is on $H_{a,1}$ is equivalent to $ag^2y_1 - gy_1^2 - ay_1 + g = 0$. Solving for a, we find

$$a = \frac{y_1(g^2-1)}{g(y_1^2-1)}.$$

We now search for conditions so that the point (g^2, y_2) will also be on the Huff curve (using this value of a). Some simple arithmetic leads to

$$g^2y_2y_1^2 - gy_2^2y_1 - g^2y_2 + y_2y_1^2 + gy_1 - y_2 = 0. \tag{4.1}$$

This is a quadratic equation in g, with discriminant

$$D := y_2^4y_1^2 - 4y_2^2y_1^4 + 6y_2^2y_1^2 - 4y_2^2 + y_1^2.$$

In order to have a rational solution, we need the discriminant to be square. Setting D to be equal a square w^2, we may view the equation as defining an elliptic curve E in y_2 and w (with parameter y_1). The curve E has a rational point with y_2-coordinate

$$y_2 = \frac{2y_1^2(y_1^2-1)}{(y_1^2+1)(y_1^4+1)}.$$

Substituting this value into the quadratic equation (4.1) in g thus allows the equation to be factored, leading to the root

$$g = -\frac{y_1^6 - y_1^4 - y_1^2 + 1}{2y_1(y_1^4+1)}, \tag{4.2}$$

or its inverse.

As we have infinitely many choices for y_1, we therefore have constructed an infinite number of geometric progressions on the Huff curve with five terms. More concretely, for any value $y_1 \neq 0, \pm 1$, we define g as in equation (4.2). We then have the following points on $H_{y_1(g^2-1)/g(y_1^2-1),1}$:

$$\left(\frac{1}{g^2}, -\frac{2y_1^2(y_1^2-1)}{y_1^6+y_1^4+y_1^2+1}\right), (1/g, -y_1), (1,1), (g, y_1), \left(g^2, \frac{2y_1^2(y_1^2-1)}{y_1^6+y_1^4+y_1^2+1}\right).$$

□

Example 4.2. Let $y_1 = -2$, then $g = 45/68$ and $a = 2599/4590$. If we consider the curve $H_{2599,4590}$, the points $(1/g^2, -24/85)$, $(1/g, 2)$, $(1,1)$, $(g, -2)$, and $(g^2, 24/85)$ are all on the curve.

The curve $H_{a,b}$ can be transformed into the Weierstrass curve $y^2 = x(x+a)(x+b)$. Checking with SAGE, the points $(g, -2)$ and $(g^2, 24/85)$ are linearly independent since the determinant of their height pairing matrix is $15.597467634 \neq 0$.

From the specialization in Example 4.2, we see that the rank of the free part of the Huff curve family in Theorem 4.1 is at least 2 over $\mathbb{Q}(y_1)$. Note the construction of the geometric progression used that if (x, y) is on the

Huff curve, then so is $(1/x, -y)$. These points are linearly dependent, as $(x, y) + (1/x - y)$ is a point of order 2. This explains why the lower bound on the rank is not higher, as might be otherwise expected if it were to be assumed the non-trivial points in the geometric progression were independent.

5. Weierstrass models

We begin by looking for geometric progressions on the cubic curve $C_{b,c,d}$: $y^2 = x^3 + bx^2 + cx + d$ which is slightly more general than the traditional short Weierstrass form written as $E : y^2 = x^3 + ax + b$. We construct infinitely many such Weierstrass curves with 6 points in geometric progression.

Under the birational transformation $(x, y) \to (ax, ay)$ with $a \neq 0$, the curve $C_{b,c,d}$ can be put into the form $C_{a,b,c',d'} : y^2 = ax^3 + bx^2 + c'x + d'$ while still preserving any geometric progression. Here $c' = c/a$ and $d' = d/a^2$. Thus, it is equivalent to find a geometric progression on a curve of the form $C_{a,b,c,d}$.

Theorem 5.1. *There are infinitely many Weierstrass curves of the form $C(b, c, d) : y^2 = x^3 + bx^2 + cx + d$ containing a geometric progression of length 8 (of the x-coordinates).*

Proof. Consider the polynomial

$$f(x) = \prod_{i=0}^{3} (x - g^{2i+1})(x - g^{-(2i+1)}),$$

which is a degree 8 polynomial in x. We now let $h(x) = x^4 + c_3x^3 + c_2x^2 + c_1x + c_0$, for some unknowns c_i, and compute $f(x) - h(x)^2$:

$$\begin{aligned} f(x) - h(x)^2 = {} & d_7(g, c_3)x^7 + d_6(g, c_2, c_3)x^6 + d_5(g, c_1, c_2, c_3)x^5 \\ & + d_4(g, c_0, c_1, c_2, c_3)x^4 + j(x). \end{aligned}$$

Here the polynomials d_k are linear in c_{k-4} for $k = 7, \ldots, 4$, and $j(x)$ is a degree 3 polynomial. Accordingly, by setting d_7, d_6, and d_5 successively to zero we can recursively solve for c_3, c_2, c_1, and c_0 in terms of g. That is, we first obtain c_3 by setting $d_7 = 0$ and then replace this value of c_3 in $d_6 = 0$ to then obtain c_2. We continue this process and recover c_1 and c_0. Then evaluating the equation $f(x) - h(x)^2 = j(x)$ at the roots $g^{\pm(2i+1)}$ yields $-j(g^{\pm(2i+1)})$ is square (for $i = 0, ..., 3$).

Therefore, on the elliptic curve $y^2 = -j(x)$ we have a geometric progression of length eight: $x = 1/g^7, 1/g^5, 1/g^3, 1/g, g, g^3, g^5, g^7$. As g is arbitrary, we see that have an infinite number of such progressions. We note that any length 8 geometric progression could be used, however the resulting formula for $j(x)$ is much simpler for the progression we chose. ∎

EXAMPLE 5.2. By setting $g = 2$, and after removing square factors, we have the curve

$$C_2 : y^2 = 126586246859593875456 00x^3 + 164489992792118352412672x^2 + 116022278635208261812736 00x + 509740584746551687568673361.$$

The curve C_2 has rational points with x-coordinates

$$1/128, 1/32, 1/8, 1/2, 2, 8, 32, 128.$$

A computation in SAGE shows that the first seven of these eight points are linearly independent, as the determinant of their height pairing matrix is non-zero.

Thus, by the Silverman specialization theorem again, we can conclude the free part of the infinite family of curves with an eight term progression given in Theorem 5.1 has rank at least 7 over $\mathbb{Q}(g)$.

6. Quartic models

A quartic genus 1 curve is defined by an equation $C : y^2 = f(x)$, where f is a polynomial of degree 4 without multiple roots. It is well known that if a rational point exists on C, then C is birationally equivalent to an elliptic curve in Weierstrass form. This section deals with finding infinite families of quartic curves containing 10-term geometric progressions.

THEOREM 6.1. *There are infinitely many quartic elliptic curves of the form $E : y^2 = ax^4 + bx^3 + cx^2 + dx + e$ containing a geometric progression of length 10 (of the x-coordinates).*

PROOF. We use the same technique as in the previous section. Consider the polynomial

$$f(x) = \prod_{i=0}^{4} (x - g^{2i+1})(x - g^{-(2i+1)}),$$

which is a degree 10 polynomial in x. We now let $h(x) = x^5 + c_4x^4 + c_3x^3 + c_2x^2 + c_1x + c_0$, for some unknowns c_i, and compute $f(x) - h(x)^2$. Similar to the proof of Theorem 5.1, d_k is linear in c_{k-5} for $k = 9, \ldots, 5$. Thus we can recursively solve for c_4, c_3, c_2, c_1, and c_0 (in terms of g), so that the polynomial $j(x) = f(x) - h(x)^2$ will be a quartic polynomial. By construction, $-j(g^{\pm(2i+1)})$ is square (for $i = 0, ..., 4$).

We thus have an infinite number of geometric progressions of length ten. We note that any length 10 geometric progression could be used, however the resulting formula for $j(x)$ is much simpler for the progression we chose. □

We omit providing an example, as the coefficients of the quartic are quite large even for small g. For $g = 2$, we computed the height pairing matrix determinant and found that nine of the points are linearly independent. We

therefore conclude that a lower bound on the rank of the free part of the family of quartic curves in Theorem 6.1 is 9 (considered over $\mathbb{Q}(g)$).

7. Conclusion

In this work, we have examined geometric progressions on elliptic curves. Future work would be to improve the length of the geometric progressions found for each of the different models of curve families. Recall one of the original motivations for finding (arithmetic) progressions on curves seemed to be to find curves with high rank. It would be interesting to see if the families described in this paper lead to elliptic curves with high rank, above that given by the lower bounds we gave. Intuitively, it would seem that the points in geometric progression might be independent with high probability.

References

[1] A. Alvarado, *An arithmetic progession on quintic curves*, J. Integer Seq. **12** (2009), Article 09.7.3, 6pp.

[2] A. Berczes and V. Ziegler, *On geometric progressions on Pell equations and Lucas sequences*, Glas. Mat. Ser. III **48(68)** (2013), 1–22.

[3] D. J. Bernstein, P. Birkner, M. Joye, T. Lange and P. Peters, *Twisted Edwards curves*, in Progress in Cryptology (AFRICACRYPT'08), Springer, Berlin, 2008, 389–405.

[4] A. Bremner, *On arithmetic progressions on elliptic curves*, Experiment. Math. **8** (1999), 409–413.

[5] A. Bremner, *Arithmetic progressions on Edwards curves*, J. Integer Seq. **16** (2013), Article 13.8.5, 5pp.

[6] A. Bremner and M. Ulas, *Rational points in geometric progressions on certain hyperelliptic curves*, Publ. Math. Debrecen **82** (2013), 669–683.

[7] G. Campbell, *A note on arithmetic progressions on elliptic curves*, J. Integer Seq. **6** (2003), Article 03.1.3, 5pp.

[8] A. Choudhry, *Arithmetic progressions on Huff curves*, J. Integer Seq. **18** (2015), Article 15.5.2, 9pp.

[9] H. M. Edwards, *A normal form for elliptic curves*, Bull. Amer. Math. Soc. (N.S.) **44** (2007), 393–422.

[10] I. García-Selfa and J. Tornero, *Searching for simultaneous arithmetic progressions on elliptic curves*, Bull. Austral. Math. Soc. **71** (2005), 417–424.

[11] E. González-Jiménez, *Markoff-Rosenberger triples in geometric progression*, Acta Math. Hungar. **142** (2014), 231–243.

[12] E. Gonzalez-Jiménez, *On arithmetic progressions on Edwards curves*, Acta Arith. **167** (2015), 117–132.

[13] G. B. Huff, *Diophantine problems in geometry and elliptic ternary forms*, Duke Math. J. **15** (1948), 443–453.

[14] M. Joye, M. Tibouchi, and D. Vergnaud, *Huff's model for elliptic curves*, in Algorithmic number theory (ANTS-IX), Springer, Berlin, 2010, 234–250.

[15] A. MacLeod, *14-term arithmetic progressions on quartic elliptic curves*, J. Integer Seq. **9** (2006), Article 06.1.2, 4pp.

[16] D. Moody, *Arithmetic progressions on Edwards curves*, J. Integer Seq. **14** (2011), Article 11.1.7, 4pp.

[17] D. Moody, *Arithmetic progressions on Huff curves*, Ann. Math. Inform. **38** (2011), 111–116.

[18] D. Moody and A. S. Zargar, *On the rank of elliptic curves with long arithmetic progressions*, to appear in Colloq. Math., 2016, DOI:10.4064/cm7036-9-2016.
[19] Sage software, Version 4.5.3, `http://sagemath.org`.
[20] J. H. Silverman, Advanced topics in the arithmetic of elliptic curves, Springer-Verlag, New-York, 1994.
[21] M. Ulas, *A note on arithmetic progressions on quartic elliptic curves*, J. Integer Seq. **8** (2005), Article 05.3.1, 5 pp.
[22] M. Ulas, *On arithmetic progressions on genus two curves*, Rocky Mountain J. Math. **39** (2009), 971–980.
[23] L. Washington, Elliptic curves: number theory and cryptography, CRC press, 2008.

A. A. Ciss
Laboratoire de Traitement de l'Information et Systèmes Intelligents,
École Polytechnique de Thiès
BP A10 Thiès
Sénégal
E-mail: `aaciss@ept.sn`

D. Moody
National Institute of Standards and Technology (NIST)
100 Bureau Drive, Gaithersburg, 20899-8930
USA
E-mail: `dustin.moody@nist.gov`

Received: `22.8.2016.`
Revised: `14.12.2016.`