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QUASI-PARTICLE BASES OF PRINCIPAL SUBSPACES OF THE AFFINE LIE ALGEBRA OF TYPE $G_2^{(1)}$

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Abstract. The aim of this work is to construct the quasi-particle basis of principal subspace of standard module of highest weight $k\Lambda_0$ of level $k \geq 1$ of affine Lie algebra of type $G_2^{(1)}$ by means of which we obtain the basis of principal subspace of generalized Verma module.

1. Introduction

Principal subspaces of standard modules of affine Lie algebras $A_1^{(1)}$ were first introduced by B. L. Feigin and A. V. Stoyanovsky in [16]. Motivated by the work of J. Lepowsky and M. Primc ([26]), Feigin and Stoyanovsky related characters of principal subspaces with Rogers-Ramanujan type identities. This connection was further studied by many authors, in particular in [3], [6, 7], [8–11], [12], [13, 14], [18], [24], [28, 29], [30, 31] and others. More recently, Slaven Kožić in [22, 23] showed that character formulas for level 1 principal subspaces associated with the integrable highest weight module of quantum affine algebra $U_q(\widehat{\mathfrak{sl}_2})$ coincide with the character formulas found in [16].

In [18], G. Georgiev constructed bases for principal subspaces of certain standard $A_l^{(1)}$-modules by using monomials of certain vertex operator coefficients corresponding to simple roots of A_l, the so-called quasi-particles (cf.

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[16]), from which were easily obtained the Rogers-Ramanujan type character formulas. In [4] and [5] we extended Georgiev's construction of quasi-particle bases for principal subspaces of standard module $L(k\Lambda_0)$ and generalized Verma module $N(k\Lambda_0)$ of highest weight $k\Lambda_0$, $k \in \mathbb{N}$ for affine Lie algebras of type $B_l^{(1)}$ and $C_l^{(1)}$, $l \geq 2$. As a consequence we proved two new series of Rogers-Ramanujan type identities obtained from the characters of principal subspaces of generalized Verma module.

In this note we construct quasi-particle bases of principal subspaces of generalized Verma module $N(k\Lambda_0)$ and its irreducible quotient in the case of affine Lie algebra of type $G_2^{(1)}$. Two main steps in the construction are similar to the case of $B_2^{(1)}$. First step is to find relations among quasi-particles from which follow the spanning set of principal subspaces and the second step is to prove that the spanning set is linearly independent by induction on the linear order on quasi-particles. The main differences with the case of $B_2^{(1)}$ are relations which describe the interaction of quasi-particles associated to different simple roots and operators which we use in the proof of linear independence, since we don't have a simple current operator as in the proof of independence for $B_2^{(1)}$.

To state our main results, denote by $W_{L(k\Lambda_0)}$ the principal subspace of level k standard module and by ch $W_{L(k\Lambda_0)}$ the character of $W_{L(k\Lambda_0)}$ and by $W_{N(k\Lambda_0)}$ the principal subspace of generalized Verma module $N(k\Lambda_0)$. Our result states:

Theorem 1.1.

$$\mathrm{ch}\ W_{L(k\Lambda_0)} = \sum_{\substack{r_1^{(1)}\geq\ldots\geq r_1^{(k)}\geq 0\\ r_2^{(1)}\geq\ldots\geq r_2^{(3k)}\geq 0}} \frac{q^{\sum_{s=1}^{k} r_1^{(s)^2}+\sum_{s=1}^{3k} r_2^{(s)^2}-\sum_{s=1}^{k} r_1^{(s)}(r_2^{(3s)}+r_2^{(3s-1)}+r_2^{(3s-2)})}}{(q)_{r_1^{(1)}-r_1^{(2)}}\cdots(q)_{r_1^{(k)}}(q)_{r_2^{(1)}-r_2^{(2)}}\cdots(q)_{r_2^{(3k)}}} y_1^{r_1}y_2^{r_2},$$

where $r_1 = \sum_{s=1}^{k} r_1^{(s)}$ *and* $r_2 = \sum_{s=1}^{3k} r_2^{(s)}$.

This new fermionic formula which follows directly from quasi-particle basis of $W_{L(k\Lambda_0)}$ is related to the study of parafermionic Rogers-Ramanujan type characters ([19]).

We use quasi-particle bases of $W_{L(k\Lambda_0)}$ in the construction of quasi-particles bases of principal subspace $W_{N(k\Lambda_0)}$ of generalized Verma module, from which follows a generalization of Euler-Cauchy identity.

THEOREM 1.2.

(1.1)

$$\prod_{m>0}\frac{1}{(1-q^my_1)}\frac{1}{(1-q^my_2)}\frac{1}{(1-q^my_1y_2)}\frac{1}{(1-q^my_1y_2^2)}\frac{1}{(1-q^my_1y_2^3)}\frac{1}{(1-q^my_1^2y_2^3)}$$
$$=\sum_{\substack{r_1^{(1)}\geq r_1^{(2)}\geq r_1^{(3)}\geq\cdots\geq 0\\ r_2^{(1)}\geq r_2^{(2)}\geq r_2^{(3)}\geq\cdots\geq 0}}\frac{q^{\sum_{s\geq1}r_1^{(s)^2}+\sum_{s\geq1}r_2^{(s)^2}-\sum_{s\geq1}r_1^{(s)}(r_2^{(3s)}+r_2^{(3s-1)}+r_2^{(3s-2)})}}{(q)_{r_1^{(1)}-r_1^{(2)}}(q)_{r_1^{(2)}-r_1^{(3)}}\cdots(q)_{r_2^{(1)}-r_2^{(2)}}(q)_{r_2^{(2)}-r_2^{(3)}}\cdots}y_1^{r_1}y_2^{r_2},$$

where $r_1=\sum_{s\geq1}r_1^{(s)}$ *and* $r_2=\sum_{s\geq1}r_2^{(s)}$. *The sum on the right side of (1.1) is over all descending infinite sequences of non-negative integers with finite support.*

2. Principal subspaces

Let $\mathfrak{g}$ be a complex simple Lie algebra of type G_2 with a triangular decomposition $\mathfrak{g}=\mathfrak{n}_-\oplus\mathfrak{h}\oplus\mathfrak{n}_+$, with the basis $\Pi=\{\alpha_1=\frac{1}{\sqrt{3}}(-2\epsilon_1+\epsilon_2+\epsilon_3),\alpha_2=\frac{1}{\sqrt{3}}(\epsilon_1-\epsilon_2)\}$ of the root system R and the corresponding set of fundamental weights $\{\omega_1=2\alpha_1+3\alpha_2,\omega_2=\alpha_1+2\alpha_2\}$, where $\epsilon_1,\epsilon_2,\epsilon_3$ are vectors of the standard basis of $\mathbb{R}^3$. Denote by $\theta=\frac{1}{\sqrt{3}}(-\epsilon_1-\epsilon_2+2\epsilon_3)$ the highest root and assume that all long roots $\alpha\in R$ are normalized by the condition $\langle\alpha,\alpha\rangle=2$, where $\langle\cdot,\cdot\rangle$ denotes the invariant nondegenarate bilinear form on $\mathfrak{g}$, which induces a bilinear form on $\mathfrak{h}^*$. Denote by Q the root lattice and by P the weight lattice of $\mathfrak{g}$. Then, $P=Q$. For later use we fix root vectors

(2.1)
$$x_{\alpha_1+\alpha_2}=[x_{\alpha_2},x_{\alpha_1}],\ x_{\alpha_1+2\alpha_2}=[x_{\alpha_2},x_{\alpha_1+\alpha_2}],$$
$$x_{\alpha_1+3\alpha_2}=[x_{\alpha_2},x_{\alpha_1+2\alpha_2}],\ x_{2\alpha_1+3\alpha_2}=[x_{\alpha_1},x_{\alpha_1+3\alpha_2}].$$

Let $\widetilde{\mathfrak{g}}$ be the associated affine Lie algebra

$$\widetilde{\mathfrak{g}}=\widehat{\mathfrak{g}}\oplus\mathbb{C}d,\qquad\widehat{\mathfrak{g}}=\mathfrak{g}\otimes\mathbb{C}\left[t,t^{-1}\right]\oplus\mathbb{C}c,$$

with commutation relations

(2.2)
$$[x(j_1),y(j_2)]=[x,y]\,(j_1+j_2)+\langle x,y\rangle\,j_1\delta_{j_1+j_2,0}c,$$
$$[c,\widetilde{\mathfrak{g}}]=0,\quad[d,x(j)]=jx(j),$$

where $x(j)=x\otimes t^j$ for $x,y\in\mathfrak{g}$, $j,j_1,j_2\in\mathbb{Z}$, (cf. [21]). We consider $\widetilde{\mathfrak{g}}$-subalgebras

$$\mathcal{L}(\mathfrak{n}_+)=\mathfrak{n}_+\otimes\mathbb{C}[t,t^{-1}],$$
$$\mathcal{L}(\mathfrak{n}_+)_{\geq0}=\mathfrak{n}_+\otimes\mathbb{C}[t],\quad\mathcal{L}(\mathfrak{n}_+)_{<0}=\mathfrak{n}_+\otimes t^{-1}\mathbb{C}[t^{-1}]$$

and

$$\mathcal{L}(\mathfrak{n}_\alpha)=\mathfrak{n}_\alpha\otimes\mathbb{C}[t,t^{-1}],$$

where

$$\mathfrak{n}_\alpha = \mathbb{C}x_\alpha$$

are one-dimensional $\mathfrak{g}$-subalgebras generated with root vectors x_α, $\alpha \in R$.

We extend our form $\langle \cdot, \cdot \rangle$ to $\widetilde{\mathfrak{h}} = \mathfrak{h} \oplus \mathbb{C}c \oplus \mathbb{C}d$. The set of simple roots of $\widetilde{\mathfrak{g}}$ is $\{\alpha_0, \alpha_1, \alpha_2\}$ and $\{\Lambda_0, \Lambda_1, \Lambda_2\}$ is the set of fundamental weights. Denote by $L(\Lambda_0)$ a standard (i.e. integrable highest weight) $\widetilde{\mathfrak{g}}$-module of level 1 with the highest weight vector $v_{L(\Lambda_0)}$.

Fix $k \in \mathbb{N}$. Denote by $N(k\Lambda_0)$ the generalized Verma module and by $L(k\Lambda_0)$ its irreducible quotient. The induced $\widetilde{\mathfrak{g}}$-module $N(k\Lambda_0)$ is defined as

$$N(k\Lambda_0) = U(\widehat{\mathfrak{g}}) \otimes_{U(\widehat{\mathfrak{g}}_{\geq 0})} \mathbb{C}v_{N(k\Lambda_0)},$$

where $\widehat{\mathfrak{g}}_{\geq 0} = \bigoplus_{n \geq 0} \mathfrak{g} \otimes t^n \oplus \mathbb{C}c$ and $\mathbb{C}v_{N(k\Lambda_0)}$ is 1-dimensional $\widehat{\mathfrak{g}}_{\geq 0}$-module, such that

$$cv_{N(k\Lambda_0)} = kv_{N(k\Lambda_0)}, \quad dv_{N(k\Lambda_0)} = 0, \quad (\mathfrak{g} \otimes t^j)v_{N(k\Lambda_0)} = 0, \quad j \geq 0.$$

Set

$$v_{N(k\Lambda_0)} = 1 \otimes v_{N(k\Lambda_0)}.$$

The generalized Verma module has a structure of a vertex operator algebra, as its irreducible quotient $L(k\Lambda_0)$ and all the level k standard modules are modules for vertex operator algebra $L(k\Lambda_0)$. The vertex operator map is determined by

$$Y(x(-1)v_{N(k\Lambda_0)}, z) = \sum_{m \in \mathbb{Z}} x(m) z^{-m-1} = x(z)$$

for $x \in \mathfrak{g}$ (cf. [25]). We will use the commutator formula among vertex operators:

$$\begin{aligned} &[Y(x_\alpha(-1)v_{N(k\Lambda_0)}, z_1), Y(x_\beta(-1)^r v_{N(k\Lambda_0)}, z_2)] \\ &= \sum_{j \geq 0} \frac{(-1)^j}{j!} \left(\frac{d}{dz_1}\right)^j z_2^{-1} \delta\left(\frac{z_1}{z_2}\right) Y(x_\alpha(j) x_\beta(-1)^r v_{N(k\Lambda_0)}, z_2), \end{aligned} \tag{2.3}$$

where $\alpha, \beta \in R$, (cf. [17]).

Denote by $v_{L(k\Lambda_0)}$ the highest weight vector of $L(k\Lambda_0)$. We define a principal subspace $W_{L(k\Lambda_0)}$ of $L(k\Lambda_0)$ (see [16, 18]) as

$$W_{L(k\Lambda_0)} = U(\mathcal{L}(\mathfrak{n}_+))v_{L(k\Lambda_0)}$$

and the principal subspace $W_{N(k\Lambda_0)}$ of the generalized Verma module $N(k\Lambda_0)$ as

$$W_{N(k\Lambda_0)} = U(\mathcal{L}(\mathfrak{n}_+))v_{N(k\Lambda_0)}.$$

Note that the map

$$f : U(\mathcal{L}(\mathfrak{n}_+)_{<0}) \to W_{N(k\Lambda_0)}, \qquad f(b) = bv_{N(k\Lambda_0)}$$

is an isomorphism of $\mathcal{L}(\mathfrak{n}_+)_{<0}$-modules. If we order basis elements of $\mathfrak{n}_+$

$$\{x_{\alpha_1}, x_{\alpha_2}, x_{\alpha_1+\alpha_2}, x_{\alpha_1+2\alpha_2}, x_{\alpha_1+3\alpha_2}, x_{2\alpha_1+3\alpha_2}\}$$

in the following way:

$$x_{\alpha_2} < x_{\alpha_1} < x_{\alpha_1+\alpha_2} < x_{\alpha_1+2\alpha_2} < x_{\alpha_1+3\alpha_2} < x_{2\alpha_1+3\alpha_2}$$

and basis elements of $\mathcal{L}(\mathfrak{n}_+)_{<0}$

$$\{x_\alpha(m) : \alpha \in R_+, m < 0\}$$

as:

$$x(m) \leq y(m') \quad \Leftrightarrow \quad x < y \ \text{ or } \ x = y \ \text{ and } \ m < m',$$

then from the Poincaré-Birkhoff-Witt theorem follows that vectors

$$x_{\alpha_2}(m_1^1) \cdots x_{\alpha_2}(m_1^{s_1}) x_{\alpha_1}(m_2^1) \cdots x_{\alpha_1}(m_2^{s_2}) \cdots x_{2\alpha_1+3\alpha_2}(m_6^1) \cdots x_{2\alpha_1+3\alpha_2}(m_6^{s_6}) v_{N(k\Lambda_0)}, \tag{2.4}$$

where $m_i^1 \leq \cdots \leq m_i^{s_i} < 0$, $s_i \geq 0$, $1 \leq i \leq 6$, form a basis of a vector space $W_{N(k\Lambda_0)}$.

In next sections, we construct bases of principal subspaces $W_{L(k\Lambda_0)}$ and $W_{N(k\Lambda_0)}$ in terms of certain coefficients of vertex operators corresponding to vectors $x_{\alpha_i}(-1)^r v_{L(k\Lambda_0)}$ (and $x_{\alpha_i}(-1)^r v_{N(k\Lambda_0)}$), where $r \geq 1$ and $\alpha_i \in \Pi$.

First, we choose a special subspace of $U(\mathcal{L}(\mathfrak{n}_+))$

$$U = U(\mathcal{L}(\mathfrak{n}_{\alpha_2}))U(\mathcal{L}(\mathfrak{n}_{\alpha_1})).$$

It is easy to see that principal subspaces are generated by operators in U acting on the highest weight vectors $v_{L(k\Lambda_0)}$ and $v_{N(k\Lambda_0)}$ (see Lemma 3.1 in [18]).

3. Quasi-particle bases of principal subspaces

We start this section with introducing all necessary notions and facts needed in the construction of quasi-particle bases of principal subspaces. Some terms and labels which we use, but are not mentioned, are the same as in our previous work, therefore, for more details we refer to [4, 5] and also to [18].

3.1. *Quasi-particle monomials.* For given $i \in \{1, 2\}$, $r \in \mathbb{N}$ and $m \in \mathbb{Z}$ define a quasi-particle of color i, charge r and energy $-m$ by

$$x_{r\alpha_i}(m) = \mathrm{Res}_z \left\{ z^{m+r-1} x_{r\alpha_i}(z) \right\},$$

where $x_{r\alpha_i}(z)$ is a vertex operator

$$x_{r\alpha_i}(z) := x_{\alpha_i}(z)^r = Y((x_{\alpha_i}(-1))^r v_{L(k\Lambda_0)}, z).$$

$x_{r\alpha_i}(z)$ is the generating function of quasi-particles of color i and charge r.

Denote by $b(\alpha_i)$ the monochromatic quasi-particle monomial, that is the product of quasi-particles of the same color i. We say that monomial b "colored" with more colors is a polychromatic monomial. As in the case of $B_2^{(1)}$, our basis monomials will be "colored" with two colors $i = 1, 2$ and our monomials will have the form

$$b = b(\alpha_2)b(\alpha_1).$$

For monomial

$$b(\alpha_2)b(\alpha_1) = x_{n_{r_2^{(1)},2}\alpha_2}(m_{r_2^{(1)},2})\cdots x_{n_{1,2}\alpha_2}(m_{1,2})$$
$$x_{n_{r_1^{(1)},1}\alpha_1}(m_{r_1^{(1)},1})\cdots x_{n_{1,1}\alpha_1}(m_{1,1}),$$

we will say it is of charge-type

$$\mathcal{R}' = \left(n_{r_2^{(1)},2}, \ldots, n_{1,2}; n_{r_1^{(1)},1}, \ldots, n_{1,1}\right),$$

where

$$0 \leq n_{r_i^{(1)},i} \leq \ldots \leq n_{1,i},$$

dual-charge-type

$$\mathcal{R} = \left(r_2^{(1)}, \ldots, r_2^{(s_2)}; r_1^{(1)}, \ldots, r_1^{(s_1)}\right),$$

where

$$r_i^{(1)} \geq r_i^{(2)} \geq \ldots \geq r_i^{(s_i)} \geq 0$$

and color-type

$$(r_2, r_1),$$

where

$$r_i = \sum_{p=1}^{r_i^{(1)}} n_{p,i} = \sum_{t=1}^{s_i} r_i^{(t)} \quad \text{and} \quad s_i \in \mathbb{N},$$

(cf. [4, 5, 18]) if for every color $\mathcal{R}$ and $\mathcal{R}'$ are mutually conjugate partitions of r_i (cf. [1]). We use the same terminology for the products of generating functions.

We assume that all monomial factors are sorted so that energies of quasi-particles of the same color and the same charge form an increasing sequence of integers from right to left. We compare charge-type $\mathcal{R}'$ and $\overline{\mathcal{R}'}$, where $\overline{\mathcal{R}'} = \left(\overline{n}_{\overline{r}_2^{(1)},2}, \ldots, \overline{n}_{1,1}\right)$, so that we compare their charges from right to left, i.e. we write $\mathcal{R}' < \overline{\mathcal{R}'}$ if there is $u \in \mathbb{N}$, such that $n_{1,i} = \overline{n}_{1,i}, n_{2,i} = \overline{n}_{2,i}, \ldots, n_{u-1,i} = \overline{n}_{u-1,i}$, and $u = \overline{r}_i^{(1)} + 1$ or $n_{u,i} < \overline{n}_{u,i}$.

We compare two monomials b and $\overline{b}$ by comparing first their charge-types $\mathcal{R}'$ and $\overline{\mathcal{R}'}$ and then their sequences of energies $\left(m_{r_2^{(1)},2}, \ldots, m_{1,1}\right)$ and $\left(\overline{m}_{\overline{r}_2^{(1)},2}, \ldots, \overline{m}_{1,1}\right)$ (in a similar way as charge-types, again starting from color $i = 1$):

$$b < \overline{b} \ \text{ if } \ \begin{cases} \mathcal{R}' < \overline{\mathcal{R}'}, \\ \mathcal{R}' = \overline{\mathcal{R}'} \quad \text{and} \quad \left(m_{r_2^{(1)},2}, \ldots, m_{1,1}\right) < \left(\overline{m}_{\overline{r}_2^{(1)},2}, \ldots, \overline{m}_{1,1}\right). \end{cases}$$

3.2. *Relations among quasi-particles.* On a standard module $L(k\Lambda_0)$, we have vertex operator algebra relations

$$x_{(k+1)\alpha_1}(z) = 0, \tag{3.1}$$

$$x_{(3k+1)\alpha_2}(z) = 0, \tag{3.2}$$

$$x_{n\alpha_i}(z)v_{L(k\Lambda_0)} \in W_{L(k\Lambda_0)}\left[[z]\right], \tag{3.3}$$

and

$$x_{n\alpha_i}(m)v_{L(k\Lambda_0)} = 0, \quad \text{for} \quad m > -n, \tag{3.4}$$

when $n \leq k$ for $i = 1$ and $n \leq 3k$ for $i = 2$, (see [25, 27]).

In reducing the set $Uv_{L(k\Lambda_0)}$ to the spanning set we use relations for a sequence of monochromatic monomial vectors (see Lemma 2.2.1 in [4], or [20, 18, 15])

$$x_{n\alpha_i}(m)x_{n'\alpha_i}(m')v_{L(k\Lambda_0)}, x_{n\alpha_i}(m-1)x_{n'\alpha_i}(m'+1)v_{L(k\Lambda_0)}, \ldots$$
$$\ldots, x_{n\alpha_i}(m-2n+1)x_{n'\alpha_i}(m'+2n-1)v_{L(k\Lambda_0)},$$

colored with color i and with charge-type (n, n'), where $n < n'$, which we express as a (finite) linear combination of monomial vectors

$$x_{n\alpha_i}(j)x_{n'\alpha_i}(j')v_{L(k\Lambda_0)} \quad \text{such that} \quad j \leq m - 2n \quad \text{and} \quad j' \geq m' + 2n \tag{3.5}$$

and monomial vectors with a factor quasi-particle $x_{(n'+1)\alpha_i}(j_1)$, $j_1 \in \mathbb{Z}$.

In the case when $n = n'$ monomials

$$x_{n\alpha_i}(m)x_{n\alpha_i}(m') \quad \text{with} \quad m' - 2n < m \leq m'$$

can be expressed as a linear combination of monomials

$$x_{n\alpha_i}(j)x_{n\alpha_i}(j') \quad \text{with} \quad j \leq j' - 2n \tag{3.6}$$

and monomials with quasi-particle $x_{(n+1)\alpha_i}(j_1)$, $j_1 \in \mathbb{Z}$ (see Corollary 2.2.2 in [4], or [20, 18, 15]).

Next, we consider products of quasi-particles colored with different colors. First, from commutation formulas (2.1) and (2.2) and induction on $n, n' \in \mathbb{N}$ follows

Lemma 3.1. *Let $n \leq 3k$, $n' \leq k$ be fixed. We have:*

a)

$$\begin{aligned} x_{\alpha_1}(0)x_{\alpha_2}^n(-1)v_{L(k\Lambda_0)} = & -nx_{\alpha_2}^{n-1}(-1)x_{\alpha_1+\alpha_2}(-1)v_{L(k\Lambda_0)} \\ & + \binom{n}{2}x_{\alpha_2}^{n-2}(-1)x_{\alpha_1+2\alpha_2}(-2)v_{L(k\Lambda_0)} \\ & - \binom{n}{3}x_{\alpha_2}^{n-3}(-1)x_{\alpha_1+3\alpha_2}(-3)v_{L(k\Lambda_0)}; \end{aligned}$$

b)

$$x_{\alpha_1}(1)x^n_{\alpha_2}(-1)v_{L(k\Lambda_0)} = \binom{n}{2}x^{n-2}_{\alpha_2}(-1)x_{\alpha_1+2\alpha_2}(-1)v_{L(k\Lambda_0)} - \binom{n}{3}x^{n-3}_{\alpha_2}(-1)x_{\alpha_1+3\alpha_2}(-2)v_{L(k\Lambda_0)};$$

c) $x_{\alpha_1}(2)x^n_{\alpha_2}(-1)v_{L(k\Lambda_0)} = -\binom{n}{3}x^{n-3}_{\alpha_2}(-1)x_{\alpha_1+3\alpha_2}(-1)v_{L(k\Lambda_0)}$;
d) $x_{\alpha_1}(j)x^n_{\alpha_2}(-1)v_{L(k\Lambda_0)} = 0$, *where* $j \geq 3$;
e) $x_{\alpha_2}(0)x^{n'}_{\alpha_1}(-1)v_{L(k\Lambda_0)} = n'x^{n'-1}_{\alpha_1}x_{\alpha_1+\alpha_2}(-1)v_{L(k\Lambda_0)}$;
f) $x_{\alpha_2}(j)x^{n'}_{\alpha_1}(-1)v_{L(k\Lambda_0)} = 0$, *where* $j \geq 1$.

The previous lemma implies relation among quasi-particles of different colors:

LEMMA 3.2. *Let* $n_1 \leq k,\ n_2 \leq 3k$. *One has*

$$(3.7)\qquad (z_1 - z_2)^{\min\{3n_1,n_2\}}x_{n_1\alpha_1}(z_1)x_{n_2\alpha_2}(z_2) = (z_1 - z_2)^{\min\{3n_1,n_2\}}x_{n_2\alpha_2}(z_2)x_{n_1\alpha_1}(z_1).$$

PROOF. Note, that from commutator formula for vertex operators (2.3), statements a), b), c) and d) of Lemma 3.1 and properties of δ-function we have

$$(3.8)\qquad (z_1 - z_2)^3x_{\alpha_1}(z_1)x_{n_2\alpha_2}(z_2) = (z_1 - z_2)^3x_{n_2\alpha_2}(z_2)x_{\alpha_1}(z_1).$$

In a similar way, using e) and f) parts of Lemma 3.1 we have

$$(3.9)\qquad (z_1 - z_2)x_{n_1\alpha_1}(z_1)x_{\alpha_2}(z_2) = (z_1 - z_2)x_{\alpha_2}(z_2)x_{n_1\alpha_1}(z_1).$$

Now, from (3.8) and (3.9) follows the lemma. ∎

By using derived relations we can define the set of quasi-particle monomials which generate our bases (acting on the highest weight vectors)

$$B_{W_{L(k\Lambda_0)}} = \bigcup_{\substack{n_{r_1^{(1)},1}\leq\cdots\leq n_{1,1}\leq k\\ n_{r_2^{(1)},2}\leq\cdots\leq n_{1,2}\leq 3k}} \left(\text{or, equivalently,} \bigcup_{\substack{r_1^{(1)}\geq\cdots\geq r_1^{(k)}\geq 0\\ r_2^{(1)}\geq\cdots\geq r_2^{(3k)}\geq 0}}\right)$$

$$\Big\{ b = b(\alpha_2)b(\alpha_1) = x_{n_{r_2^{(1)},2}\alpha_2}(m_{r_2^{(1)},2})\cdots x_{n_{1,2}\alpha_2}(m_{1,2})x_{n_{r_1^{(1)},1}\alpha_1}(m_{r_1^{(1)},1})\cdots x_{n_{1,1}\alpha_1}(m_{1,1}) :$$

$$\left|\begin{array}{l} m_{p,1} \leq -n_{p,1} - \sum_{p>p'>0} 2\ \min\{n_{p,1}, n_{p',1}\},\ 1 \leq p \leq r_1^{(1)};\\ m_{p+1,1} \leq m_{p,1} - 2n_{p,1} \text{ if } n_{p+1,1} = n_{p,1},\ 1 \leq p \leq r_1^{(1)} - 1;\\ m_{p,2} \leq -n_{p,2} + \sum_{q=1}^{r_1^{(1)}} \min\{3n_{q,1}, n_{p,2}\} - \sum_{p>p'>0} 2\ \min\{n_{p,2}, n_{p',2}\},\\ \qquad 1 \leq p \leq r_2^{(1)};\\ m_{p+1,2} \leq m_{p,2} - 2n_{p,2} \text{ if } n_{p,2} = n_{p+1,2},\ 1 \leq p \leq r_2^{(1)} - 1 \end{array}\right\}.$$

The condition on energies of quasi-particles colored with color $i = 2$ contains a part which follows from relation (3.7). The other conditions on energies which follow from relations (3.1)–(3.6) are similar to difference conditions as in the case of $B_2^{(1)}$.

Now, we can state the Proposition 3.3, whose proof follows closely [18].

PROPOSITION 3.3. *The set*

$$\mathfrak{B}_{W_{L(k\Lambda_0)}} = \left\{ b v_{L(k\Lambda_0)} : b \in B_{W_{L(k\Lambda_0)}} \right\}$$

spans the principal subspace $W_{L(k\Lambda_0)}$.

In the rest of this section we consider the proof of linear independence of the set $\mathfrak{B}_{W_{L(k\Lambda_0)}}$. First, we introduce the properties of operators on a standard module level 1, which we will use in our proof.

3.3. *Projection* $\pi_{\mathcal{R}}$. Let $k > 1$. We realize the principal subspace $W_{L(k\Lambda_0)}$ as a subspace of the tensor product $W_{L(\Lambda_0)}^{\otimes k} \subset L(\Lambda_0)^{\otimes k}$, where

$$v_{L(k\Lambda_0)} = \underbrace{v_{L(\Lambda_0)} \otimes \cdots \otimes v_{L(\Lambda_0)}}_{k \text{ factors}}$$

is the highest weight vector.

For a chosen dual-charge-type

$$\mathfrak{R} = \left(r_2^{(1)}, \ldots, r_2^{(3k)}; r_1^{(1)}, \ldots, r_1^{(k)} \right),$$

denote by $\pi_{\mathfrak{R}}$ the projection of principal subspace $W_{L(k\Lambda_0)}$ to the subspace

$$W_{L(\Lambda_0)_{(\mu_2^{(k)}; r_1^{(k)})}} \otimes \cdots \otimes W_{L(\Lambda_0)_{(\mu_2^{(1)}; r_1^{(1)})}},$$

where $W_{L(\Lambda_0)_{(\mu_2^{(t)}; r_1^{(t)})}}$ is a $\mathfrak{h}$-weight subspace of weight $\mu_2^{(t)}\alpha_2 + r_1^{(t)}\alpha_1 \in Q$ with

$$\mu_2^{(t)} = r_2^{(3t)} + r_2^{(3t-1)} + r_2^{(3t-2)},$$

for every $1 \leq t \leq k$.

We shall denote by the same symbol $\pi_{\mathfrak{R}}$ the generalization of this projection to the space of formal series with coefficients in $W_{L(\Lambda_0)}^{\otimes k}$. Let

$$x_{n_{r_2^{(1)},2}\alpha_2}(z_{r_2^{(1)},2}) \cdots x_{n_{1,2}\alpha_2}(z_{1,2}) x_{n_{r_1^{(1)},1}\alpha_1}(z_{r_1^{(1)},1}) \cdots x_{n_{1,1}\alpha_1}(z_{1,1}) \tag{3.10}$$

be a generating function of the chosen dual-charge-type $\mathfrak{R}$ and the corresponding charge-type $\mathfrak{R}'$. Then, from relations (3.1) and (3.2) and definition of the action of Lie algebra on the modules, follows that the projection of the generating function (3.10) is

$$
\begin{aligned}
&\pi_{\mathfrak{R}} x_{n_{r_2^{(1)},2}\alpha_2}(z_{r_2^{(1)},2}) \cdots x_{n_{1,1}\alpha_1}(z_{1,1})\ v_{L(k\Lambda_0)} \\
&= \mathrm{C}\ x_{n^{(k)}_{r_2^{(3k-2)},2}\alpha_2}(z_{r_2^{(3k-2)},2}) \cdots x_{n^{(k)}_{r_2^{(3k-1)},2}\alpha_2}(z_{r_2^{(3k-1)},2}) \cdots \\
&\qquad \cdots x_{n^{(k)}_{r_2^{(3k)},2}\alpha_2}(z_{r_2^{(3k)},2}) \cdots \\
&\qquad \cdots x_{n^{(k)}_{1,2}\alpha_2}(z_{1,2}) x_{n^{(k)}_{r_1^{(k)},1}\alpha_1}(z_{r_1^{(k)},1}) \cdots x_{n^{(k)}_{1,1}\alpha_1}(z_{1,1})\ v_{L(\Lambda_0)} \qquad (3.11)\\
&\otimes \cdots \otimes \\
&\otimes x_{n^{(1)}_{r_2^{(1)},2}\alpha_2}(z_{r_2^{(1)},2}) \cdots x_{n^{(1)}_{r_2^{(2)},2}\alpha_2}(z_{r_2^{(2)},2}) \cdots x_{n^{(1)}_{r_2^{(3)},2}\alpha_2}(z_{r_2^{(3)},2}) \cdots \\
&\qquad \cdots x_{n^{(1)}_{1,2}\alpha_2}(z_{1,2}) \cdots x_{n^{(1)}_{r_1^{(1)},1}\alpha_1}(z_{r_1^{(1)},1}) \cdots x_{n^{(1)}_{1,1}\alpha_1}(z_{1,1})\ v_{L(\Lambda_0)},
\end{aligned}
$$

where $\mathrm{C} \in \mathbb{C}^*$,

$$0 \leq n^{(t)}_{p,2} \leq 3, \quad n^{(1)}_{p,2} \geq n^{(2)}_{p,2} \geq \ldots \geq n^{(k-1)}_{p,2} \geq n^{(k)}_{p,2},\ n_{p,2} = \sum_{t=1}^{k} n^{(t)}_{p,2},$$

for every every p, $1 \leq p \leq r_2^{(1)}$, so that at most one $n^{(t)}_{p,2}$ $(1 \leq t \leq k)$ can be 1 or 2 and

$$0 \leq n^{(t)}_{p,1} \leq 1, \quad 1 \leq t \leq k,\ n^{(1)}_{p,1} \geq n^{(2)}_{p,1} \geq \ldots \geq n^{(k-1)}_{p,1} \geq n^{(k)}_{p,1},\ n_{p,1} = \sum_{t=1}^{k} n^{(t)}_{p,1},$$

for every every p, $1 \leq p \leq r_1^{(1)}$.

EXAMPLE 3.4. In the case when $k = 2$ the projection $\pi_{\mathfrak{R}}$, where $\mathfrak{R} = (6,5,4,3,2,1;3,2)$, of generating function

$$x_{\alpha_2}(z_{6,2}) x_{2\alpha_2}(z_{5,2}) x_{3\alpha_2}(z_{4,2}) x_{4\alpha_2}(z_{3,2}) x_{5\alpha_2}(z_{2,2})$$
$$x_{6\alpha_2}(z_{1,2}) x_{\alpha_1}(z_{3,1}) x_{2\alpha_1}(z_{2,1}) x_{2\alpha_1}(z_{1,1})$$

on

$$W_{L(\Lambda_0)_{(6;2)}} \otimes W_{L(\Lambda_0)_{(15;3)}}$$

can be represented graphically as in the Figure 1, where at most one generating function of color $i = 1$ is placed on every tensor factor and at most three generating functions of color $i = 2$ are placed on every tensor factor.

We define the projection of monomial vector $bv_{L(k\Lambda_0)}$, with $b \in B_{W_{L(k\Lambda_0)}}$ colored with color-type (r_2, r_1), charge-type $\mathfrak{R}'$ and dual-charge-type $\mathfrak{R}$

$$b = x_{n_{r_2^{(1)},2}\alpha_2}(m_{r_2^{(1)},2}) \cdots x_{n_{1,2}\alpha_2}(m_{1,2}) x_{n_{r_1^{(1)},1}\alpha_1}(m_{r_1^{(1)},1}) \cdots x_{n_{1,1}\alpha_1}(m_{1,1}) \qquad (3.12)$$

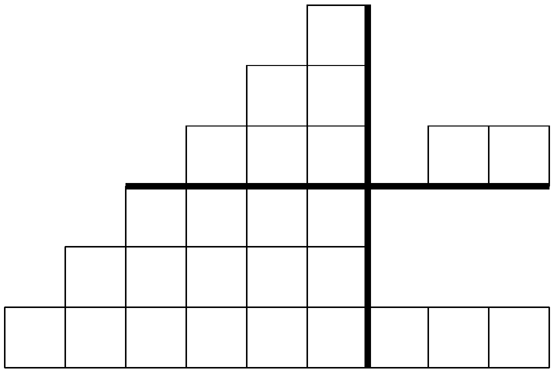

FIGURE 1. Example 3.4

as a coefficient of the projection of the generating function (3.11) which we denote as

$$\pi_{\mathfrak{R}} b v_{L(k\Lambda_0)}.$$

If $\bar{b} \in B_{W_{L(k\Lambda_0)}}$ is a monomial of charge-type $(\bar{n}_{\bar{r}_2^{(1)},2}, \ldots, \bar{n}_{1,2}; \bar{n}_{\bar{r}_1^{(1)},1}, \ldots, \bar{n}_{1,1})$, dual-charge-type $\bar{\mathfrak{R}} = \left(\bar{r}_2^{(1)}, \ldots, \bar{r}_2^{(3k)}; \bar{r}_1^{(1)}, \ldots, \bar{r}_1^{(k)}\right)$ and such that

$$b < \bar{b},$$

then, from the definition of projection, follows that

$$\pi_{\mathfrak{R}} \bar{b} v_{L(k\Lambda_0)} = 0.$$

We will use this property of projection $\pi_{\mathfrak{R}}$ in the proof of linear independence.

3.4. *Operator A_θ.* Denote by A_θ the coefficient of an intertwining operator $x_\theta(z)$

$$A_\theta = \mathrm{Res}_z z^{-1}\ x_\theta(z) = x_\theta(-1)$$

which commutes with the action of $\mathcal{L}(\mathfrak{n}_+)$ and such that

$$A_\theta v_{L(\Lambda_0)} = x_\theta(-1) v_{L(\Lambda_0)}. \tag{3.13}$$

We act with operator

$$1 \otimes \cdots \otimes A_\theta \otimes \underbrace{1 \otimes \cdots \otimes 1}_{s-1 \text{ factors}}, \quad s \leq k$$

on the vector $b v_{L(k\Lambda_0)} \in \mathcal{B}_{W_{L(k\Lambda_0)}}$, where quasi-particle monomial b is as in (3.12). From the definition of projection, it follows that vector

$$(1 \otimes \cdots \otimes 1 \otimes A_\theta \otimes 1 \otimes \cdots \otimes 1)\left(\pi_{\mathfrak{R}} b v_{L(k\Lambda_0)}\right)$$

is the coefficient of

$$(1 \otimes \cdots \otimes A_\theta \otimes 1 \otimes \cdots \otimes 1)\pi_{\mathfrak{R}} x_{n_{r_2^{(1)},2}\alpha_2}(z_{r_2^{(1)},2}) \cdots x_{s\alpha_1}(z_{1,1}) v_{L(k\Lambda_0)}. \tag{3.14}$$

From (3.13) it follows that in the s-th tensor row of (3.14) we have

$$\begin{aligned} &\otimes x_{n^{(s)}_{r_2^{(3s-2)},2}\alpha_2}(z_{r_2^{(3s-2)},2}) \cdots x_{n^{(s)}_{r_2^{(3s-1)},2}\alpha_2}(z_{r_2^{(3s-1)},2}) \cdots \\ &\cdots x_{n^{(s)}_{r_2^{(3s)},2}\alpha_2}(z_{r_2^{(3s)},2}) \cdots x_{n^{(s)}_{1,2}\alpha_l}(z_{1,2}) \\ &x_{n^{(s)}_{r_1^{(s)},1}\alpha_1}(z_{r_1^{(s)},1}) \cdots x_{\alpha_1}(z_{1,1}) x_\theta(-1) v_{L(\Lambda_0)} \otimes \cdots , \end{aligned} \tag{3.15}$$

where $0 \leq n^{(s)}_{p,1} \leq 1$, for $1 \leq p \leq r_1^{(s)}$ and $0 \leq n^{(s)}_{p,2} \leq 3$, for $1 \leq p \leq r_2^{(3s-2)}$.

3.5. *Operators* e_α. For every root $\alpha \in R$, we define on the level 1 standard module $L(\Lambda_0)$, the "Weyl group translation" operator e_α by

$$\begin{aligned} e_\alpha = &\exp\ x_{-\alpha}(1) \exp\ (-x_\alpha(-1)) \exp\ x_{-\alpha}(1) \exp\ x_\alpha(0) \\ &\exp\ (-x_{-\alpha}(0)) \exp\ x_\alpha(0), \end{aligned}$$

(for properly normalized root vectors, cf. [21]). Then on $L(\Lambda_0)$ we have

$$e_\alpha v_{L(\Lambda_0)} = -x_\alpha(-1) v_{L(\Lambda_0)}, \tag{3.16}$$

$$x_\beta(j) e_\alpha = e_\alpha x_\beta(j - \beta(\alpha^\vee)), \quad \beta \in R, \quad j \in \mathbb{Z}. \tag{3.17}$$

For $\alpha = \theta$, from (3.16) and (3.17), it follows that we can (3.15) write as

$$\begin{aligned} &\cdots \otimes x_{n^{(s)}_{r_l^{(3s-2)},2}\alpha_2}(z_{r_2^{(3s-2)},2}) \cdots x_{n^{(s)}_{r_2^{(3s-1)},2}\alpha_2}(z_{r_2^{(3s-1)},2}) \cdots \\ &\cdots x_{n^{(s)}_{r_2^{(3s)},2}\alpha_2}(z_{r_2^{(3s)},2}) \cdots x_{n^{(s)}_{1,2}\alpha_l}(z_{1,2}) \\ &x_{n^{(s)}_{r_1^{(k)},1}\alpha_1}(z_{r_1^{(k)},1}) z_{r_1^{(k)},1} \cdots x_{\alpha_1}(z_{1,1}) z_{1,1} v_{L(\Lambda_0)} \otimes \cdots . \end{aligned}$$

By taking the corresponding coefficients, we have

$$(1 \otimes \cdots \otimes 1 \otimes A_\theta \otimes 1 \otimes \cdots \otimes 1)\pi_{\mathfrak{R}} b v_{L(k\Lambda_0)} = (1 \otimes \cdots \otimes 1 \otimes e_\theta \otimes 1 \otimes \cdots \otimes 1)\pi_{\mathfrak{R}} b^+ v_{L(k\Lambda_0)}$$

where

$$b^+ = b^+(\alpha_2) b^+(\alpha_1) = b(\alpha_2) x_{n_{r_1^{(1)},1}\alpha_1}(m_{r_1^{(1)},1} + 1) \cdots x_{s\alpha_1}(m_{1,1} + 1).$$

Now, let $\alpha = \alpha_1$. We consider the projection $\pi_{\mathfrak{R}} b v_{L(k\Lambda_0)}$ of the monomial vector $b v_{L(k\Lambda_0)}$ where $b \in B_{W_{L(k\Lambda_0)}}$ is a monomial

$$\begin{aligned} b &= b(\alpha_2) b(\alpha_1) x_{s\alpha_1}(-s) \\ &= x_{n_{r_2^{(1)},2}\alpha_2}(m_{r_2^{(1)},2}) \cdots x_{n_{1,2}\alpha_2}(m_{1,2}) \\ &\quad x_{n_{r_1^{(1)},1}\alpha_1}(m_{r_1^{(1)},1}) \cdots x_{n_{2,1}\alpha_1}(m_{2,1}) x_{s\alpha_1}(-s), \end{aligned} \tag{3.18}$$

of dual-charge-type

$$\mathfrak{R} = \left(r_2^{(1)}, \ldots, r_2^{(3k)}; r_1^{(1)}, \ldots, r_1^{(s)}, 0 \ldots, 0\right).$$

The projection is a coefficient of the generating function

$$\begin{aligned}
&\pi_{\mathfrak{R}} x_{n_{r_2^{(1)},2}\alpha_2}(z_{r_2^{(1)},2}) \cdots x_{n_{1,2}\alpha_2}(z_{1,2}) x_{n_{r_1^{(1)},1}\alpha_1}(z_{r_1^{(1)},1}) \cdots x_{n_{2,1}\alpha_1}(z_{2,1}) \\
&\qquad \left(v_{L(\Lambda_0)} \otimes \cdots \otimes v_{L(\Lambda_0)} \otimes x_{\alpha_1}(-1) v_{L(\Lambda_0)} \otimes \cdots \otimes x_{\alpha_1}(-1) v_{L(\Lambda_0)}\right) \\
&= C x_{n^{(k)}_{r_2^{(3k-2)},2}\alpha_2}(z_{r_2^{(3k-2)},2}) \cdots x_{n^{(k)}_{r_2^{(3k-1)},2}\alpha_2}(z_{r_2^{(3k-1)},2}) \\
&\qquad \cdots x_{n^{(k)}_{r_2^{(3k)},2}\alpha_2}(z_{r_2^{(3k)},2}) \cdots x_{n^{(k)}_{1,2}\alpha_2}(z_{1,2}) v_{L(\Lambda_0)} \\
&\quad \otimes \cdots \otimes \\
&\quad \otimes x_{n^{(s)}_{r_2^{(3s-2)},2}\alpha_2}(z_{r_2^{(3s-2)},2}) \cdots x_{n^{(s)}_{r_2^{(3s-1)},2}\alpha_2}(z_{r_2^{(3s-1)},2}) \\
&\qquad \cdots x_{n^{(s)}_{r_2^{(3s)},2}\alpha_2}(z_{r_2^{(3s)},2}) \cdots x_{n^{(s)}_{1,2}\alpha_2}(z_{1,2}) \\
&\qquad x_{n^{(s)}_{r_1^{(s)},1}\alpha_1}(z_{r_1^{(s)},1}) \cdots x_{n^{(s)}_{2,1}\alpha_1}(z_{2,1}) e_{\alpha_1} v_{L(\Lambda_0)} \\
&\quad \otimes \cdots \otimes \\
&\quad \otimes x_{n^{(1)}_{r_2^{(1)},2}\alpha_2}(z_{r_2^{(1)},2}) \cdots x_{n^{(1)}_{r_2^{(2)},2}\alpha_2}(z_{r_2^{(2)},2}) \cdots x_{n^{(1)}_{r_2^{(3)},2}\alpha_2}(z_{r_2^{(3)},2}) \\
&\qquad \cdots x_{n^{(1)}_{2,2}\alpha_2}(z_{2,2}) x_{n^{(1)}_{1,2}\alpha_2}(z_{1,2}) \\
&\qquad x_{n^{(1)}_{r_1^{(1)},1}\alpha_1}(z_{r_1^{(1)},1}) \cdots x_{n^{(1)}_{2,1}\alpha_1}(z_{2,1}) e_{\alpha_1} v_{L(\Lambda_0)}.
\end{aligned}$$

Now, if we shift $1 \otimes \cdots \otimes e_{\alpha_1} \otimes e_{\alpha_1} \otimes \cdots \otimes e_{\alpha_1}$ all the way to left using commutation relations (3.17) we get

$$\pi_{\mathfrak{R}} b v_{L(k\Lambda_0)} = (1 \otimes \cdots \otimes e_{\alpha_1} \otimes e_{\alpha_1} \otimes \cdots \otimes e_{\alpha_1}) \pi_{\mathfrak{R}^-} b' v_{k\Lambda_0},$$

where b' is quasi-particle monomial

$$\begin{aligned}
b' &= b'(\alpha_2) b'(\alpha_1) \\
&= x_{n_{r_2^{(1)},2}\alpha_2}(m_{r_2^{(1)},2} - n^{(1)}_{r_2^{(1)},2} - \cdots - n^{(s)}_{r_2^{(1)},2}) \\
&\quad \cdots x_{n_{1,2}\alpha_2}(m_{1,2} - n^{(1)}_{1,2} - \cdots - n^{(s)}_{1,2}) \\
&\quad x_{n_{r_1^{(1)},1}\alpha_1}(m_{r_1^{(1)},1} + 2n_{r_1^{(1)}}) \cdots x_{n_{2,1}\alpha_1}(m_{2,1} + 2n_{1,2}) \\
&= x_{n_{r_2^{(1)},2}\alpha_2}(m'_{r_2^{(1)},2}) \cdots x_{n_{1,2}\alpha_2}(m'_{1,2}) \\
&\quad x_{n_{r_1^{(1)},1}\alpha_1}(m'_{r_1^{(1)},1}) \cdots x_{n_{2,1}\alpha_1}(m'_{2,1}),
\end{aligned} \tag{3.19}$$

$0 \le n_{p,2}^{(t)} \le 3$, $0 \le n_{p,1}^{(t)} \le 1$, $1 \le p \le r_i^{(s)}, 1 \le t \le s$ of dual-charge-type

$$\mathfrak{R}^- = \left(r_2^{(1)}, \ldots, r_2^{(3s)}; r_1^{(1)} - 1, \ldots, r_1^{(s)} - 1\right).$$

Proposition 3.5. *Monomial b' (3.19) is an element of the set $B_{W_{L(k\Lambda_0)}}$.*

Proof. We will prove that $m'_{p,2}$, $2 \le p \le r_1^{(1)}$ and $1 \le p \le r_2^{(1)}$ satisfy the same difference conditions as energies of quasi-particle monomials from the set $B_{W_{L(k\Lambda_0)}}$. We will consider only energies $m'_{p,2}$, since for the color $i = 1$ the proof is similar as in the case of $B_2^{(1)}$. We have two cases:

1) if $n_{p,2} \ge 3s$, then we have:

$$\begin{aligned} m'_{p,2} &= m_{p,2} - 3s \\ &\le -n_{p,2} + \sum_{q=1}^{r_1^{(1)}} \min\{3n_{q,1}, n_{p,2}\} - \sum_{p>p'>0} 2\ \min\{n_{p,2}, n_{p',2}\} - 3s; \\ m'_{p+1,2} &= m_{p+1,2} - 3s \\ &\le m_{p,2} - 2n_{p,2} - 3s \\ &= m'_{p,2} - 2n_{p,2} \quad \text{when} \quad n_{p,2} = n_{p+1,2}; \end{aligned}$$

2) if $n_{p,2} < 3s$, then we have:

$$\begin{aligned} m'_{p,2} &= m_{p,2} - n_{p,2} \\ &\le -n_{p,2} + \sum_{q=1}^{r_1^{(1)}} \min\{3n_{q,1}, n_{p,2}\} - \sum_{p>p'>0} 2\ \min\{n_{p,2}, n_{p',2}\} - n_{p,2} \\ &= -n_{p,2} + \sum_{q=2}^{r_1^{(1)}} \min\{3n_{q,1}, n_{p,2}\} - \sum_{p>p'>0} 2\ \min\{n_{p,2}, n_{p',2}\}; \\ m'_{p+1,2} &= m_{p+1,2} - n_{p,2} \\ &\le m_{p,2} - 2n_{p,2} - n_{p,2} \\ &= m'_{p,2} - 2n_{p,2} \quad \text{when} \quad n_{p,2} = n_{p+1,2}. \end{aligned}$$

□

3.6. *Proof of linear independence.* We prove linear independence of the set $\mathfrak{B}_{W_{L(k\Lambda_0)}}$ by induction on charge-type of monomials from the set $B_{W_{L(k\Lambda_0)}}$. Then from the Proposition 3.3 will follow

Theorem 3.6. *The set $\mathfrak{B}_{W_{L(k\Lambda_0)}}$ is a basis of the principal subspace $W_{L(k\Lambda_0)}$.*

PROOF. First consider a finite linear combination

$$\sum_a c_a b_a v_{L(k\Lambda_0)} = 0 \tag{3.20}$$

of monomial vectors $b_a v_{L(k\Lambda_0)} \in \mathfrak{B}_{W_{L(k\Lambda_0)}}$ of the same color-type (r_2, r_1). Denote by $b = b(\alpha_2)b(\alpha_1)x_{n_{1,1}\alpha_1}(j)$ the smallest monomial in (3.20) such that $c_a \neq 0$. Assume that b is of charge-type

$$\mathfrak{R}' = \left(n_{r_2^{(1)},2}, \ldots, n_{1,2}; n_{r_2^{(1)},1}, \ldots, n_{1,1}\right) \tag{3.21}$$

and a dual-charge-type

$$\mathfrak{R} = \left(r_2^{(1)}, \ldots, r_2^{(3k)}; r_1^{(1)}, \ldots, r_1^{(n_{1,1})}\right),$$

which determines the projection $\pi_{\mathfrak{R}}$ on the vector space

$$W_{L(\Lambda_0)_{(\mu_2^{(k)};0)}} \otimes \cdots \otimes W_{L(\Lambda_0)_{(\mu_2^{(n_{1,1}+1)};0)}} \otimes W_{L(\Lambda_0)_{(\mu_2^{(n_{1,1})};r_1^{(n_{1,1})})}} \otimes \cdots \otimes W_{L(\Lambda_0)_{(\mu_2^{(1)};r_1^{(1)})}},$$

where

$$\mu_l^{(t)} = r_2^{(3t-1)} + r_2^{(3t)} + r_2^{(3t-2)}, \quad 1 \leq t \leq k.$$

$\pi_{\mathfrak{R}}$ maps to zero all vectors $b_a v_{L(k\Lambda_0)}$ in (3.20) with monomials b_a of larger charge-type's than $\mathfrak{R}'$. Now, in

$$\sum_a c_a \pi_{\mathfrak{R}} b_a v_{L(k\Lambda_0)} = 0 \tag{3.22}$$

we have a projection of $b_a v_{L(k\Lambda_0)}$, where b_a are of charge-type $\mathfrak{R}'$.

On (3.22), we act with operators $1 \otimes \cdots \otimes A_\theta \otimes \underbrace{1 \otimes \cdots \otimes 1}_{n_{1,1}-1 \text{ factors}}$ and commute to the left with operators $1 \otimes \cdots \otimes e_\theta \otimes \underbrace{1 \otimes \cdots \otimes 1}_{n_{1,1}-1 \text{ factors}}$ until we get

$$\sum_a c_a \pi_{\mathfrak{R}} b_a(\alpha_2) b_a(\alpha_1) x_{n_{1,1}\alpha_1}(-n_{1,1}) v_{L(k\Lambda_0)} = 0. \tag{3.23}$$

Note, that in (3.23) we only have monomial vectors of charge-type $\mathfrak{R}'$ with quasi-particle $x_{n_{1,1}\alpha_1}(-n_{1,1})$, since operators used above at some point annihilate all other monomial vectors with $x_{n_{1,1}\alpha_1}(m_{1,1}),\ m_{1,1} > -j$.

From the consideration in previous subsection it follows that (3.23) can be written as

$$1 \otimes \cdots \otimes e_{\alpha_1} \otimes \cdots \otimes e_{\alpha_1} \left(\sum_a c_a \pi_{\mathfrak{R}^-} b'_a(\alpha_2) b'_a(\alpha_1) v_{L(k\Lambda_0)}\right) = 0, \tag{3.24}$$

and after dropping out the invertible operator $1 \otimes \cdots \otimes e_{\alpha_1} \otimes \cdots \otimes e_{\alpha_1}$ as

$$\sum_a c_a \pi_{\mathfrak{R}^-} b'_a(\alpha_2) b'_a(\alpha_1) v_{L(k\Lambda_0)} = 0,$$

where $b'_a(\alpha_2)b'_a(\alpha_1) \in B_{W_{L(k\Lambda_0)}}$ are quasi-particle monomials od dual-charge type

$$\mathfrak{R}^- = \left(r_2^{(1)}, \ldots, r_2^{(k)}; r_1^{(1)} - 1, \ldots, r_1^{(n_{1,1})} - 1\right),$$

with smaller charge-type from $\mathfrak{R}'$.

We repeat the described processes, until we get

$$\sum_a c_a \pi_{\mathfrak{R}^-} b_a(\alpha_2) v_{L(k\Lambda_0)} = 0, \tag{3.25}$$

where monomial vectors $b_a(\alpha_2)v_{L(k\Lambda_0)}$ are colored only with color $i = 2$.

Similar as in the case of $B_2^{(1)}$ we will see vectors $b_a(\alpha_2)v_{L(k\Lambda_0)}$ in (3.25) as elements of

$$\underbrace{W_{L^A(3\Lambda_0)} \otimes \cdots \otimes W_{L^A(3\Lambda_0)}}_{k \text{ factors}}, \tag{3.26}$$

where $W_{L^A(3\Lambda_0)}$ is the principal subspace of level 3 standard $\widetilde{sl}_2(\alpha_2)$-module $L^A(3\Lambda_0)$ with the highest weight vector $v_{L(\Lambda_0)}$. Denote by $\pi'_{\mathfrak{R}^-}$ the projection of (3.26) on

$$W_{L^A(\Lambda_0)_{(r_2^{(3k)})}} \otimes W_{L(\Lambda_0)_{(r_2^{(3k-1)})}} \otimes \cdots \otimes W_{L(\Lambda_0)_{(r_2^{(2)})}} \otimes W_{L(\Lambda_0)_{(r_2^{(1)})}}, \tag{3.27}$$

where $W_{L(\Lambda_0)_{(r_2^{(t)})}}$, $1 \leq t \leq 3k$ is a $\mathfrak{h}$-weighted subspace of $W_{L^A(3\Lambda_0)}$ of weight $r_2^{(t)}\alpha_2$. From the condition (3.2), follows that monomial vectors

$$\pi'_{\mathfrak{R}^-}\left(\pi_{\mathfrak{R}^-} b_a(\alpha_2) v_{L(k\Lambda_0)}\right) \tag{3.28}$$

are elements of vector space (3.27). Now, using Georgiev's argument from [18] follows $c_a = 0$. □

4. Characters of principal subspaces

From Theorem 3.6 we easily obtain the character of the principal subspace $W_{L(k\Lambda_0)}$,

$$\text{ch } W_{L(k\Lambda_0)} := \sum_{m, r_1, r_2 \geq 0} \dim W_{L(k\Lambda_0)_{(m,r_1,r_2)}} q^m y_1^{r_1} y_2^{r_2}, \tag{4.1}$$

where $W_{L(k\Lambda_0)_{(m,r_1,r_2)}}$ is a weight subspace spanned by monomial vectors of weight $-m$ and color-type (r_1, r_2) (see [4, 5, 18]).

If we write conditions on energies of quasi-particles of a basis $\mathfrak{B}_{W_{L(k\Lambda_0)}}$ in terms of the dual-charge-type (and the corresponding charge-type)

$$\left(r_2^{(1)}, r_2^{(2)}, \ldots, r_2^{(3k)}; r_1^{(1)}, r_1^{(2)}, \ldots, r_1^{(k)}\right):$$

$$\sum_{p=1}^{r_2^{(1)}}\sum_{q=1}^{r_1^{(1)}}\min\{n_{p,2},3n_{q,1}\}=\sum_{s=1}^{k}r_1^{(s)}(r_2^{(2s)}+r_2^{(2s-1)}+r_2^{(2s-2)}),\tag{4.2}$$

$$\sum_{p=1}^{r_1^{(1)}}(\sum_{p>p'>0}2\min\{n_{p,1},n_{p',1}\}+n_{p,1})=\sum_{s=1}^{k}r_1^{(s)^2},\tag{4.3}$$

$$\sum_{p=1}^{r_2^{(1)}}(\sum_{p>p'>0}2\min\{n_{p,2},n_{p',2}\}+n_{p,2})=\sum_{s=1}^{3k}r_2^{(s)^2},\tag{4.4}$$

then, we have the following result.

THEOREM 4.1.
ch $W_{L(k\Lambda_0)}$

$$=\sum_{\substack{r_1^{(1)}\geq\ldots\geq r_1^{(k)}\geq 0\\ r_2^{(1)}\geq\ldots\geq r_2^{(3k)}\geq 0}}\frac{q^{\sum_{s=1}^{k}r_1^{(s)^2}+\sum_{s=1}^{3k}r_2^{(s)^2}-\sum_{s=1}^{k}r_1^{(s)}(r_2^{(3s)}+r_2^{(3s-1)}+r_2^{(3s-2)})}}{(q)_{r_1^{(1)}-r_1^{(2)}}\cdots(q)_{r_1^{(k)}}(q)_{r_2^{(1)}-r_2^{(2)}}\cdots(q)_{r_2^{(3k)}}}y_1^{r_1}y_2^{r_2},$$

where $(q)_0=1$, $(q)_r=(1-q)(1-q^2)\cdots(1-q^r)$ *for* $r>0$, $r_1=\sum_{s=1}^{k}r_1^{(s)}$ *and* $r_2=\sum_{s=1}^{3k}r_2^{(s)}$.

At the end we state the theorem in which we describe the basis of the principal subspace $W_{N(k\Lambda_0)}$:

THEOREM 4.2. *The set* $\mathfrak{B}_{W_{N(k\Lambda_0)}}=\left\{bv_{N(k\Lambda_0)}:b\in B_{W_{N(k\Lambda_0)}}\right\}$, *where*

$$B_{W_{N(k\Lambda_0)}}=\bigcup_{\substack{n_{r_1^{(1)},1}\leq\ldots\leq n_{1,1}\\ n_{r_2^{(1)},2}\leq\ldots\leq n_{1,2}}}\left(\text{or, equivalently,}\quad\bigcup_{\substack{r_1^{(1)}\geq r_1^{(2)}\geq\cdots\geq 0\\ r_2^{(1)}\geq r_2^{(2)}\geq\cdots\geq 0}}\right)$$

$$\Big\{b=b(\alpha_2)b(\alpha_1)$$
$$=x_{n_{r_2^{(1)},2}\alpha_2}(m_{r_2^{(1)},2})\cdots x_{n_{1,2}\alpha_2}(m_{1,2})x_{n_{r_1^{(1)},1}\alpha_1}(m_{r_1^{(1)},1})\cdots x_{n_{1,1}\alpha_1}(m_{1,1}):$$

$$\left|\begin{array}{l} m_{p,1}\leq -n_{p,1}-\sum_{p>p'>0}2\ \min\{n_{p,1},n_{p',1}\},\ 1\leq p\leq r_1^{(1)};\\ m_{p+1,1}\leq m_{p,1}-2n_{p,1}\ \text{if}\ n_{p+1,1}=n_{p,1},\ 1\leq p\leq r_1^{(1)}-1;\\ m_{p,2}\leq -n_{p,2}+\sum_{q=1}^{r_1^{(1)}}\min\{3n_{q,1},n_{p,2}\}-\sum_{p>p'>0}2\ \min\{n_{p,2},n_{p',2}\},\\ \qquad 1\leq p\leq r_2^{(1)};\\ m_{p+1,2}\leq m_{p,2}-2n_{p,2}\ \text{if}\ n_{p,2}=n_{p+1,2},\ 1\leq p\leq r_2^{(1)}-1\end{array}\right.\Bigg\}$$

is a basis of the principal subspace $W_{N(k\Lambda_0)}$.

The proof of the Theorem 4.2 is similar as in the case of $W_{L(k\Lambda_0)}$ (see [4]), from which we can as before obtain the character of $W_{N(k\Lambda_0)}$:

THEOREM 4.3.

$$\mathrm{ch}\ W_{N(k\Lambda_0)} = \sum_{\substack{r_1^{(1)}\geq\ldots\geq r_1^{(u)}\geq 0\\ r_2^{(1)}\geq\ldots\geq r_2^{(3v)}\geq 0\\ u,v\geq 0}} \frac{q^{\sum_{s=1}^{u} {r_1^{(s)}}^2+\sum_{s=1}^{3v} {r_2^{(s)}}^2-\sum_{s\geq 1} r_1^{(s)}(r_2^{(3s)}+r_2^{(3s-1)}+r_2^{(3s-2)})}}{(q)_{r_1^{(1)}-r_1^{(2)}}\cdots(q)_{r_1^{(u)}}(q)_{r_2^{(1)}-r_2^{(2)}}\cdots(q)_{r_2^{(3v)}}} y_1^{r_1}y_2^{r_2},$$

where $r_1=\sum_{s=1}^{u} r_1^{(s)}$ *and* $r_2=\sum_{s=1}^{3v} r_2^{(s)}$.

From (2.4) and previous theorem follows a generalization of Euler-Cauchy theorem (cf. (2.2.8) and (2.2.9) in [1] and (4.1) in [2]):

THEOREM 4.4.

$$\prod_{m>0}\frac{1}{(1-q^my_1)}\frac{1}{(1-q^my_2)}\frac{1}{(1-q^my_1y_2)}\frac{1}{(1-q^my_1y_2^2)}\frac{1}{(1-q^my_1y_2^3)}\frac{1}{(1-q^my_1^2y_2^3)}$$
$$= \sum_{\substack{r_1^{(1)}\geq r_1^{(2)}\geq r_1^{(3)}\geq\ldots\geq 0\\ r_2^{(1)}\geq r_2^{(2)}\geq r_2^{(3)}\geq\ldots\geq 0}} \frac{q^{\sum_{s\geq 1} {r_1^{(s)}}^2+\sum_{s\geq 1} {r_2^{(s)}}^2-\sum_{s\geq 1} r_1^{(s)}(r_2^{(3s)}+r_2^{(3s-1)}+r_2^{(3s-2)})}}{(q)_{r_1^{(1)}-r_1^{(2)}}(q)_{r_1^{(2)}-r_1^{(3)}}\cdots(q)_{r_2^{(1)}-r_2^{(2)}}(q)_{r_2^{(2)}-r_2^{(3)}}\cdots} y_1^{r_1}y_2^{r_2}, \tag{4.5}$$

where $r_1=\sum_{s\geq 1} r_1^{(s)}$ *and* $r_2=\sum_{s\geq 1} r_2^{(s)}$. *The sum on the right side of (4.5) is over all descending infinite sequences of non-negative integers with finite support.*

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