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FINITE NONABELIAN p-GROUPS OF EXPONENT $> p$ WITH A SMALL NUMBER OF MAXIMAL ABELIAN SUBGROUPS OF EXPONENT $> p$

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Abstract. Y. Berkovich has proposed to classify nonabelian finite p-groups G of exponent $> p$ which have exactly p maximal abelian subgroups of exponent $> p$ and this was done here in Theorem 1 for $p = 2$ and in Theorem 2 for $p > 2$. The next critical case, where G has exactly $p + 1$ maximal abelian subgroups of exponent $> p$ was done only for the case $p = 2$ in Theorem 3.

Let G be a nonabelian finite p-group of exponent $> p$. If S is a minimal nonabelian subgroup in G, then S has exactly $p + 1$ maximal subgroups $S_1, S_2, \ldots, S_{p+1}$ and they are abelian and they lie in $p + 1$ pairwise distinct maximal abelian subgroups in G. If at least two of $S_i's$ are elementary abelian, then S is generated by its elements of order p and then (by Lemma 65.1 in [2]) $S \cong \mathrm{D}_8$ or $S \cong \mathrm{S}(p^3)$ (the nonabelian group of order p^3 and exponent $p > 2$). If all minimal nonabelian subgroups of G are generated by its elements of order p, then by Theorem 10.33 in [1] (for $p = 2$) and Proposition 7 in [3] (for $p > 2$), G has only one maximal abelian subgroup A of exponent $> p$, where A is of index p in G and $A = \mathrm{H}_p(G)$ (Hughes subgroup). However, if a minimal nonabelian subgroup of G has at most one elementary abelian maximal subgroup, then G has at least p maximal abelian subgroups of exponent $> p$.

From the above follows that a nonabelian p-group G of exponent $> p$ has either exactly one maximal abelian subgroup of exponent $> p$ or G has at least p of them. Therefore Y. Berkovich has proposed to classify nonabelian finite p-groups of exponent $> p$ which have exactly p maximal abelian subgroups of exsponent $> p$ and this was done here in Theorem 1 for $p = 2$ and in Theorem

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2 for $p > 2$. By the above, such a group G possesses a minimal nonabelian subgroup S which is not isomorphic to D_8 or $S(p^3)$. Also, such an S has exactly one maximal subgroup X which is elementary abelian so that $\Phi(S) = Z(S)$ is elementary abelian and $|S : \Phi(S)| = p^2$. Let $a \in S - X$ and $b \in X - \Phi(S)$ so that $o(a) \leq p^2$, $o(b) = p$ and $S = \langle a, b\rangle$, where $\Phi(S) = \langle a^p, [a, b]\rangle$. If $|\Phi(S)| = p$, then $|S| = p^3$ and $S \cong M_{p^3}$ (the nonabelian group of order p^3 and exponent p^2, where $p > 2$). If $|\Phi(S)| = p^2$, then $S \cong M_p(2, 1, 1)$, where

$$M_p(2, 1, 1) = \langle a, b \mid a^{p^2} = b^p = 1,\ [a, b] = c,\ c^p = [c, a] = [c, b] = 1\rangle.$$

Suppose that G possesses a non-normal maximal abelian subgroup H of exponent $> p$. Set $K = N_G(H)$ so that $|G : K| = p$, $H < K$ and $H^G \leq K$. All elements in $G - K$ are of order p. If $p = 2$, then K is abelian (by a result of Burnside), a contradiction. Hence in this case we must have $p > 2$. For any $g \in G - K$, $H^g \leq K$ and so H and H^g normalize each other.

Y. Berkovich has proposed to consider also the next critical case, where G has exactly $p + 1$ maximal abelian subgroups of exponent $> p$. However, we have been able to classify such p-groups only in case $p = 2$ in Theorem 3.

THEOREM 1. *Let G be a nonabelian 2-group with exactly 2 maximal abelian subgroups of exponent > 2. Then $G = M \times V$, where*

$$M \cong M_2(2, 1, 1) = \langle a, b \mid a^4 = b^2 = 1,\ [a, b] = c,\ c^2 = [c, a] = [c, b] = 1\rangle$$

and $exp(V) \leq 2$.

PROOF. Let G be a nonabelian 2-group with exactly 2 maximal abelian subgroups of exponent > 2. Let H_1 and H_2 be the two maximal abelian subgroups of exponent > 2, where we know that H_1 and H_2 are normal in G. If $H_1H_2 < G$, then all elements in $G - (H_1H_2)$ are involutions and then (by a result of Burnside) H_1H_2 would be abelian, a contradiction. Hence $H_1H_2 = G$ and $H_1 \cap H_2 = Z(G)$ so that G is of class 2 and all elements in $G-(H_1 \cup H_2)$ are involutions. Indeed, all elements of order > 2 lie in H_1 or H_2 (by our hypothesis). If $g \in G-(H_1 \cup H_2)$, then a maximal abelian subgroup H containing $\langle g\rangle$ is elementary abelian implying that $Z(G)$ is elementary abelian. Since $H \trianglelefteq G$, Lemma 57.1 in [2] implies that for any $x \in G-H$ there is $h \in H$ such that $\langle x, h\rangle$ is minimal nonabelian. Since $\langle x, h\rangle \cong D_8$ or $M_2(2, 1, 1)$, it follows that $\exp(\langle x, h\rangle) = 4$ and so $o(x) \leq 4$. We have proved that $\exp(G) = 4$. For any $x, y \in G$, $[x^2, y] = [x, y]^2 = 1$ and so we get $\mho_1(G) \leq Z(G)$.

Suppose that both H_1 and H_2 are not maximal subgroups in G. Then $|H_i : Z(G)| \geq 4$ for $i = 1, 2$ and let $h_i \in H_i - Z(G)$ be an element of order 4 ($i = 1, 2$) so that $1 \neq h_i^2 \in Z(G)$. Let H_i^* be a maximal subgroup of H_i which contains $Z(G)\langle h_i\rangle$, $i = 1, 2$. Then $M_1 = H_1H_2^*$ and $M_2 = H_2H_1^*$ are distinct maximal subgroups of G containing H_1 and H_2, respectively. Since all elements in $G - (H_1 \cup H_2)$ are involutions, it follows that all elements in

$G - (M_1 \cup M_2)$ are involutions. Let $g \in G - (M_1 \cup M_2)$ and $m \in M_1 \cap M_2$. Then g and $gm \in G - (M_1 \cup M_2)$ are involutions and so we get

$$1 = (gm)^2 = gmgm = g^2 m^g m = m^g m \text{ and so } m^g = m^{-1}.$$

It follows that g inverts each element in $M_1 \cap M_2$ so that a result of Burnside implies that $M_1 \cap M_2$ is abelian. In particular, $\langle h_1, h_2 \rangle$ is abelian. Let Y be a maximal abelian subgroup in G containing $\langle h_1, h_2 \rangle$. By our hypothesis, $Y = H_1$ or $Y = H_2$, a contradiction. We have proved that we may assume $|G : H_1| = 2$ and so H_1 is a maximal subgroup in G.

Let H_1^* be a maximal subgroup of H_1 containing $\Omega_1(H_1)$. Then $M_2 = H_2 H_1^*$ is a maximal subgroup of G and all elements in $G - (H_1 \cup M_2)$ are involutions. If $g \in G - (H_1 \cup M_2)$, then for any $x \in H_1^* = H_1 \cap M_2$, $gx \in G - (H_1 \cup M_2)$ is an involution. This implies $x^g = x^{-1}$ and so g inverts each element in H_1^*. In particular, g centralizes $\Omega_1(H_1)$. It follows that $\Omega_1(H_1) \leq \mathrm{Z}(G)$ and so $\Omega_1(H_1) = \mathrm{Z}(G) = H_1 \cap H_2$ and therefore all elements in $H_1 - \mathrm{Z}(G)$ are of order 4.

Suppose that $\mathrm{Z}(G)$ is not a maximal subgroup in H_1. Note that all elements in $G - (H_1 \cup H_2)$ are involutions and all elements in $H_2 - H_1$ and in $H_1 - \mathrm{Z}(G)$ are of order 4. Let $v \in H_1 - \mathrm{Z}(G)$ so that $v^2 \in \mathrm{Z}(G)$ and let H_1^{**} be a maximal subgroup of H_1 containing $\mathrm{Z}(G)\langle v \rangle$ so that $M_2^* = H_1^{**} H_2$ is a maximal subgroup in G. If $g \in G - (H_1 \cup M_2^*)$, then g and $gv \in G - (H_1 \cup M_2^*)$ are involutions implying that $v^g = v^{-1}$. Then each element in $G - H_1$ also inverts $\langle v \rangle$. Hence each element in $G - H_1$ inverts each element of order 4 in H_1 and since it also centralizes $\mathrm{Z}(G)$, it follows that each element in $G - H_1$ inverts each element in H_1. But then G is quasidihedral and so in particular all elements in $G - H_1$ must be involutions, a contradiction. We have proved that $\mathrm{Z}(G) = H_1 \cap H_2$ is a maximal subgroup in H_1 and so H_2 is also a maximal subgroup in G.

If each minimal nonabelian subgroup in G is isomorphic to D_8, then by Theorem 10.33 in [1] our group G is quasidihedral and so G has only one maximal abelian subgroup of exponent > 2, a contradiction. Hence G possesses a minimal nonabelian subgroup

$$M \cong \mathrm{M}_2(2,1,1) = \langle a, b \mid a^4 = b^2 = 1,\ [a,b] = c,\ c^2 = [c,a] = [c,b] = 1 \rangle.$$

Then M covers G/H_1 and $H_1/\mathrm{Z}(G)$ and $M \cap H_1$ is abelian of type $(4,2)$, where we have $M \cap \mathrm{Z}(G) \cong \mathrm{E}_4$. Indeed, if M does not cover G/H_1 or $H_1/\mathrm{Z}(G)$, then M would be abelian, a contradiction. Let V be a complement of $M \cap \mathrm{Z}(G)$ in $\mathrm{Z}(G)$. Then $G = M \times V$ and our theorem is proved. ∎

THEOREM 2. *Let G be a nonabelian p-group of exponent $> p$, where $p > 2$. Suppose that G has exactly p maximal abelian subgroups $H_1, H_2, \ldots, H_p$ of exponent $> p$. Then $exp(G) = p^2$, $\mathrm{Z}(G)$ is elementary abelian, each H_i normalizes each H_j $(i,j = 1,2,\ldots,p)$, $H = H_1 H_2 \cdots H_p = \mathrm{H}_p(G)$ (Hughes subgroup) and $\mho_1(G) \leq \mathrm{Z}(H) = H_1 \cap H_2 \cdots \cap H_p$.*

Proof. Let G be a p-group, $p > 2$, satisfying the assumptions of Theorem 2. It is easy to see that G possesses at least one minimal nonabelian subgroup M which is isomorphic to M_{p^3} or $\mathrm{M}_p(2,1,1)$. Suppose that this is false. Then all minimal nonabelian subgroups of G are isomorphic to $\mathrm{S}(p^3)$ and so by Proposition 7 in [3] G has an abelian subgroup A of exponent $> p$ and index p such that $A = \mathrm{H}_p(G)$. But then G has only one maximal abelian subgroup of exponent $> p$, a contradiction. Hence there is such M as above. Any two maximal subgroups of M lie in two distinct maximal abelian subgroups in G. In this way we get p pairwise distinct maximal abelian subgroups in G of exponent $> p$ and one maximal abelian subgroup which is elementary abelian. In particular, $\mathrm{Z}(G)$ is elementary abelian.

We want to show that $\exp(G) = p^2$. Let $H_1, H_2, \ldots, H_p$ be the set of all p maximal abelian subgroups in G which are of exponent $> p$. Set $\exp(G) = p^e$, where $e \geq 2$ and let g be an element of order p^e so that $g \in H = H_1 H_2 \cdots H_p$, where we know that each H_i normalizes each H_j (see the paragraph preceding Theorem 1). If g is not contained in all H_i $(i = 1, 2, \ldots, p)$, say $g \notin H_1$, then by Lemma 57.1 in [2], there is $h_1 \in H_1$ such that $\langle g, h_1 \rangle$ is minimal nonabelian. Since all minimal nonabelian subgroups of G are of exponent $\leq p^2$, we get $e = 2$. So suppose that $g \in H_i$ for all $i = 1, 2, \ldots, p$. In particular, $g \in H_1 \cap H_2$. Since $\langle H_2 - H_1 \rangle = H_2$, there is $h \in H_2 - H_1$ such that $\mathrm{o}(h) = p^e$. By Lemma 57.1 in [2], there is $k \in H_1$ such that $\langle h, k \rangle$ is minimal nonabelian. This implies again $e = 2$. We have proved that $\exp(G) = p^2$. If $H < G$, then all elements in $G - H$ are of order p and so $H = \mathrm{H}_p(G)$. Now, $\mathrm{Z}(H)$ centralizes all H_i and so $\mathrm{Z}(H) \leq H_1 \cap H_2 \cdots \cap H_p$. But $H_1 \cap H_2 \cdots \cap H_p \leq \mathrm{Z}(H)$ and so we get $\mathrm{Z}(H) = H_1 \cap H_2 \cdots \cap H_p$.

Let g be any element of order p^2 in G. Then $g \in H = H_1 H_2 \cdots H_p$, where $H_i \trianglelefteq H$ for all $i = 1, 2, \ldots, p$. We have either $g \in H_i$ (and then also $g^p \in H_i$) or (by Lemma 57.1 in [2]) there is $h_i \in H_i$ such that $M = \langle g, h_i \rangle$ is minimal nonabelian, where $M \cong \mathrm{M}_{p^3}$ or $M \cong \mathrm{M}_p(2,1,1)$. Then we know that M contains exactly one maximal subgroup X of exponent p^2 such that $X \leq H_i$. This implies that $g^p \in X \leq H_i$. Hence in any case we get $g^p \in H_i$ for all $i = 1, 2, \ldots, p$. Hence $g^p \in H_1 \cap H_2 \cdots \cap H_p = \mathrm{Z}(H)$ and so $\mho_1(G) \leq \mathrm{Z}(H)$. Our theorem is proved. ∎

Theorem 3. *Let G be a nonabelian 2-group with exactly 3 maximal abelian subgroups H_1, H_2, H_3 of exponent > 2. Then $G = H_1 H_2 H_3$ and $\mathrm{Z}(G) = H_1 \cap H_2 \cap H_3$.*

(a) *If H_1 is conjugate in G to (say) H_2, then $\exp(H_1) = 4$, H_3 is of index 2 in G with $\exp(H_3) \leq 8$, $\mathrm{Z}(G)$ is elementary abelian and G has a maximal subgroup which is quasidihedral of exponent 4.*

(b) *If all H_i are normal in G, $i = 1, 2, 3$, then G is of class 2, $\mho_1(G) \leq \mathrm{Z}(G)$ and so G' is elementary abelian.*

Proof. Let G be a nonabelian 2-group with exactly 3 maximal abelian subgroups H_1, H_2, H_3 of exponent > 2. Set $H = \langle H_1, H_2, H_3\rangle$ so that $H \trianglelefteq G$. If $H < G$, then all elements in $G - H$ are involutions. But then (by a result of Burnside) H is abelian, a contradiction. Hence we have $G = \langle H_1, H_2, H_3\rangle$ and then obviously $\mathrm{Z}(G) = H_1 \cap H_2 \cap H_3$.

(i) First we consider the case where some H_i are not normal in G.

Then we may assume that H_1 and H_2 are conjugate in G and then $H_3 \trianglelefteq G$. We set $K = \mathrm{N}_G(H_1)$ so that $|G : K| = 2$, $H_1 < K$ and $K = \mathrm{N}_G(H_2)$. For any $g \in G - K$, $H_2 = H_1^g$ and $H_1H_2 = H_1^G$. Then H_3 covers $G/(H_1H_2)$ so that $G = (H_1H_2)H_3$. All elements in $G-(K\cup H_3)$ are involutions and so for each involution $i \in G - (K \cup H_3)$, a maximal abelian subgroup in G containing i is elementary abelian. In particular, $\mathrm{Z}(G)$ is elementary abelian.

Set $G_1 = H_1H_3$ and let $g \in H_3 - K$ so that $H_2 = H_1^g \le G_1$. It follows $G_1 = G$ and set $H_3^* = H_3 \cap K$ so that H_3^* normalizes H_1. We have $H_1 \cap H_3 = \mathrm{Z}(G)$ is elementary abelian and also $H_2 \cap H_3 = \mathrm{Z}(G)$. Then $K = H_1H_3^*$ and

$$K' \le H_1 \cap H_3^* = \mathrm{Z}(G) \le \mathrm{Z}(K)$$

so that K is of class 2 and K' is elementary abelian. For any $k_1, k_2 \in K$ follows $[k_1^2, k_2] = [k_1, k_2]^2 = 1$ and so $\mho_1(K) \le \mathrm{Z}(K)$. We have $\mathrm{Z}(K) < H_1$ and if $\mathrm{Z}(K) > \mathrm{Z}(G)$, then $\mathrm{Z}(K)H_3^*$ is contained in a maximal abelian subgroup in G distinct from H_1, H_2 and H_3 and so $\mathrm{Z}(K)H_3^*$ must be elementary abelian. We have proved that in any case $\mathrm{Z}(K)$ is elementary abelian and so $\exp(K) = 4$ and $4 \le \exp(H_3) \le 8$.

Assume, by way of contradiction, that $\mathrm{Z}(K) > \mathrm{Z}(G)$. Since $\mathrm{Z}(K) < H_1$, it follows that $L = \mathrm{Z}(K)H_3$ is a proper subgroup of G. We know that all elements in $G-(K\cup L)$ are involutions. Let $i \in G-(K\cup L)$ and $x \in K\cap L$. Then $ix \in G - (K \cup L)$ and so

$$1 = (ix)^2 = ixix \text{ implying } x^i = x^{-1}.$$

Since i inverts each element in $K\cap L$, it follows that i centralizes $\mathrm{Z}(K)$ (noting that $\mathrm{Z}(K)$ is elementary abelian). But then $\mathrm{Z}(K) \le \mathrm{Z}(G)$, a contradiction. We have proved that $\mathrm{Z}(K) = \mathrm{Z}(G)$ and so in particular, $\mho_1(H_1) \le \mathrm{Z}(G)$.

Suppose, by way of contradiction, that H_3 is not a maximal subgroup in G. Let v be an element of order 4 in H_1 so that $v^2 \in \mathrm{Z}(K) = \mathrm{Z}(G)$ and we set $R = H_3\langle v\rangle$. Since $|R : H_3| = 2$, it follows that R is a proper subgroup of G and all elements in $G - (K \cup R)$ are involutions. If $i \in G - (K \cup R)$ and $y \in K \cap R$, then $iy \in G - (K \cup R)$ so that iy is an involution implying $y^i = y^{-1}$. Thus i inverts each element in $K\cap R = \langle v\rangle H_3^*$ implying that $K\cap R$ is abelian. Let X be a maximal abelian subgroup of G containing $K \cap R$. Since X is obviously distinct from each H_i, $i = 1, 2, 3$, and $\exp(X) > 2$, we have a contradiction. We have proved that H_3 is a maximal subgroup in G.

All elements in $G - (K \cup H_3)$ are involutions, where K and H_3 are two distinct maximal subgroups in G. Then each involution $i \in G - (K \cup H_3)$

inverts each element in $K \cap H_3 = H_3^*$. In particular, i centralizes $\Omega_1(H_3^*)$ and so $\Omega_1(H_3^*) = H_1 \cap H_3 = \mathrm{Z}(G)$. Since $H_1 \cap H_3 < H_3^*$, it follows that $\exp(H_3^*) = 4$. Then $H_3^*\langle i \rangle$ is quasidihedral of exponent 4 and $H_3^*\langle i \rangle$ is a maximal subgroup in G. Finally, $H_1 \cap H_3^* = \mathrm{Z}(G)$ is a maximal subgroup of H_1 and so $\exp(H_1) = 4$ and $G = H_1 H_3 = H_1 H_2 H_3$. We have proved all properties of G stated in part (a) of our theorem.

(ii) Now assume that all H_i are normal in G, $i = 1, 2, 3$.
Then we have again $G = H_1 H_2 H_3$.

(ii1) First suppose that H_1, H_2 and H_3 do not cover G.
Then $G - (H_1 \cup H_2 \cup H_3)$ is not empty so that all elements in $G - (H_1 \cup H_2 \cup H_3)$ are involutions. Let $i \in G - (H_1 \cup H_2 \cup H_3)$ and let A be a maximal abelian subgroup in G containing i so that A is distinct from H_1, H_2 and H_3 implying that A must be elementary abelian. Since $\mathrm{Z}(G) < A$, it follows that $\mathrm{Z}(G)$ is elementary abelian.

It is easy to see that $\exp(G) = 4$. Suppose that $g \in G$ with $\mathrm{o}(g) \geq 8$. For any $i \in \{1, 2, 3\}$, we have either $g \in H_i$ (and then also $g^2 \in H_i$) or $g \in G - H_i$. In the second case Lemma 57.1 in [2] implies that there is $h \in H_i$ such that $M = \langle g, h \rangle$ is minimal nonabelian. Since $\exp(M) \geq 8$, each of the three maximal subgroups M_i $(i = 1, 2, 3)$ of M are of exponent > 2 and they lie in three pairwise distinct maximal abelian subgroups H_1, H_2, H_3 of exponent > 2 in G. Hence for an $j \in \{1, 2, 3\}$, we have $M_j \leq H_i$ and then $g^2 \in M_j \leq H_i$. We have proved that in any case $g^2 \in H_i$ for each $i \in \{1, 2, 3\}$ and so $g^2 \in H_1 \cap H_2 \cap H_3 = \mathrm{Z}(G)$. But $\mathrm{Z}(G)$ is elementary abelian and so $\mathrm{o}(g^2) \leq 2$, a contradiction. We have proved that $\exp(G) = 4$.

Suppose that there is $h \in G$ of order 4 such that $h^2 \notin \mathrm{Z}(G)$. Since all elements of order 4 in G are contained in $H_1 \cup H_2 \cup H_3$, we may assume that $h \in H_1$. Then interchanging H_2 and H_3 (if necessary), we may assume that $h^2 \notin H_2$. Set $K_0 = H_1 H_2$ so that $\mathrm{Z}(K_0) = H_1 \cap H_2$ and $h^2 \notin \mathrm{Z}(K_0)$. We have $K_0' \leq H_1 \cap H_2 = \mathrm{Z}(K_0)$ and so K_0 is of class 2. Suppose, by way of contradiction, that $\exp(\mathrm{Z}(K_0)) = 4$. Let $k \in K_0 - (H_1 \cup H_2)$ and let B be a maximal abelian subgroup of G containing $\mathrm{Z}(K_0)\langle k \rangle$ so that we must have $B = H_3$. But then $H_3 \geq \mathrm{Z}(K_0)$ and so $\mathrm{Z}(K_0) = H_1 \cap H_2 \cap H_3 = \mathrm{Z}(G)$, a contradiction. Hence $\mathrm{Z}(K_0)$ is elementary abelian. But then for all $x \in K_0$, $[h^2, x] = [h, x]^2 = 1$ and so $h^2 \in \mathrm{Z}(K_0)$, a final contradiction. We have proved that $\mho_1(G) \leq \mathrm{Z}(G)$ implying that G' is elementary abelian and so we have obtained some 2-groups from part (b) of our theorem.

(ii2) Now assume that $G = H_1 \cup H_2 \cup H_3$, i.e., H_1, H_2, H_3 cover G.
Let $i \neq j$ with $i, j \in \{1, 2, 3\} = \{i, j, k\}$. If $H_i H_j < G$, then $H_k \geq G - (H_i H_j)$ and since $\langle G - (H_i H_j) \rangle = G$, G would be abelian, a contradiction. Thus

$$H_i H_j = G,\ H_i \cap H_j = \mathrm{Z}(G),\ H_k \geq G - (H_i \cup H_j) \text{ and } H_k \geq \mathrm{Z}(G).$$

Because $i \neq j$ are arbitrary elements in $\{1, 2, 3\}$, we also get

$$H_i \cap H_k = H_j \cap H_k = \mathrm{Z}(G) \text{ and so } H_k = (G - (H_i \cup H_j)) \cup \mathrm{Z}(G).$$

Also, $G' \leq H_i \cap H_j = \mathrm{Z}(G)$ and so G is of class 2.

If $\mathrm{Z}(G)$ is elementary abelian, then for any $x, y \in G$, $[x^2, y] = [x, y]^2 = 1$ and so $\mho_1(G) \leq \mathrm{Z}(G)$. So assume that $\exp(\mathrm{Z}(G)) > 2$. In this case each maximal abelian subgroup of G contains $\mathrm{Z}(G)$ and so must be equal to one of H_1, H_2, H_3. Let $g \in G$. Then either $g \in H_i$ (and then also $g^2 \in H_i$) or $g \in G - H_i$. In the second case, by Lemma 57.1 in [2], there is $h \in H_i$ such that $M = \langle g, h \rangle$ is minimal nonabelian. Then three maximal subgroups S_1, S_2, S_3 of M lie in three pairwise distinct maximal abelian subgroups in G which are equal to H_1, H_2 or H_3. Hence we may assume $S_1 \leq H_i$ and so $g^2 \in H_i$. Thus in any case, $g^2 \in H_1 \cap H_2 \cap H_3 = \mathrm{Z}(G)$ and so we get again $\mho_1(G) \leq \mathrm{Z}(G)$. For any $x, y \in G$, $[x, y]^2 = [x^2, y] = 1$ and so G' is elementary abelian. We have obtained the groups from part (b) of our theorem and we are done. □

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