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DUAL FRAMES COMPENSATING FOR ERASURES

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Abstract. We discuss the problem of recovering signal from frame coefficients with erasures. Such problems arise naturally from applications where some of the coefficients could be corrupted or erased during the data transmission. Provided that the erasure set satisfies the minimal redundancy condition, we construct a suitable synthesizing dual frame which enables us to perfectly reconstruct the original signal without recovering the lost coefficients. Such dual frames which compensate for erasures are described from various viewpoints.

1. Introduction

Frames are often used in process of encoding and decoding signals. It is the redundancy property of frames that makes them robust to erasures and corrupted data. A number of articles have been written on methods for reconstruction from frame coefficients with erasures and related problems.

Recall that a sequence $(x_n)_{n=1}^{\infty}$ in a Hilbert space H is a *frame* for H if there exist positive constants A and B, that are called frame bounds, such that

$$A\|x\|^2 \leq \sum_{n=1}^{\infty} |\langle x, x_n\rangle|^2 \leq B\|x\|^2, \quad \forall x \in H. \tag{1.1}$$

If $A = B$ we say that a frame is *tight* and, in particular, if $A = B = 1$ so that

$$\sum_{n=1}^{\infty} |\langle x, x_n\rangle|^2 = \|x\|^2, \quad \forall x \in H, \tag{1.2}$$

we say that $(x_n)_{n=1}^{\infty}$ is a *Parseval frame.* A sequence $(x_n)_{n=1}^{\infty}$ in H is a *Bessel sequence* if it satisfies the second inequality in (1.1).

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For each Bessel sequence $(x_n)_{n=1}^{\infty}$ in H one defines the *analysis operator* $U : H \to \ell^2$ by $Ux = (\langle x, x_n \rangle)_n$, $x \in H$. It is evident that U is bounded. Its adjoint operator U^*, which is called the *synthesis operator*, is given by $U^*((c_n)_n) = \sum_{n=1}^{\infty} c_n x_n$, $(c_n)_n \in \ell^2$. Moreover, if $(x_n)_{n=1}^{\infty}$ is a frame, the analysis operator U is also bounded from below, the synthesis operator U^* is a surjection and the product U^*U (sometimes called the *frame operator*) is an invertible operator on H. It turns out that the sequence $(y_n = (U^*U)^{-1}x_n)_{n=1}^{\infty}$ is also a frame for H that is called the *canonical dual* frame and satisfies the reconstruction formula

$$x = \sum_{n=1}^{\infty} \langle x, x_n \rangle y_n, \quad \forall x \in H. \tag{1.3}$$

In general, the canonical dual is not the only frame for H which provides us with the reconstruction in terms of the frame coefficients $\langle x, x_n \rangle$. Any frame $(v_n)_{n=1}^{\infty}$ for H that satisfies

$$x = \sum_{n=1}^{\infty} \langle x, x_n \rangle v_n, \quad \forall x \in H \tag{1.4}$$

is called a *dual frame* for $(x_n)_{n=1}^{\infty}$.

In the present paper we work with infinite frames for infinite-dimensional separable Hilbert spaces and our frames will be denoted as $(x_n)_n$, $(y_n)_n$, etc. Accordingly, by writing $\sum_{n=1}^{\infty} c_n x_n$, $\sum_{n=1}^{\infty} c_n y_n$, ... with $(c_n)_n \in \ell^2$, we indicate that the corresponding summations consist of infinitely many terms. However, all the results that follow (including the proofs) are valid for finite frames in finite-dimensional spaces.

Frames were first introduced by Duffin and Schaeffer in [7]. The readers are referred to some standard references, e.g. [5, 6, 9, 12] for more information about frame theory and their applications.

In applications, we first compute the frame coefficients $\langle x, x_n \rangle$ of a signal x (analyzing or encoding x) and then apply (1.3) or (1.4) to reconstruct (synthesizing or decoding) x using a suitable dual frame. During the processing the frame coefficients or data transmission some of the coefficients could get lost. Thus, a natural question arises: how to reconstruct the original signal in a best possible way with erasure-corrupted frame coefficients? Recently many researchers have been working on different approaches to this and related problems. In particular, we refer the readers to [2–4, 8, 10, 11, 13–15] and references therein.

It turns out that the perfect reconstruction is possible as long as erased coefficients are indexed by a set that satisfies the *minimal redundancy condition* ([13]; see Definition 2.1 below). Most approaches assume a pre-specified dual frame and hence aim to recover the missing coefficients using the non-erased ones. Alternatively, one may try to find an alternate dual frame, depending on the set of erased coefficients, in order to compensate for errors.

Here we use this second approach. Assuming that the set of indices E for which the coefficients $\langle x, x_n \rangle$, $n \in E$, are erased is finite and satisfies the minimal redundancy condition, we construct a frame $(v_n)_n$ dual to $(x_n)_n$ such that

$$v_n = 0, \quad \forall n \in E. \tag{1.5}$$

Obviously, such an "E^c-supported" frame $(v_n)_n$ (with E^c denoting the complement of E in the index set), enables the perfect reconstruction using (1.4) without knowing or recovering the lost coefficients $\langle x, x_n \rangle$, $n \in E$. Such dual frames are the central object of our study in the present article.

In Section 2 we construct, for each finite set E satisfying the minimal redundancy condition, a dual frame with property (1.5). The construction is enabled by a parametrization of dual frames by oblique projections to the range of the analysis operator that is obtained in [1]. Moreover, Theorem 2.5 provides a concrete procedure for computing the elements of such a dual frame in terms of the canonical dual. It turns out that the computation boils down to solving certain system of linear equations. Our discussion also includes several characterizations of finite sets which satisfy the minimal redundancy condition (see Proposition 2.4, Proposition 2.8 and Lemma 2.13).

In Theorem 2.12 we provide another description of the dual frame constructed in Theorem 2.5. Then we introduce in Theorem 2.14 a finite iterative algorithm for computing the elements of the constructed dual frame. Finally, in our Theorem 2.15 we improve a result from [13] which provides us with an alternative technique for obtaining our dual.

At the end of this introductory section we establish the rest of our notation. The linear span of a set X will be denoted by $\operatorname{span} X$, and its closure by $\overline{\operatorname{span}}\, X$. The set of all bounded operators on a Hilbert space H is denoted by $\mathbb{B}(H)$ (or $\mathbb{B}(H,K)$ if two different spaces are involved). For $x, y \in H$ we denote by $\theta_{x,y}$ a rank one operator on H defined by $\theta_{x,y}(v) = \langle v, y \rangle x$, $v \in H$. The null-space and the range of a bounded operator T will be denoted by $\mathrm{N}(T)$ and $\mathrm{R}(T)$, respectively. By $X \dot{+} Y$ we denote a direct sum of (sub)spaces X and Y. Finally, we denote by $(e_n)_n$ the canonical basis in ℓ^2.

2. Results

We begin with the definition of the minimal redundancy condition as formulated in [13].

Definition 2.1. *Let $(x_n)_n$ be a frame for a Hilbert space H. We say that a finite set of indices E satisfies the minimal redundancy condition for $(x_n)_n$ if $\overline{\operatorname{span}}\,\{x_n : n \in E^c\} = H$.*

Remark 2.2. If a finite set E satisfies the minimal redundancy condition for a frame $(x_n)_n$ for H it is a non-trivial fact, though relatively easy to prove, that the reduced sequence $(x_n)_{n\in E^c}$ is still a frame for H. In fact, if H is finite-dimensional, there is nothing to prove since in this situation frames are just spanning sets. We refer the reader to [13] for a proof in the infinite-dimensional case.

We also note: if E satisfies the minimal redundancy condition for a frame $(x_n)_n$ with the analysis operator U, then E has the same property for all frames of the form $(Tx_n)_n$ where $T \in \mathbb{B}(H)$ is a surjection. In particular, this applies to the canonical dual $(y_n)_n, y_n = (U^*U)^{-1}x_n$, $n \in \mathbb{N}$, and to the associated Parseval frame $((U^*U)^{-\frac{1}{2}}x_n)_n$.

Remark 2.3. Suppose that $(x_n)_n$ is a frame for H for which a finite set E satisfies the minimal redundancy condition. Then, clearly, there exists a frame $(v_n)_n$ for H dual to $(x_n)_n$ such that $v_n = 0$ for all $n \in E$. Indeed, since $(x_n)_{n\in E^c}$ is a frame for H (as noted in the preceding Remark 2.2), by taking an arbitrary dual frame $(v_n)_{n\in E^c}$ of $(x_n)_{n\in E^c}$ and putting $v_n = 0$ for $n \in E$, we get a dual frame $(v_n)_n$ of $(x_n)_n$ with the desired property.

However, from the application point of view this is not enough; what we really need is a concrete construction of such a dual $(v_n)_n$.

We start with some alternative descriptions of the minimal redundancy condition. Note that the equivalence (a)⇔(d) in our Proposition 2.4 below is proved in Lemma 2.3 from [10] for finite frames using a different technique.

We first need some additional notation.

Consider an arbitrary frame $(x_n)_n$ for H with the analysis operator U and a finite set of indices $E = \{n_1, n_2, \ldots, n_k\}$. Obviously, sequences $(x_n)_{n\in E^c}$ and $(x_n)_{n\in E}$ are Bessel. Denote the corresponding analysis operators by U_{E^c} and U_E, respectively. Notice that $(x_n)_{n\in E}$ is finite, so U_E takes values in $\mathbb{C}^k$. It is evident that the corresponding frame operators are given by $U_{E^c}^*U_{E^c}x = \sum_{n\in E^c}\langle x, x_n\rangle x_n$, $U_E^*U_Ex = \sum_{n\in E}\langle x, x_n\rangle x_n$, $x \in H$, and hence

$$U_{E^c}^*U_{E^c} = U^*U - U_E^*U_E. \tag{2.1}$$

Further, if $(y_n)_n$ is the canonical dual of $(x_n)_n$, its analysis operator V is of the form $V = U(U^*U)^{-1}$. The analysis operators of Bessel sequences $(y_n)_{n\in E^c}$ and $(y_n)_{n\in E}$ will be denoted by V_{E^c} and V_E, respectively. Observe that $V_{E^c} = U_{E^c}(U^*U)^{-1}$ and $V_E = U_E(U^*U)^{-1}$. Since $V^*U = I$, we obtain (in the same way as (2.1))

$$V_{E^c}^*U_{E^c} = I - V_E^*U_E. \tag{2.2}$$

Proposition 2.4. *Let $(x_n)_n$ be a frame for a Hilbert space H with the analysis operator U and the canonical dual $(y_n)_n$. Let $E = \{n_1, n_2, \ldots, n_k\}$, $k \in \mathbb{N}$, be a finite set of indices. The following statements are equivalent:*

(a) *E satisfies the minimal redundancy condition for $(x_n)_n$.*

(b) $\mathrm{R}(U) \cap \mathit{span}\{e_n : n \in E\} = \{0\}$.
(c) $I - V_E^* U_E \in \mathbb{B}(H)$ *is invertible.*
(d) *The matrix*

$$\begin{bmatrix} \langle y_{n_1}, x_{n_1} \rangle & \langle y_{n_2}, x_{n_1} \rangle & \dots & \langle y_{n_k}, x_{n_1} \rangle \\ \langle y_{n_1}, x_{n_2} \rangle & \langle y_{n_2}, x_{n_2} \rangle & \dots & \langle y_{n_k}, x_{n_2} \rangle \\ \vdots & \vdots & & \vdots \\ \langle y_{n_1}, x_{n_k} \rangle & \langle y_{n_2}, x_{n_k} \rangle & \dots & \langle y_{n_k}, x_{n_k} \rangle \end{bmatrix} - I$$

is invertible.

Proof. We can assume without loss of generality that $E = \{1, 2, \dots, k\}$.

(a)⇔(b) Suppose that we have $s \in \mathrm{R}(U) \cap \mathrm{span}\{e_n : n \in E\}$, $s \neq 0$. Equivalently, there exists $x \in H$, $x \neq 0$, such that $x \perp x_n$ for all $n \in E^c$. By continuity of the inner product, this is equivalent to $x \perp \overline{\mathrm{span}}\{x_n : n \in E^c\}$. So, the intersection $\mathrm{R}(U) \cap \mathrm{span}\{e_n : n \in E\}$ is non-trivial if and only if the sequence $(x_n)_{n \in E^c}$ is not fundamental in H.

(a)⇔(c) By Remark 2.2, E satisfies the minimal redundancy condition for $(x_n)_n$ if and only if $(x_n)_{n \in E^c}$ is a frame for H, if and only if the operator $U_{E^c}^* U_{E^c}$ is invertible. Since

$$I - V_E^* U_E \overset{(2.2)}{=} V_{E^c}^* U_{E^c} = (U^* U)^{-1} U_{E^c}^* U_{E^c},$$

this is further equivalent with invertibility of $I - V_E^* U_E \in \mathbb{B}(H)$.

(c)⇔(d) First, $I - V_E^* U_E \in \mathbb{B}(H)$ is invertible if and only if its adjoint $I - U_E^* V_E \in \mathbb{B}(H)$ is invertible. By a well known result, $I - U_E^* V_E \in \mathbb{B}(H)$ is invertible if and only if $I - U_E V_E^* \in \mathbb{B}(\mathbb{C}^k)$ is invertible. It only remains to notice that the matrix of $U_E V_E^* - I$ in the canonical basis of $\mathbb{C}^k$ is precisely

$$\begin{bmatrix} \langle y_1, x_1 \rangle & \langle y_2, x_1 \rangle & \dots & \langle y_k, x_1 \rangle \\ \langle y_1, x_2 \rangle & \langle y_2, x_2 \rangle & \dots & \langle y_k, x_2 \rangle \\ \vdots & \vdots & & \vdots \\ \langle y_1, x_k \rangle & \langle y_2, x_k \rangle & \dots & \langle y_k, x_k \rangle \end{bmatrix} - I.$$

∎

We now prove our main result. In the theorem that follows we give a concrete description of a dual frame with the desired property (as in Remark 2.3) in terms of the canonical dual.

Theorem 2.5. *Let $(x_n)_n$ be a frame for a Hilbert space H with the canonical dual $(y_n)_n$. Suppose that a finite set of indices $E = \{n_1, n_2, \dots, n_k\}, k \in \mathbb{N}$, satisfies the minimal redundancy condition for $(x_n)_n$. For each $n \in E^c$ let*

$(\alpha_{n1}, \alpha_{n2}, \ldots, \alpha_{nk})$ *be a (unique) solution of the system*

(2.3)
$$\left(\begin{bmatrix} \langle y_{n_1}, x_{n_1} \rangle & \langle y_{n_2}, x_{n_1} \rangle & \ldots & \langle y_{n_k}, x_{n_1} \rangle \\ \langle y_{n_1}, x_{n_2} \rangle & \langle y_{n_2}, x_{n_2} \rangle & \ldots & \langle y_{n_k}, x_{n_2} \rangle \\ \vdots & \vdots & & \vdots \\ \langle y_{n_1}, x_{n_k} \rangle & \langle y_{n_2}, x_{n_k} \rangle & \ldots & \langle y_{n_k}, x_{n_k} \rangle \end{bmatrix} - I \right) \begin{bmatrix} \alpha_{n1} \\ \alpha_{n2} \\ \vdots \\ \alpha_{nk} \end{bmatrix} = \begin{bmatrix} \langle y_n, x_{n_1} \rangle \\ \langle y_n, x_{n_2} \rangle \\ \vdots \\ \langle y_n, x_{n_k} \rangle \end{bmatrix}.$$

Put

$$(2.4)\quad v_{n_1} = v_{n_2} = \ldots = v_{n_k} = 0,\ v_n = y_n - \sum_{i=1}^{k} \alpha_{ni} y_{n_i}, \quad n \neq n_1, n_2, \ldots, n_k.$$

Then $(v_n)_n$ *is a frame for* H *dual to* $(x_n)_n$*.*

Proof. Denote by U the analysis operator of $(x_n)_n$. Recall from Corollary 2.4 in [1] that all dual frames of $(x_n)_n$ are parameterized by bounded oblique projections to $\mathrm{R}(U)$ or, equivalently, by closed direct complements of $\mathrm{R}(U)$ in ℓ^2. More precisely, a frame $(v_n)_n$ with the analysis operator V is dual to $(x_n)_n$ if and only if V^* is of the form $V^* = (U^*U)^{-1}U^*F$ where $F \in \mathbb{B}(\ell^2)$ is the oblique projection to $\mathrm{R}(U)$ parallel to some closed subspace Y of ℓ^2 such that $\ell^2 = \mathrm{R}(U) \dot{+} Y$. In particular, the canonical dual frame $(y_n)_n$ corresponds to the orthogonal projection $F = U(U^*U)^{-1}U^*$ to $\mathrm{R}(U)$.

Hence, to obtain a dual frame $(v_n)_n$ with the required property $v_n = 0$ for $n \in E$, we only need to find a closed direct complement Y of $\mathrm{R}(U)$ in ℓ^2 such that $e_n \in Y$ for all $n \in E$. Then we will have

$$Fe_n = 0, \quad \forall n \in E,$$

and, consequently,

$$v_n = V^* e_n = (U^*U)^{-1}U^*Fe_n = 0, \quad \forall n \in E.$$

Since E satisfies the minimal redundancy condition for $(x_n)_n$, Proposition 2.4 tells us that $\mathrm{R}(U) \cap \mathrm{span}\{e_n : n \in E\} = \{0\}$. Denote by Z the orthogonal complement of $\mathrm{R}(U) \dot{+} \mathrm{span}\{e_n : n \in E\}$. (Indeed, this is a closed subspace, being a sum of two closed subspaces, one of which is finite-dimensional.) In other words, let

$$(2.5)\qquad \ell^2 = \Big(\mathrm{R}(U) \dot{+} \mathrm{span}\{e_n : n \in E\}\Big) \oplus Z.$$

This may be rewritten in the form

$$(2.6)\qquad \ell^2 = \mathrm{R}(U) \dot{+} \left(\mathrm{span}\{e_n : n \in E\} \oplus Z\right).$$

Put

$$(2.7)\qquad Y = \mathrm{span}\{e_n : n \in E\} \oplus Z.$$

Clearly, Y is a closed direct complement of $\mathrm{R}(U)$ in ℓ^2 with the desired property.

Assume, without loss of generality, that $E = \{1, 2, \ldots, k\}$. Recall that the synthesis operator of our desired dual $(v_n)_n$ is $V^* = (U^*U)^{-1}U^*F$, so v_n's are given by

$$v_n = (U^*U)^{-1}U^*Fe_n, \quad \forall n \in \mathbb{N}. \tag{2.8}$$

We want to express $(v_n)_n$ in terms of the canonical dual frame $(y_n)_n$. Recall that

$$y_n = (U^*U)^{-1}U^*e_n, \quad \forall n \in \mathbb{N}. \tag{2.9}$$

Let $p_n \in \mathrm{R}(U)$ and $a_n \in \mathrm{R}(U)^\perp$ be such that

$$e_n = p_n + a_n, \quad \forall n \in \mathbb{N}. \tag{2.10}$$

Since $a_n \in \mathrm{R}(U)^\perp = \mathrm{N}(U^*)$, we can rewrite (2.9) in the form

$$y_n = (U^*U)^{-1}U^*p_n, \quad \forall n \in \mathbb{N}. \tag{2.11}$$

Recall now that $U(U^*U)^{-1}U^*$ is the orthogonal projection onto $\mathrm{R}(U)$. Hence, by applying U to (2.11) we get

$$Uy_n = p_n, \quad \forall n \in \mathbb{N}. \tag{2.12}$$

On the other hand, using (2.6), we can find $r_n \in \mathrm{R}(U), b_n \in \mathrm{span}\{e_1, e_2, \ldots, e_k\}$ and $c_n \in Z$ such that

$$e_n = r_n + b_n + c_n, \quad \forall n \in \mathbb{N}. \tag{2.13}$$

Since F is the oblique projection to $\mathrm{R}(U)$ along $\mathrm{span}\,\{e_1, e_2, \ldots, e_k\} \oplus Z$, we have

$$Fe_n = r_n, \quad \forall n \in \mathbb{N}. \tag{2.14}$$

Observe that

$$b_n = e_n,\ r_n = 0,\ c_n = 0, \quad \forall n = 1, 2, \ldots, k. \tag{2.15}$$

Since each b_n belongs to $\mathrm{span}\,\{e_1, e_2, \ldots, e_k\}$, there exist coefficients α_{ni} such that

$$b_n = \sum_{i=1}^{k} \alpha_{ni} e_i, \quad \forall n \in \mathbb{N}. \tag{2.16}$$

Note that (2.15) implies

$$\alpha_{ni} = \delta_{ni}, \quad \forall n, i = 1, 2, \ldots, k. \tag{2.17}$$

We now have for all $n \in \mathbb{N}$

$$\begin{aligned} e_n &\stackrel{(2.13)}{=} r_n + b_n + c_n \stackrel{(2.16)}{=} r_n + \sum_{i=1}^{k} \alpha_{ni} e_i + c_n \\ &\stackrel{(2.10)}{=} r_n + \sum_{i=1}^{k} \alpha_{ni}(p_i + a_i) + c_n \\ &= \left(r_n + \sum_{i=1}^{k} \alpha_{ni} p_i \right) + \left(\sum_{i=1}^{k} \alpha_{ni} a_i + c_n \right). \end{aligned}$$

Observe that $\left(r_n + \sum_{i=1}^{k} \alpha_{ni} p_i\right) \in \mathrm{R}(U)$, while $\left(\sum_{i=1}^{k} \alpha_{ni} a_i + c_n\right) \in \mathrm{R}(U)^{\perp}$. Thus, comparing this last equality with (2.10) we obtain

$$r_n = p_n - \sum_{i=1}^{k} \alpha_{ni} p_i, \quad a_n = \sum_{i=1}^{k} \alpha_{ni} a_i + c_n, \quad \forall n \in \mathbb{N}. \tag{2.18}$$

Finally, we conclude that for all $n \in \mathbb{N}$

$$\begin{aligned} v_n &\stackrel{(2.8)}{=} (U^*U)^{-1} U^* F e_n \stackrel{(2.14)}{=} (U^*U)^{-1} U^* r_n \\ &\stackrel{(2.18)}{=} (U^*U)^{-1} U^* \left(p_n - \sum_{i=1}^{k} \alpha_{ni} p_i \right) \stackrel{(2.11)}{=} y_n - \sum_{i=1}^{k} \alpha_{ni} y_i. \end{aligned} \tag{2.19}$$

Note that (2.19) and (2.17) show that $v_1 = v_2 = \ldots = v_k = 0$, as required.

So far we have described our desired dual frame $(v_n)_n$ in terms of the canonical dual $(y_n)_n$. Obviously, to obtain v_n's one has to compute all the coefficients α_{ni}, $i = 1, 2, \ldots, k$, $n \geq k+1$. To do that, let us first note the following useful consequence of the preceding computation. We claim that

$$\langle v_n, x_i \rangle = -\alpha_{ni}, \quad \forall i = 1, 2, \ldots, k, \quad \forall n \geq k+1. \tag{2.20}$$

Indeed, for $i = 1, 2, \ldots, k$ and $n \geq k+1$ we have

$$\begin{aligned} \langle v_n, x_i \rangle &= \langle v_n, U^* e_i \rangle \stackrel{(2.8)}{=} \langle U(U^*U)^{-1} U^* F e_n, e_i \rangle \\ &= \langle F e_n, e_i \rangle \stackrel{(2.14)}{=} \langle r_n, e_i \rangle \\ &\stackrel{(2.13)}{=} \langle e_n - b_n - c_n, e_i \rangle \quad (\text{since } i < n \text{ and } c_n \perp e_i) \\ &= -\langle b_n, e_i \rangle \stackrel{(2.16)}{=} -\alpha_{ni}. \end{aligned}$$

For each $n \geq k+1$ we can rewrite (2.20), using (2.19), as

$$\left\langle y_n - \sum_{j=1}^{k} \alpha_{nj} y_j, x_i \right\rangle = -\alpha_{ni}, \quad \forall i = 1, 2, \ldots, k$$

or, equivalently,

$$\sum_{j=1}^{k} \langle y_j, x_i \rangle \alpha_{nj} - \alpha_{ni} = \langle y_n, x_i \rangle, \quad \forall i = 1, 2, \ldots, k.$$

The above equalities can be regarded as a system of k equations in unknowns $\alpha_{n1}, \alpha_{n2}, \ldots, \alpha_{nk}$ that can be written in the matrix form as

(2.21)

$$\left(\begin{bmatrix} \langle y_1, x_1 \rangle & \langle y_2, x_1 \rangle & \ldots & \langle y_k, x_1 \rangle \\ \langle y_1, x_2 \rangle & \langle y_2, x_2 \rangle & \ldots & \langle y_k, x_2 \rangle \\ \vdots & \vdots & & \vdots \\ \langle y_1, x_k \rangle & \langle y_2, x_k \rangle & \ldots & \langle y_k, x_k \rangle \end{bmatrix} - I \right) \begin{bmatrix} \alpha_{n1} \\ \alpha_{n2} \\ \vdots \\ \alpha_{nk} \end{bmatrix} = \begin{bmatrix} \langle y_n, x_1 \rangle \\ \langle y_n, x_2 \rangle \\ \vdots \\ \langle y_n, x_k \rangle \end{bmatrix},$$

where I denotes the unit $k \times k$ matrix. By Proposition 2.4 the matrix of the above system is invertible; hence, the system has a unique solution $(\alpha_{n1}, \alpha_{n2}, \ldots, \alpha_{nk})$ for each $n \geq k+1$. ∎

REMARK 2.6. (a) Clearly, if $(x_n)_n$ is a Parseval frame, our constructed dual frame $(v_n)_n$ is expressed in terms of the original frame members x_n's.

(b) Note that the matrix of the system (2.3) is independent not only of n, but also of all $x \in H$. Thus, the inverse matrix can be computed in advance, without knowing for which x the coefficients $\langle x, x_{n_1} \rangle$, $\langle x, x_{n_2} \rangle, \ldots, \langle x, x_{n_k} \rangle$ will be lost.

(c) The frame $(v_n)_n$ from Theorem 2.5 coincides with the canonical dual if and only if $x_{n_1} = x_{n_2} = \ldots = x_{n_k} = 0$.

(d) The existence of frames dual to $(x_n)_n$ with pre-determined elements indexed by indices from E is also proved in Theorem 5.2 from [13], but only for finite frames in finite-dimensional spaces. Besides, there such dual frames are not described explicitly.

Let us note the following obvious corollary to Theorem 2.5 in the case $m = 1$ holds.

COROLLARY 2.7. *Let $(x_n)_n$ be a frame for a Hilbert space H with the analysis operator U and the canonical dual $(y_n)_n$. Suppose that a set $E = \{m\}$ satisfies the minimal redundancy condition for $(x_n)_n$. Let $v_m = 0$ and*

$$v_n = y_n + \frac{\langle y_n, x_m \rangle}{1 - \langle y_m, x_m \rangle} y_m, \quad \forall n \neq m. \tag{2.22}$$

Then $(v_n)_n$ is a frame for H dual to $(x_n)_n$.

It is also useful to note another characterization of the minimal redundancy condition:

PROPOSITION 2.8. *Let $(x_n)_n$ be a frame for a Hilbert space H, let $E = \{n_1, n_2, \ldots, n_k\}$, $k \in \mathbb{N}$ be a finite set of indices. Then E satisfies the minimal* *redundancy condition for $(x_n)_n$ if and only if there exists a frame $(z_n)_n$ for H dual to $(x_n)_n$ such that*

$$\begin{bmatrix} \langle z_{n_1}, x_{n_1}\rangle & \langle z_{n_2}, x_{n_1}\rangle & \dots & \langle z_{n_k}, x_{n_1}\rangle \\ \langle z_{n_1}, x_{n_2}\rangle & \langle z_{n_2}, x_{n_2}\rangle & \dots & \langle z_{n_k}, x_{n_2}\rangle \\ \vdots & \vdots & & \vdots \\ \langle z_{n_1}, x_{n_k}\rangle & \langle z_{n_2}, x_{n_k}\rangle & \dots & \langle z_{n_k}, x_{n_k}\rangle \end{bmatrix} - I \tag{2.23}$$

is an invertible matrix.

Proof. If E satisfies the minimal redundancy condition for $(x_n)_n$ then, by Proposition 2.4, the canonical dual frame of (x_n) can be taken for (z_n).

Let us prove the converse. We again assume that $E = \{1, 2, \dots, k\}$.

Denote by W the analysis operator of the frame $(z_n)_n$. As before, let $U_E, W_E, U_{E^c}, W_{E^c}$ be analysis operators of Bessel sequences $(x_n)_{n\in E}$, $(z_n)_{n\in E}$, $(x_n)_{n\in E^c}$, $(z_n)_{n\in E^c}$, respectively. The matrix (2.23) represents the operator $U_E W_E^* - I \in \mathbb{B}(\mathbb{C}^k)$ in the canonical basis for $\mathbb{C}^k$, so, by our assumption, this operator is invertible. This in turn implies that the operator $U_E^* W_E - I \in \mathbb{B}(H)$ is also invertible. Since (x_n) and (z_n) are dual to each other, we have $I = U^*W$, wherefrom we get $I = U_E^* W_E + U_{E^c}^* W_{E^c}$. Then $I - U_E^* W_E = U_{E^c}^* W_{E^c}$. Thus, $U_{E^c}^* W_{E^c}$ is invertible and hence $U_{E^c}^*$ is surjective. This proves that $(x_n)_{n\in E^c}$ is a frame for H, i.e. E has the minimal redundancy condition for (x_n). ∎

Remark 2.9. In the light of the preceding proposition and Proposition 2.4 one may ask: if the set $E = \{n_1, n_2, \dots, n_k\}$ satisfies the minimal redundancy condition for a frame $(x_n)_n$, can we conclude that the matrix from (2.23) is invertible for *each* dual frame $(z_n)_n$?

The answer is negative which is demonstrated by the following simple example. Take an orthonormal basis $\{\epsilon_1, \epsilon_2\}$ of a two-dimensional space H and consider a frame $(x_n)_{n=1}^3 = (\epsilon_1, \epsilon_1 + \epsilon_2, \epsilon_2)$ and its dual $(\epsilon_1, 0, \epsilon_2)$. Then the set $E = \{1\}$ satisfies the minimal redundancy condition for $(x_n)_{n=1}^3$ but the corresponding matrix from the preceding proposition is equal to 0.

However, it should be mentioned that in general, if E satisfies the minimal redundancy property for $(x_n)_n$, there are dual frames to $(x_n)_n$ different from the canonical dual frame for which the matrix from (2.23) is invertible. For example, this matrix is invertible for every dual $(v_n)_n$ as in Theorem 2.5, i.e. such that $v_n = 0$ for all $n \in E$ (and, unless $x_n = 0$ for all n in E, such dual frames differ from the canonical dual of $(x_n)_n$).

Let us now turn to the examples.

Example 2.10. Let $(\epsilon_n)_n$ be an orthonormal basis for a Hilbert space H. Let

$$x_1 = \frac{1}{3}\epsilon_1,\ x_2 = \frac{2}{3}\epsilon_1 - \frac{1}{\sqrt{2}}\epsilon_2,\ x_3 = \frac{2}{3}\epsilon_1 + \frac{1}{\sqrt{2}}\epsilon_2, x_n = \epsilon_{n-1},\ \forall n \geq 4.$$

One easily checks that $(x_n)_n$ is a Parseval frame for H; thus, here we have $y_n = x_n$ for all $n \in \mathbb{N}$. It is obvious that the set $E = \{1\}$ has the minimal redundancy property for $(x_n)_n$. We shall use Corollary 2.7 to construct a dual frame $(v_n)_n$ for $(x_n)_n$ such that $v_1 = 0$.

It follows from (2.22) that

$$v_n = x_n + \frac{9}{8}\langle x_n, x_1\rangle x_1, \quad \forall n \geq 2.$$

Since $\langle x_2, x_1\rangle = \langle x_3, x_1\rangle = \frac{2}{9}$, and $\langle x_n, x_1\rangle = 0$ for $n \geq 4$, we find

$$v_1 = 0,\ v_2 = x_2 + \frac{1}{4}x_1,\ v_3 = x_3 + \frac{1}{4}x_1,\ v_n = x_n,\ \forall n \geq 4,$$

that is,

$$v_1 = 0,\ v_2 = \frac{3}{4}\epsilon_1 - \frac{1}{\sqrt{2}}\epsilon_2,\ v_3 = \frac{3}{4}\epsilon_1 + \frac{1}{\sqrt{2}}\epsilon_2,\ v_n = \epsilon_{n-1},\ \forall n \geq 4.$$

EXAMPLE 2.11. Let (ϵ_1, ϵ_2) be an orthonormal basis of a 2-dimensional Hilbert space H. Consider a frame $(x_n)_{n=1}^4$ where

$$x_1 = \frac{1}{2}\epsilon_1,\ x_2 = \frac{1}{2}\epsilon_2,\ x_3 = \frac{1}{2}\epsilon_1 - \frac{1}{2}\epsilon_2,\ x_4 = \frac{1}{2}\epsilon_1 + \frac{1}{2}\epsilon_2.$$

One easily verifies that $(x_n)_{n=1}^4$ is a tight frame with $U^*U = \frac{3}{4}I$, so the members of the canonical dual $(y_n)_{n=1}^4$ are given by

$$y_1 = \frac{2}{3}\epsilon_1,\ y_2 = \frac{2}{3}\epsilon_2,\ y_3 = \frac{2}{3}\epsilon_1 - \frac{2}{3}\epsilon_2,\ y_4 = \frac{2}{3}\epsilon_1 + \frac{2}{3}\epsilon_2.$$

Obviously, the set $E = \{1, 2\}$ satisfies the minimal redundancy condition for $(x_n)_{n=1}^4$. To obtain the dual frame $(v_n)_{n=1}^4$ from Theorem 2.5 we have to solve the system

$$\left(\begin{bmatrix} \langle y_1, x_1\rangle & \langle y_2, x_1\rangle \\ \langle y_1, x_2\rangle & \langle y_2, x_2\rangle \end{bmatrix} - I\right)\begin{bmatrix} \alpha_{n1} \\ \alpha_{n2} \end{bmatrix} = \begin{bmatrix} \langle y_n, x_1\rangle \\ \langle y_n, x_2\rangle \end{bmatrix}, \quad n = 3, 4.$$

Since

$$\begin{bmatrix} \langle y_1, x_1\rangle & \langle y_2, x_1\rangle \\ \langle y_1, x_2\rangle & \langle y_2, x_2\rangle \end{bmatrix} - I = \begin{bmatrix} -\frac{2}{3} & 0 \\ 0 & -\frac{2}{3} \end{bmatrix},$$

we have

$$\begin{bmatrix} \alpha_{n1} \\ \alpha_{n2} \end{bmatrix} = \begin{bmatrix} -\frac{3}{2} & 0 \\ 0 & -\frac{3}{2} \end{bmatrix}\begin{bmatrix} \langle y_n, x_1\rangle \\ \langle y_n, x_2\rangle \end{bmatrix}, \quad n = 3, 4.$$

From this one easily finds

$$\alpha_{31} = -\frac{1}{2},\ \alpha_{32} = \frac{1}{2},\ \alpha_{41} = -\frac{1}{2},\ \alpha_{42} = -\frac{1}{2}.$$

Theorem 2.5 gives us now

$$v_1 = 0,\ v_2 = 0,\ v_3 = y_3 + \frac{1}{2}y_1 - \frac{1}{2}y_2,\ v_4 = y_4 + \frac{1}{2}y_1 + \frac{1}{2}y_2,$$

that is,

$$v_1 = 0,\, v_2 = 0,\, v_3 = \epsilon_1 - \epsilon_2,\, v_4 = \epsilon_1 + \epsilon_2.$$

In the rest of the paper we give another description of the dual frame $(v_n)_n$ constructed in Theorem 2.5. In the theorem that follows we show that $(v_n)_{n\in E^c}$ is in fact the canonical dual of $(x_n)_{n\in E^c}$.

THEOREM 2.12. *Let $(x_n)_n$ be a frame for a Hilbert space H with the canonical dual $(y_n)_n$. Suppose that a finite set $E = \{n_1, n_2, \ldots, n_k\}$, $k \in \mathbb{N}$, satisfies the minimal redundancy condition for $(x_n)_n$. If $(v_n)_n$ is a dual frame for $(x_n)_n$ defined by (2.4) then $(v_n)_{n\in E^c}$ is the canonical dual of $(x_n)_{n\in E^c}$.*

PROOF. Let us keep notation from Theorem 2.5 and its proof. Again, we assume without loss of generality that $E = \{1, 2, \ldots, k\}$.

Let U_1 be the analysis operator of the frame $(0)_{n\in E} \cup (x_n)_{n\in E^c}$. Then

$$\mathrm{R}(U) \dot{+} \operatorname{span}\{e_n : n \in E\} = \mathrm{R}(U_1) \oplus \operatorname{span}\{e_n : n \in E\}.$$

For $Y = \operatorname{span}\{e_n : n \in E\} \oplus Z$ we now have

$$\ell^2 = \mathrm{R}(U) \dot{+} Y \quad \text{and} \quad \ell^2 = \mathrm{R}(U_1) \oplus Y. \tag{2.24}$$

Let F_1 be the orthogonal projection on $\mathrm{R}(U_1)$ (along Y), and F the oblique projection on $\mathrm{R}(U)$ along Y.

Let $(\alpha_n) \in \ell^2$ be arbitrary. Then

$$(\alpha_n) = Ux + y \tag{2.25}$$

for some $x \in H$ and $y \in Y$. Denote $(\beta_n) = (\langle x, x_1\rangle, \ldots, \langle x, x_k\rangle, 0, 0, \ldots)$. Then $(\beta_n) \in \operatorname{span}\{e_n : n \in E\} \subseteq Y$, so, if we take $y_1 = (\beta_n) + y$, then $y_1 \in Y$ and

$$(\alpha_n) = U_1 x + y_1. \tag{2.26}$$

Since both sums in (2.24) are direct and U, U_1 are injective, the vectors x, y, y_1 from (2.25) and(2.26) are unique for each given $(\alpha_n) \in \ell^2$.

This means that for every $(\alpha_n) \in \ell^2$ there is a unique $x \in H$ such that $F(\alpha_n) = Ux$ and $F_1(\alpha_n) = U_1 x$. In particular, for every $n \in \mathbb{N}$ let $z_n \in H$ be a unique vector such that $Fe_n = Uz_n$ and $F_1 e_n = U_1 z_n$. Then

$$\begin{aligned} v_n &= (U^*U)^{-1}U^*Fe_n = (U^*U)^{-1}U^*Uz_n = z_n \\ &= (U_1^*U_1)^{-1}U_1^*U_1 z_n = (U_1^*U_1)^{-1}U_1^*F_1 e_n \end{aligned}$$

for all $n \in \mathbb{N}$. Since F_1 is the orthogonal projection on $\mathrm{R}(U_1)$, the frame $((U_1^*U_1)^{-1}U_1^*F_1e_n)_n$ is the canonical dual of $(0)_{n\in E} \cup (x_n)_{n\in E^c}$, that is

$$v_n = (U_1^*U_1)^{-1}x_n, \quad \forall n \in E^c.$$

Observe that $U_1^*U_1 = U_{E^c}^*U_{E^c}$, so

$$v_n = (U_{E^c}^*U_{E^c})^{-1}x_n, \quad \forall n \in E^c. \tag{2.27}$$

which means that $(v_n)_{n\in E^c}$ is the canonical dual of $(x_n)_{n\in E^c}$. $\square$

Denote again by V the analysis operator of the canonical dual $(y_n)_n$ and consider the corresponding operators V_{E^c} and V_E. Recall that $V = U(U^*U)^{-1}$ and $V_E = U_E(U^*U)^{-1}$, $V_{E^c} = U_{E^c}(U^*U)^{-1}$. We now have

$$\begin{aligned}(I - V_E^* U_E)^{-1}(U^*U)^{-1} &\overset{(2.2)}{=} (V_{E^c}^* U_{E^c})^{-1}(U^*U)^{-1} \\ &= ((U^*U)^{-1}U_{E^c}^* U_{E^c})^{-1}(U^*U)^{-1} = (U_{E^c}^* U_{E^c})^{-1}.\end{aligned}$$

Using this, we may rewrite (2.27) as

$$v_n = (I - V_E^* U_E)^{-1}(U^*U)^{-1} x_n = (I - V_E^* U_E)^{-1} y_n, \quad \forall n \in E^c.$$

If $E = \{1, 2, \ldots, k\}$, $k \in \mathbb{N}$, then we have

$$V_E^* U_E x = \sum_{i=1}^{k} \langle x, x_i \rangle y_i = \sum_{i=1}^{k} \theta_{y_i, x_i}(x), \quad \forall x \in H. \tag{2.28}$$

Thus, in applications we need an efficient procedure for computing the inverse $(I - \sum_{i=1}^{k} \theta_{y_i, x_i})^{-1}$; then the desired dual $(v_n)_n$ will be obtained using

$$v_n = (I - \sum_{i=1}^{k} \theta_{y_i, x_i})^{-1} y_n, \quad \forall n \in E^c. \tag{2.29}$$

Observe that the case $k = 1$ is trivial. Namely, if $x, y \in H$ are such that $I - \theta_{y,x}$ is invertible, then $\langle y, x \rangle \neq 1$. (Indeed, $\langle y, x \rangle = 1$ would imply $(I - \theta_{y,x})y = y - \langle y, x \rangle y = 0$ and, by invertibility of $I - \theta_{y,x}$, we would have $y = 0$ which contradicts the equality $\langle y, x \rangle = 1$.) Moreover, if $(I - \theta_{y,x})^{-1}$ exists, then it is given by

$$(I - \theta_{y,x})^{-1} = I + \frac{1}{1 - \langle y, x \rangle} \theta_{y,x}. \tag{2.30}$$

In the theorem that follows we demonstrate an iterative procedure for computing all inverses $(I - \sum_{i=1}^{n} \theta_{y_i, x_i})^{-1}$, $n = 1, 2, \ldots, k$, by expressing them as a product of exactly n simple inverses $(I - \theta_{y,x})^{-1}$ which one obtains using (2.30).

But we first need the following simple observation.

Lemma 2.13. *Let $(x_n)_n$ be a frame for a Hilbert space H with the canonical dual $(y_n)_n$. Then the set $E = \{1, 2, \ldots, k\}, k \in \mathbb{N}$, satisfies the minimal redundancy condition for $(x_n)_n$ if and only if the operators*

$$I - \sum_{i=1}^{n} \theta_{y_i, x_i}, \quad n = 1, \ldots, k$$

are all invertible.

Proof. If E has the minimal redundancy property then every $F \subseteq E$ has this property as well. By Proposition 2.4 and (2.28) it follows that $I - V_F^* U_F = I - \sum_{i \in F} \theta_{y_i, x_i}$ are invertible for all $F \subseteq E$. It remains to take $F = \{1, \ldots, n\}$ for $n = 1, \ldots, k$. The converse is obvious. ∎

THEOREM 2.14. *Let $(x_n)_n$ be a frame for a Hilbert space H with the canonical dual $(y_n)_n$. Suppose that a set $E = \{1, 2, \dots, k\}$, $k \in \mathbb{N}$, satisfies the minimal redundancy condition for $(x_n)_n$. Let $\overline{y}_1, \dots, \overline{y}_n$ be defined as*

$$\begin{aligned} \overline{y}_1 &= y_1, \\ \overline{y}_n &= (I - \theta_{\overline{y}_{n-1},x_{n-1}})^{-1} \dots (I - \theta_{\overline{y}_1,x_1})^{-1} y_n, \quad n = 2, \dots, k. \end{aligned} \tag{2.31}$$

Then $\overline{y}_1, \dots, \overline{y}_n$ are well defined and

$$(I - \sum_{i=1}^{n} \theta_{y_i,x_i})^{-1} = (I - \theta_{\overline{y}_n,x_n})^{-1} \dots (I - \theta_{\overline{y}_1,x_1})^{-1}, \quad n = 1, \dots, k. \tag{2.32}$$

PROOF. In order to see that $\overline{y}_1, \dots, \overline{y}_k$ are well defined we have to prove that the operators $I - \theta_{\overline{y}_n,x_n}$ for $n = 1, \dots, k$ are invertible. By Lemma 2.13, $I - \sum_{i=1}^n \theta_{y_i,x_i}$ are invertible for all $n = 1, 2, \dots, k$.

We prove our statement by induction.

For $n = 1$ we have $I - \theta_{\overline{y}_1,x_1} = I - \theta_{y_1,x_1}$ which is an invertible operator by Lemma 2.13, and formula (2.32) is trivially satisfied.

Assume now that for some $n < k$ the operators $I - \theta_{\overline{y}_1,x_1}, \dots, I - \theta_{\overline{y}_n,x_n}$ are invertible and that (2.32) is satisfied.

Observe the equality

$$T\theta_{y,x} = \theta_{Ty,x} \tag{2.33}$$

which holds for all $x, y \in H$ and $T \in \mathbb{B}(H)$. Now we have

$$\begin{aligned} I - \theta_{\overline{y}_{n+1},x_{n+1}} &\overset{(2.31)}{=} I - \theta_{(I-\theta_{\overline{y}_n,x_n})^{-1}\cdots(I-\theta_{\overline{y}_1,x_1})^{-1}y_{n+1},x_{n+1}} \\ &\overset{(2.33)}{=} I - (I - \theta_{\overline{y}_n,x_n})^{-1} \cdots (I - \theta_{\overline{y}_1,x_1})^{-1} \theta_{y_{n+1},x_{n+1}} \\ &\overset{(2.32)}{=} I - (I - \sum_{i=1}^{n} \theta_{y_i,x_i})^{-1} \theta_{y_{n+1},x_{n+1}} \\ &= (I - \sum_{i=1}^{n} \theta_{y_i,x_i})^{-1} (I - \sum_{i=1}^{n} \theta_{y_i,x_i}) \\ &\quad - (I - \sum_{i=1}^{n} \theta_{y_i,x_i})^{-1} \theta_{y_{n+1},x_{n+1}} \\ &= (I - \sum_{i=1}^{n} \theta_{y_i,x_i})^{-1} (I - \sum_{i=1}^{n} \theta_{y_i,x_i} - \theta_{y_{n+1},x_{n+1}}) \\ &= (I - \sum_{i=1}^{n} \theta_{y_i,x_i})^{-1} (I - \sum_{i=1}^{n+1} \theta_{y_i,x_i}) \\ &\overset{(2.32)}{=} (I - \theta_{\overline{y}_n,x_n})^{-1} \cdots (I - \theta_{\overline{y}_1,x_1})^{-1} (I - \sum_{i=1}^{n+1} \theta_{y_i,x_i}). \end{aligned}$$

This proves that $I-\theta_{\overline{y}_{n+1},x_{n+1}}$ is invertible (as a product of invertible operators). Also, it follows from the final equality that

$$(I-\sum_{i=1}^{n+1}\theta_{y_i,x_i})^{-1}=(I-\theta_{\overline{y}_{n+1},x_{n+1}})^{-1}(I-\theta_{\overline{y}_n,x_n})^{-1}\cdots(I-\theta_{\overline{y}_1,x_1,})^{-1}.$$

□

We conclude the paper with the result that improves Theorem 6.2 from [13] by removing the linear independence assumption. In this way we provide a closed-form formula for the inverse $(I-\sum_{n=1}^{k}\theta_{y_k,x_k})^{-1}$ which can (alternatively) be used, via (2.29), for obtaining our dual frame $(v_n)_n$.

THEOREM 2.15. *Let $x_1,\dots,x_k$ and $y_1,\dots,y_k$ be vectors in a Hilbert space H such that the operator $R=I-\sum_{j=1}^{k}\theta_{y_j,x_j}\in\mathbb{B}(H)$ is invertible. Then*

$$R^{-1}=I+\sum_{i,j=1}^{k}c_{ij}\theta_{y_i,x_j},$$

where the coefficient matrix $C:=(c_{ij})$ is given by

$$C=-\begin{bmatrix}\langle y_1,x_1\rangle-1 & \langle y_2,x_1\rangle & \dots & \langle y_k,x_1\rangle\\ \langle y_1,x_2\rangle & \langle y_2,x_2\rangle-1 & \dots & \langle y_k,x_2\rangle\\ \vdots & \vdots & & \vdots\\ \langle y_1,x_k\rangle & \langle y_2,x_k\rangle & \dots & \langle y_k,x_k\rangle-1\end{bmatrix}^{-1}. \tag{2.34}$$

PROOF. Let $U,V\in\mathbb{B}(H,\mathbb{C}^k)$ be the analysis operators of Bessel sequences $(x_n)_{n=1}^k$, $(y_n)_{n=1}^k$, respectively. Then $R=I-V^*U$, and since R is invertible, UV^*-I is also an invertible operator.

The matrix representation of the operator $UV^*-I\in\mathbb{B}(\mathbb{C}^k)$ with respect to the canonical basis of $\mathbb{C}^k$ is precisely the matrix

$$\begin{bmatrix}\langle y_1,x_1\rangle-1 & \langle y_2,x_1\rangle & \dots & \langle y_k,x_1\rangle\\ \langle y_1,x_2\rangle & \langle y_2,x_2\rangle-1 & \dots & \langle y_k,x_2\rangle\\ \vdots & \vdots & & \vdots\\ \langle y_1,x_k\rangle & \langle y_2,x_k\rangle & \dots & \langle y_k,x_k\rangle-1\end{bmatrix}.$$

As a matrix representation of an invertible operator, this is an invertible matrix.

Now one proceeds exactly as in the proof of Theorem 6.2 from [13]. □

REFERENCES

[1] D. Bakić and T. Berić, *On excesses of frames*, Glas. Mat. Ser. III **50(70)** (2015), 415–427.

[2] P. Boufounos and A. V. Oppenheim, *Compensation of coefficients erasures in frame representations*, Proc. IEEE Int. Conf. Acoustics, Speech, and Signal Processing (ICASSP), May 14-19, 2006.

[3] B. G. Bodman and V. I. Paulsen, *Frames, graphs and erasures*, Linear Algebra Appl. **404** (2005), 118–146.
[4] P. G. Casazza and J. Kovačević, *Equal-norm tight frames with erasures*, Adv. Comput. Math. **18** (2003), 387–430.
[5] P. G. Casazza and J. Kovačević, Finite frames, in *Applied and numerical harmonic analysis* (Theory and Applications), Springer, 2013.
[6] O. Christensen, An introduction to frames and Riesz bases, Birkhäuser, Boston, 2003.
[7] R. J. Duffin and A. C. Schaeffer, *A class of nonharmonic Fourier series*, Trans. Amer. Math. Soc. **72** (1952), 341–366.
[8] V. K. Goyal, J. Kovačević and J. A. Kelner, *Quantized frame expansions with erasures*, Appl. Comput. Harmon. Anal. **10** (2001), 203–233.
[9] D. Han and D. R. Larson, *Frames, bases and group representations*, Mem. Amer. Math. Soc. **147** (2000), no. 697, 1–94.
[10] D. Han and W. Sun, *Reconstruction of signals from frame coefficients with erasures at unknown locations*, IEEE Trans. Inform. Theory **60** (2014), 4013–4025.
[11] R. Holmes and V. I. Paulsen, *Optimal frames for erasures*, Linear Algebra Appl. **377** (2004), 31–51.
[12] J. Kovačević and A. Chebira, *Life beyond bases: The advent of frames*, IEEE Signal Process. Mag. **24** (2007), 86–104.
[13] D. Larson and S. Scholze, *Signal reconstruction from frame and sampling erasures*, J. Fourier Anal. Appl. **21** (2015), 1146–1167.
[14] J. Lopez and D. Han, *Optimal dual frames for erasures*, Linear Algebra Appl. **432** (2010), 471–482.
[15] J. Leng and D. Han, *Optimal dual frames for erasures II*, Linear Algebra Appl. **435** (2011), 1464–1472.

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