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A delayed differentiation multi-product FPR model with scrap and a multi-delivery policy – I: Using single-machine production scheme

Yuan-Shyi Peter Chiu(1), **You-Tsung Hsieh**(1), **Jyun-Sian Kuo**(1), **Singa Wang Chiu**(2)

(1) *Department of Industrial Engineering & Management, Chaoyang University of Technology, Wufong District, Taichung City 413, TAIWAN*

(2) *Department of Business Administration, Chaoyang University of Technology, Wufong District, Taichung City 413, TAIWAN*
e-mail: swang@cyut.edu.tw

SUMMARY

This study examines a delayed differentiation multi-product single-machine finite production rate (FPR) model with scrap and a multi-delivery policy. The classic FPR model considers a single product, single stage production with all items manufactured being of perfect quality and product demand satisfied by a continuous inventory issuing policy. However, in real-life production-shipment integrated systems, multi-product production is usually adopted by vendors to maximize machine utilization, and generation of scrap items appear to be inevitable with uncontrollable factors in production. Further, distribution of finished products is often done through a periodic or multi-delivery policy rather than a continuous issuing policy. It is also assumed that these multiple products share a common intermediate part. In this situation, the producer would often be interested in evaluating a two-stage production scheme with the first stage producing common parts for all products and the second stage separately fabricating the end products to lower overall production-inventory costs and shorten the replenishment cycle time. Redesigning a multi-product FPR system to delay product differentiation to the final stage of production has proved to be an effective supply chain strategy from an inventory-reduction standpoint. Using mathematical modelling, we derive the optimal replenishment cycle time and delivery policy. A numerical example is provided to demonstrate its practical usage and compare our result to that obtained from the traditional single-stage multi-product FPR model.

KEY WORDS: *finite production rate model, delayed product differentiation, two-stage production, multi-product system, common intermediate part, multi-delivery, scrap.*

1. INTRODUCTION

The classic FPR model considers single product, single stage production with all items manufactured being of perfect quality and the product demand satisfied by a continuous inventory issuing policy [1-2]. However, in real-life production-shipment integrated systems, *multi-product* production is usually adopted by the vendors in order to maximize machine utilization [3-10]. Gordon and Surkis [3] determined control policies for a multi-item inventory environment where items are ordered from a single supplier and the demand for items is subject to severe fluctuations. Their model balanced the stock carrying and stock-out costs, a simulation procedure was adopted to determine the appropriate value of their inventory factor in the model. Lotfi and Chen [4] considered that the multi-item capacitated production planning problem consists of scheduling the size and the timing of the production for several items over a finite horizon so as to meet known future demand without incurring backlogs. Their objective was to minimize the total cost of production, holding and resource over the horizon subject to a constraint on total production capacity in each period. Ketzenberg et al. [5] developed a heuristic for a common production/inventory problem characterized by multiple products, stochastic seasonal demand, lost sales, and a constraint on overall production. They proposed a heuristic to compare with those in current use as well as optimal solutions under a variety of conditions. In testing data using dynamic programming as a benchmark, their heuristic resulted either near optimal and superior to existing heuristics. Other studies related to the multi-product systems can also be found elsewhere [6-10].

When *common intermediate part* exists in the multi-item production system, producers will be interested in evaluating different production schemes, such as *redesign of the production process as a two-stage system with delay product differentiation* in order to lower overall production-inventory costs and/or shorten the production run time. The respective literatures are as follows [11-18]. Collier [11] developed the relationship between aggregate safety stock inventory levels and component part commonality. He used a simulation experiment to support the functional form of this relationship in an uncertain operating environment. The resulting equations can be used by managers to assess the trade-offs between aggregate safety stock levels, service level, and the degree of component part standardization. Swaminathan and Tayur [12] stated that in an attempt to reduce cost while maintaining good customer service, some of the leading manufacturers in the computer industry are delaying product differentiation while managing broader product lines. However, they indicated that finding the optimal configurations and inventory levels of the vanilla boxes could be a challenging task. Accordingly, they modelled a two-stage integer program with recourse. By utilizing structural decomposition of the problem and derivative methods, they provided an effective solution procedure. In addition, they compared the performance of the vanilla assembly process to make-to-stock and assemble-to-order processes and provided managerial insights on the conditions under which one might be better than the others. Graman [13] developed a single-period, two-product, order-up-to cost model to aid in setting the levels of finished-goods inventory and postponement capacity. Minimum-cost optimal solutions to inventory levels and capacity were obtained by solving the derived analytical expressions using a non-linear programming formulation. Other works related to the delay product differentiation issues may also be referred to [14-18].

Due to different uncontrollable factors in production process, generation of defective items is inevitable in real-life manufacturing environments. Rosenblatt and Lee [19] proposed an EPQ model that deals with imperfect quality. They assumed that at some random point in time the process might shift from an in-control to an out-of-control state, followed by the production of a fixed percentage of defective items. Approximate solutions for obtaining an optimal lot size were derived accordingly. Jamal et al. [20] examined the optimal production batch size with rework process at a single-stage production system. Both cases of rework being completed within the same production cycle and rework being done after N cycles are examined. Mathematical models for each case were developed, and the optimal batch sizes and total system costs were determined. Additional studies have also been carried out to address difference issues of imperfect production situations [21-25].

Unlike the continuous inventory issuing policy assumed in the classic FPR model, in real supply chains environment, the distribution of finished products has often been done by the use of periodic or multi-delivery policy rather than a continuous issuing policy. Schwarz [26] first examined a one-warehouse *N*-retailer deterministic production-shipment system with the objective of determining the stocking policy which minimizes average system cost per unit time over the infinite time horizon. Banerjee [27] developed a joint economic-lot-size model for a case where a vendor produces an order for a purchaser on a lot-for-lot basis under deterministic condition, with the objective of minimizing the joint total relevant cost. Abdul-Jalbar et al. [28] studied a multistage distribution/inventory system with a central warehouse and *N* retailers. They assumed that the customer demand arrives at each retailer at a constant rate and the retailers replenish their inventories from the warehouse, which in turn orders from an outside supplier. It is assumed that shortages are not allowed and lead times are negligible. The goal is to determine policies which minimize the overall cost in the system. Additional references may also be found elsewhere [29-34]. Motivated by the concept of *delayed differentiation*, this study proposes a multi-product two-stage imperfect production-shipment system by a single-machine production scheme. From the aforementioned literature reviews, one notes that little attention has been paid to the investigation of joint effects of delayed product differentiation, multi-delivery policy, and random scrap on the optimal replenishment cycle time and shipment policy for this specific multi-product FPR model, hence the present study intends to bridge the gap.

2. PROBLEM DESCRIPTION AND MODELLING

This study focuses on a delayed differentiation multi-product FPR model with scrap and a multi-delivery policy using single-machine production scheme. Description of the proposed model is as follows. Consider a production process produces L products in turn on a single machine in order to maximize its utilization. In additions, these multiple products share a *common intermediate part*. In this situation, the producer would often be interested in evaluating a two-stage production scheme with the first stage producing common parts for all products and the second stage separately fabricating the end products to lower overall production-inventory costs and shorten the replenishment cycle time. In the first stage, common intermediate part is produced at the production rate $P_{1,0}$, followed by product differentiation for L items in the second stage; wherein the customized product i is produced at a production rate of $P_{1,i}$ (where $i = 1, 2, \ldots, L$) under common production cycle approach (see Figure 1).

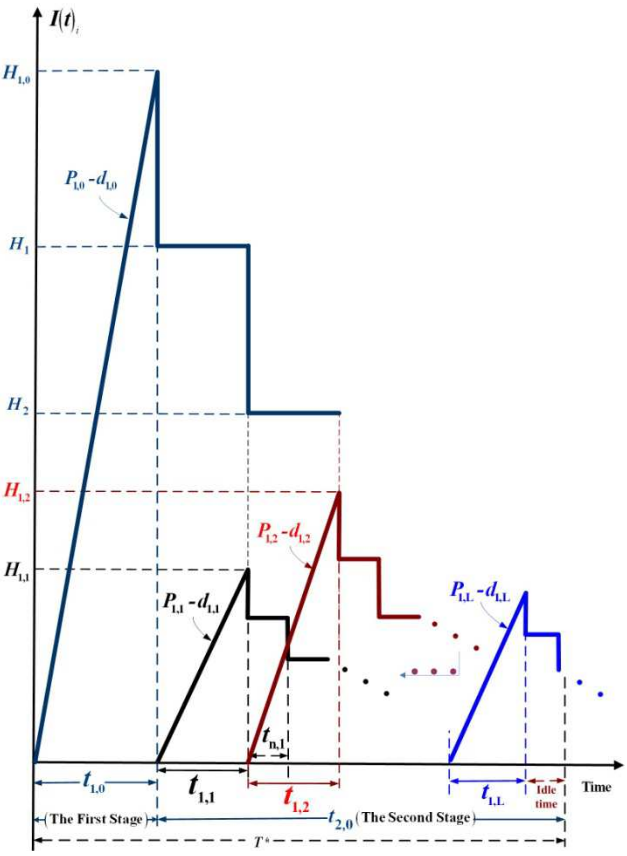

Fig. 1 *On-hand inventory level of perfect quality common intermediate parts and finished products for a single production line multi-item two-stage imperfect production system with delayed product differentiation*

During production time in each stage, there is x_i portion of defective items produced randomly at a production rate $d_{1,I}$; the random defective items produced during a production run in each stage are considered to be scrap, and they will be removed in the end of production. The constant production rate $P_{1,i}$ is larger than the sum of demand rate λ_i and production rate of defective items $d_{1,i}$. That is: $(P_{1,i}-d_{1,i}-\lambda_i)>0$ for $i=0, 1, 2, \ldots, L$ or $(1-x_i-\lambda_i/P_{1,i})>0$; where $i=0$ denote the *common intermediate part* and $d_{1,i}=P_{1,i}(x_i)$. Unlike classic EPQ model assuming a continuous issuing policy for satisfying demand, this study considers a multi-delivery policy and assumes that the delivery of finished items (in the second stage) starts at the end of regular production when the whole production lot is quality assured for each product *i*. Fixed quantity *n* instalments of the finished batch are delivered to the customer, at a fixed interval of time during the production downtime $t_{2,i}$ (see Figure 1).

The cost parameters considered in the proposed model include: production setup cost K_i, unit holding cost $h_{1,i}$, unit production cost C_i, disposal cost per scrap item $C_{S,i}$, fixed delivery cost $K_{1,i}$ per shipment, and delivery cost $C_{T,i}$ per item shipped to customers. Additional notation used is listed as follows.

i = index for customized product number, where $i=1, 2, \ldots, L$, with $i=0$ stands for the common intermediate part produced in the first stage;

T = rotation cycle length, one of the decision variables;

Q_i = production lot size for product *i* in a cycle;

$t_{1,i}$ = production uptime for product i in a cycle;

$t_{2,i}$ = delivery time for product i in a cycle, except when $i=0$ it enters the product differentiation stage;

$t_{n,i}$ = a fixed interval of time between each instalment of finished items of product i to be delivered to its buyer during production downtime $t_{2,i}$;

$H_{1,i}$ = maximum level of finished product i in the end of production, except $i=0$ stands for the maximum level of common intermediate part;

H_i = inventory level of common intermediate part during the production time of product i, where $i=1, 2, ..., L$;

n = the number of fixed quantity instalments of the finished batch to be delivered to customers in each cycle, other decision variables;

$I(t)_i$ = on-hand inventory level of perfect quality product i at time t;

$I_d(t)_I$ = on-hand inventory level of defective product i at time t, with $i=0$ stands for the inventory level of defective common intermediate parts;

$I_c(t)_i$ = on-hand inventory level of finished product i at time t at the customer's side;

$h_{3,i}$ = unit holding cost for stocks stored at the customer's side;

$h_{4,i}$ = unit holding cost for safety stocks stored at the producer's side;

I_i = the left-over number of finished items of product i in each $t_{n,i}$, at the customer's side;

D_i = number of finished items of product i to be distributed to the customer in each shipment;

α = completion rate of common intermediate part as compared to the finished product;

$TC(T,n)$ = total production-inventory-delivery costs per cycle for the proposed model;

$E[TCU(T,n)]$ = the long-run average costs per unit time for the proposed model.

The on-hand inventory level of common intermediate parts waiting to be finished during the second stage is depicted in Figure 2. The on-hand inventory level of defective item during the cycle is illustrated in Figure 3.

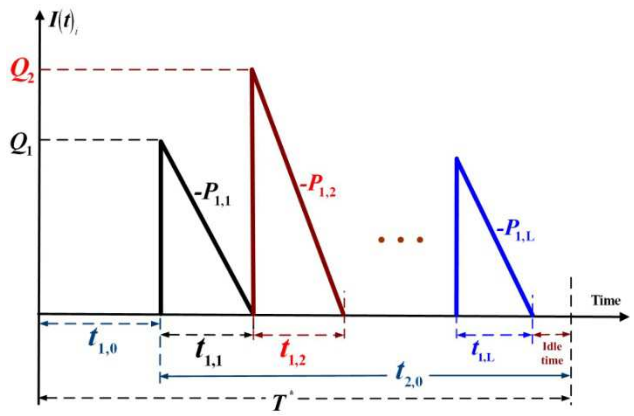

Fig. 2 *On-hand inventory level of common intermediate parts waiting to be finished during the second stage*

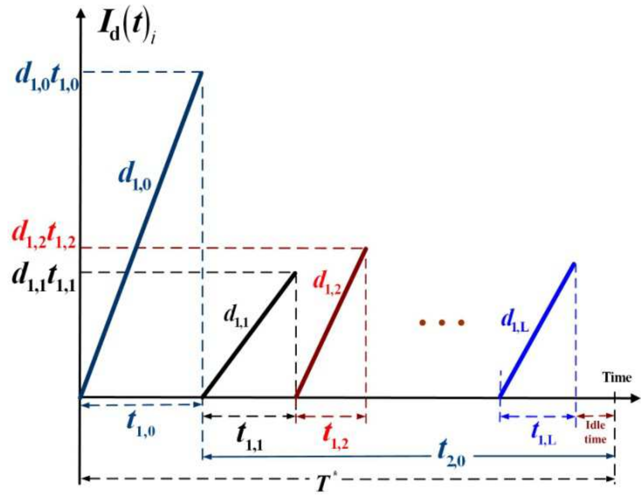

Fig. 3 *On-hand inventory level of defective items during the production cycle*

By Figures 1, 2, and 3, the following formulations can be obtained directly:

$$T = t_{1,i} + t_{2,i} \quad for\ i = 1, 2, ..., L \tag{1}$$

$$H_{1,i} = \left(P_{1,i} - d_{1,i}\right) t_{1,i} \quad for\ i = 1, 2, ..., L \tag{2}$$

$$t_{1,i} = \frac{H_{1,i}}{P_{1,i} - d_{1,i}} \quad for\ i = 1, 2, ..., L \tag{3}$$

$$t_{2,i} = n t_{n,i} \quad for\ i = 1, 2, ..., L \tag{4}$$

$$H_{1,0} = \sum_{i=1}^{L} Q_i \tag{5}$$

$$Q_i = P_{1,i}\left(t_{1,i}\right), \ for\ i = 1, 2, ..., L \tag{6}$$

$$t_{1,i} = \frac{Q_i}{P_{1,i}}, \ for\ i = 1, 2, ..., L \tag{7}$$

In order to meet the demand of the customized product i during a production cycle, the following equations must be satisfied:

$$Q_i = \frac{\lambda_i T}{1 - E\left[x_i\right]}, \ for\ i = 1, 2, ..., L \tag{8}$$

Formulations in the first stage for production of common intermediate parts begins with obtaining requirements for the common product, from Eq. (8) and Figures 1, 2, and 3 we have:

$$Q_0 = \frac{\sum_{i=1}^{L} Q_i}{1 - E\left[x_0\right]} \tag{9}$$

$$\lambda_0 = \frac{\sum_{i=1}^{L} Q_i}{T}, \ for\ i = 1, 2, ..., L \tag{10}$$

$$Q_0 = \frac{\lambda_0 T}{1 - E[x_0]} \tag{11}$$

$$t_{1,0} = \frac{H_{1,0}}{P_{1,0} - d_{1,0}} \tag{12}$$

$$H_{1,0} = (P_{1,0} - d_{1,0}) t_{1,0} \tag{13}$$

$$H_1 = H_{1,0} - Q_1 \tag{14}$$

$$H_i = H_{(i-1)} - Q_i \text{ , for } i = 2, 3, ..., L \tag{15}$$

$$H_L = H_{(L-1)} - Q_L = 0 \tag{16}$$

$$T = t_{1,0} + t_{2,0} \tag{17}$$

The prerequisite assumption of the proposed model is that the production facility should have *sufficient capacity* to produce the items to satisfy the demand for all L customized products (including the defective (scrap) items) during the processes. Therefore, the following Eq. must be satisfied:

$$\left(t_{1,0} + \sum_{i=1}^{L} t_{1,i} \right) < T \text{ or } \left(\frac{Q_0}{P_{1,0}} + \sum_{i=1}^{L} \frac{Q_i}{P_{1,i}} \right) < T \tag{18}$$

or:

$$\left(\frac{\lambda_0}{[1 - E[x_0]] P_{1,0}} + \sum_{i=1}^{L} \frac{\lambda_i}{[1 - E[x_i]] P_{1,i}} \right) < 1 \tag{19}$$

The on-hand inventory of finished customized products at the customer's side during the cycle is depicted in Figure 4. From Figures 1 and 4, one can observe the following:

$$D_i = \frac{H_{1,i}}{n} \text{ , for } i = 1 ... L \tag{20}$$

$$I_i = D_i - (\lambda_i)(t_{n,i}) \tag{21}$$

$$nI_i = \lambda_i (t_{1,i}) \tag{22}$$

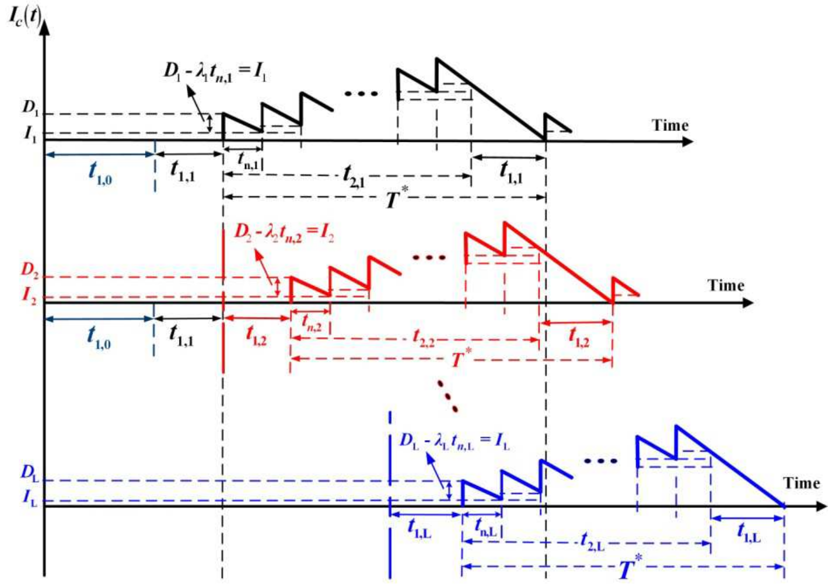

Fig. 4 *On-hand inventory of finished customized products at the customer's side during the production cycle*

2.1 THE PRODUCTION-INVENTORY-DELIVERY RELATED COST IN A PRODUCTION CYCLE

First, the inventory holding cost for the *common intermediate part* in the first stage waiting to be worked on (see Figure 1) is:

$$h_{1,0}\left[\sum_{i=1}^{L}\left[H_i\left(t_{1,i}\right)\right]\right] \quad \textbf{(23)}$$

It is noted that the inventory holding cost of the *common intermediate part* waiting to be finished for each product *i* in the second stage (see Figure 1) is:

$$\sum_{i=1}^{L}\left\{h_{1,i}\left[\frac{Q_i}{2}\left(t_{1,i}\right)\right]\right\} \quad \textbf{(24)}$$

The inventory holding cost for finished product *i* waiting to be delivered in the second stage is [31]:

$$\sum_{i=1}^{L}\left\{h_{1,i}\left(\frac{n-1}{2n}\right)H_{1,i}t_{2,i}\right\} \quad \textbf{(25)}$$

The inventory holding cost for finished product *i* kept at customers' sides during the production cycle is [33]:

$$\sum_{i=1}^{L}\left\{h_{3,i}\left[\frac{n\left(D_i-I_i\right)t_{n,i}}{2}+\frac{n\left(n+1\right)}{2}I_i t_{n,i}+\frac{nI_i\left(t_{1,i}\right)}{2}\right]\right\} \quad \textbf{(26)}$$

Finally, during a production cycle the total production-inventory-delivery cost for the proposed model includes variable production cost, setup cost, disposal cost, and holding costs for *common intermediate part* (in the first stage), all variable production costs, setup costs, disposal costs, and holding costs for *multiple customized products* (in the second stage); the fixed and variable delivery costs, and the holding costs for the stocks kept at customers' side. Therefore, one has $TC(T,n)$ as:

$$TC(T,n)=$$
$$=C_0Q_0+K_0+C_{S,0}\left[x_0Q_0\right]+h_{1,0}\left[\frac{H_{1,0}t_{1,0}}{2}+\sum_{i=1}^{L}\left[H_i\left(t_{1,i}\right)\right]+\frac{d_{1,0}t_{1,0}}{2}\left(t_{1,0}\right)\right]+h_{4,0}\left(x_0Q_0\right)T+$$
$$+\sum_{i=1}^{L}\left\{\begin{array}{l}C_iQ_i+K_i+C_{S,i}\left[x_iQ_i\right]+nK_{1,i}+C_{T,i}\left[Q_i\left(1-x_i\right)\right]+h_{4,i}\left(x_iQ_i\right)T+\\+h_{1,i}\left[\frac{Q_i}{2}\left(t_{1,i}\right)+\frac{H_{1,i}t_{1,i}}{2}+\left(\frac{n-1}{2n}\right)H_{1,i}t_{2,i}+\frac{d_{1,i}t_{1,i}}{2}\left(t_{1,i}\right)\right]+\\+h_{3,i}\left[\frac{n\left(D_i-I_i\right)t_{n,i}}{2}+\frac{n\left(n+1\right)}{2}I_it_{n,i}+\frac{nI_i\left(t_{1,i}\right)}{2}\right]\end{array}\right\} \tag{27}$$

By substituting Eqs. (1) to (22) in Eq. (27) and taking the randomness of defective rate during production into account, and with further derivation, the long-run average costs per unit time for the proposed model $E[TCU(T,n)]$ can be derived as follows:

$$E\left[TCU\left(T,n\right)\right]=$$
$$=\left\{\begin{array}{l}C_0\lambda_0E_{00}+\frac{K_0}{T}+C_{S,0}\lambda_0E_{10}+\frac{h_{1,0}\lambda_0^2T}{2}\left(E_{00}\right)^2\left[\frac{1}{P_{1,0}}\right]+T\cdot h_{1,0}\sum_{i=1}^{L}\left(\frac{\lambda_i}{P_{1,i}}E_{0i}\sum_{i=1}^{L}\left(\lambda_iE_{0i}\right)\right)+\\+T\cdot h_{1,0}\sum_{i=1}^{L}\left(-\frac{\lambda_i}{P_{1,i}}\cdot E_{0i}\cdot\sum_{j=1}^{i}\left(\lambda_jE_{0j}\right)\right)+T\cdot h_{4,0}\cdot\lambda_0\cdot E_{10}\end{array}\right\}+$$
$$+\sum_{i=1}^{L}\left\{\begin{array}{l}\left[C_i\lambda_i\cdot E_{0i}+\frac{K_i}{T}+C_{s,i}\lambda_i\cdot E_{1i}+\frac{nK_{1,i}}{T}+C_{T,i}\lambda_i\right]+T\cdot h_{4,i}\cdot\lambda_i\cdot E_{1i}+\\+\frac{h_{1,i}T{\lambda_i}^2}{2}\left\{+\left(1+\frac{1}{n}\right)\left[\frac{1}{P_{1,i}}\cdot E_{0i}^2\right]+\left(1-\frac{1}{n}\right)\left[\frac{1-E\left[x_i\right]}{\lambda_i}E_{0i}+\frac{1}{P_{1,i}}\cdot E_{0i}\cdot E_{1i}\right]\right\}+\\+\frac{h_{3,i}T{\lambda_i}^2}{2}\left\{\frac{2}{P_{1,i}}\cdot E_{0i}-\frac{1}{\lambda_i}+\left(1+\frac{1}{n}\right)\left[\frac{1-E\left[x_i\right]}{\lambda_i}\cdot E_{0i}-\frac{1}{P_{1,i}}\cdot E_{0i}^2+\frac{1}{P_{1,i}}\cdot E_{0i}\cdot E_{1i}\right]\right\}\end{array}\right\} \tag{28}$$

where:

$$E_{0j}=\frac{1}{\left(1-E[x_j]\right)}\ for\ j=1,...,i\ ;\quad E_{00}=\frac{1}{\left(1-E[x_0]\right)}\ ;\ E_{10}=\frac{E[x_0]}{\left(1-E[x_0]\right)}$$
$$E_{0i}=\frac{1}{\left(1-E[x_i]\right)}\ for\ i=1,2,...,L\ ;\ E_{1i}=\frac{E[x_i]}{\left(1-E[x_i]\right)}\quad for\ i=1,2,...,L \tag{29}$$

3. OPTIMAL PRODUCTION-SHIPMENT POLICY

In order to derive the optimal production-shipment policy for the proposed model, we need to first prove the convexity of the cost function $E[TCU(T, n)]$. The Hessian matrix equations [35] are employed to verify whether Eq. (30) holds:

$$\begin{bmatrix} T & n \end{bmatrix} \cdot \begin{bmatrix} \dfrac{\partial^2 E[TCU(T,n)]}{\partial T^2} & \dfrac{\partial^2 E[TCU(T,n)]}{\partial T \partial n} \\ \dfrac{\partial^2 E[TCU(T,n)]}{\partial T \partial n} & \dfrac{\partial^2 E[TCU(T,n)]}{\partial n^2} \end{bmatrix} \cdot \begin{bmatrix} T \\ n \end{bmatrix} > 0 \qquad (30)$$

From Eq. (29), we obtain the following:

$$\frac{\partial E[TCU(T,n)]}{\partial T} = \left\{ \begin{array}{l} \dfrac{-K_0}{T^2} + \dfrac{h_{1,0}\lambda_0^2}{2}(E_{00})^2 \left[\dfrac{1}{P_{1,0}}\right] + h_{1,0}\displaystyle\sum_{i=1}^{L}\left(\dfrac{\lambda_i}{P_{1,i}}E_{0i}\sum_{i=1}^{L}(\lambda_i E_{0i})\right) + \\ + h_{1,0}\displaystyle\sum_{i=1}^{L}\left(-\dfrac{\lambda_i}{P_{1,i}} \cdot E_{0i} \cdot \sum_{j=1}^{i}(\lambda_j E_{0j})\right) + h_{4,0} \cdot \lambda_0 \cdot E_{10} \end{array} \right\} +$$

$$+ \sum_{i=1}^{L} \left\{ \begin{array}{l} \left[\dfrac{-K_i}{T^2} - \dfrac{nK_{1,i}}{T^2}\right] + h_{4,i} \cdot \lambda_i \cdot E_{1i} + \\ + \dfrac{h_{1,i}\lambda_i^2}{2}\left\{ +\left(1+\dfrac{1}{n}\right)\left[\dfrac{1}{P_{1,i}} \cdot E_{0i}^2\right] + \left(1-\dfrac{1}{n}\right)\left[\dfrac{1-E[x_i]}{\lambda_i}E_{0i} + \dfrac{1}{P_{1,i}} \cdot E_{0i} \cdot E_{1i}\right] \right\} + \\ + \dfrac{h_{3,i}\lambda_i^2}{2}\left\{ \dfrac{2}{P_{1,i}} \cdot E_{0i} - \dfrac{1}{\lambda_i} + \left(1+\dfrac{1}{n}\right)\left[\dfrac{1-E[x_i]}{\lambda_i} \cdot E_{0i} - \dfrac{1}{P_{1,i}} \cdot E_{0i}^2 + \dfrac{1}{P_{1,i}} \cdot E_{0i} \cdot E_{1i}\right] \right\} \end{array} \right\} \qquad (31)$$

$$\frac{\partial^2 E[TCU(T,n)]}{\partial T^2} = \frac{2K_0}{T^3} + \sum_{i=1}^{L}\left\{\frac{2K_i}{T^3} + \frac{2nK_{1,i}}{T^3}\right\} \qquad (32)$$

$$\frac{\partial E[TCU(T,n)]}{\partial n} = \sum_{i=1}^{L}\left\{\frac{K_{1,i}}{T} + \frac{T\lambda_i^2}{2n^2}\left\{(h_{1,i} - h_{3,i})\left[\frac{1-E[x_i]}{\lambda_i}E_{0i} + \frac{1}{P_{1,i}} \cdot E_{0i} \cdot E_{1i} - \frac{1}{P_{1,i}} \cdot E_{0i}^2\right]\right\}\right\} \qquad (33)$$

$$\frac{\partial^2 E[TCU(T,n)]}{\partial n^2} = \sum_{i=1}^{L}\left\{\frac{T\lambda_i^2}{n^3}\left\{(h_{3,i} - h_{1,i})\left[\frac{1-E[x_i]}{\lambda_i}E_{0i} + \frac{1}{P_{1,i}} \cdot E_{0i} \cdot E_{1i} - \frac{1}{P_{1,i}} \cdot E_{0i}^2\right]\right\}\right\} \qquad (34)$$

$$\frac{\partial^2 E[TCU(T,n)]}{\partial T \partial n} = \sum_{i=1}^{L}\left\{-\frac{K_{1,i}}{T^2} + \frac{\lambda_i^2}{2n^2}\left\{(h_{1,i} - h_{3,i})\left[\frac{1-E[x_i]}{\lambda_i}E_{0i} + \frac{1}{P_{1,i}} \cdot E_{0i} \cdot E_{1i} - \frac{1}{P_{1,i}} \cdot E_{0i}^2\right]\right\}\right\} \qquad (35)$$

By substituting Eqs. (32), (34), and (35) in Eq. (30), we obtain:

$$\begin{bmatrix} T & n \end{bmatrix} \cdot \begin{bmatrix} \dfrac{\partial^2 E[TCU(T,n)]}{\partial T^2} & \dfrac{\partial^2 E[TCU(T,n)]}{\partial T \partial n} \\ \dfrac{\partial^2 E[TCU(T,n)]}{\partial T \partial n} & \dfrac{\partial^2 E[TCU(T,n)]}{\partial n^2} \end{bmatrix} \cdot \begin{bmatrix} T \\ n \end{bmatrix} = \frac{2K_0}{T} + \sum_{i=1}^{L}\frac{2K_i}{T} > 0 \qquad (36)$$

Equation (36) results as positive, because K_0, K_i, and T are all positive. Hence, $E[TCU(T, n)]$ is a strictly convex function for all T and n different from zero. Therefore, the convexity of $E[TCU(T, n)]$ is proved, and there exists a minimum of $E[TCU(T, n)]$.

In order to concurrently determine the production-shipment policy for the proposed multi-product single-machine FPR model, we can solve the linear system of Eqs. (31) and (33) by setting these partial derivatives equal to zero. With further derivations we obtain:

$$T^* = \sqrt{\frac{K_0 + \sum_{i=1}^{L}\left[K_i + nK_{1,i}\right]}{\pi_0 + \sum_{i=1}^{L}\left\{\begin{array}{l}\frac{h_{1,i}\lambda_i^2}{2}\left\{\left(1+\frac{1}{n}\right)\left[\frac{1}{P_{1,i}}E_{0i}^2\right]+\left(1-\frac{1}{n}\right)\left[\frac{1-E[x_i]}{\lambda_i}E_{0i}+\frac{E_{0i}E_{1i}}{P_{1,i}}\right]\right\}+ \\ +\frac{h_{3,i}\lambda_i^2}{2}\left\{\begin{array}{l}\frac{2}{P_{1,i}}\cdot E_{0i}-\frac{1}{\lambda_i} \\ +\left(1+\frac{1}{n}\right)\left[\frac{1-E[x_i]}{\lambda_i}\cdot E_{0i}-\frac{E_{0i}^2}{P_{1,i}}+\frac{E_{0i}E_{1i}}{P_{1,i}}\right]\end{array}\right\}+h_{4,i}\lambda_i E_{1i}\end{array}\right\}}} \tag{37}$$

$$n^* = \sqrt{\frac{\left(K_0+\sum_{i=1}^{L}\left[K_i\right]\right)\cdot\sum_{i=1}^{L}\left\{\frac{\lambda_i^2}{2}\left(h_{3,i}-h_{1,i}\right)\left[\frac{1-E[x_i]}{\lambda_i}E_{0i}+\frac{1}{P_{1,i}}\cdot E_{0i}E_{1i}-\frac{1}{P_{1,i}}\cdot E_{0i}^2\right]\right\}}{\left(\sum_{i=1}^{L}\left(K_{1,i}\right)\right)\left\{\pi_0+\sum_{i=1}^{L}\left\{\begin{array}{l}\frac{h_{1,i}\lambda_i^2}{2}\left\{\frac{1}{P_{1,i}}\cdot E_{0i}^2+\frac{1-E[x_i]}{\lambda_i}E_{0i}+\frac{E_{0i}\cdot E_{1i}}{P_{1,i}}\right\}+h_{4,i}\lambda_i E_{1i}+ \\ +\frac{h_{3,i}\lambda_i^2}{2}\left\{\frac{2}{P_{1,i}}E_{0i}-\frac{1}{\lambda_i}+\frac{1-E[x_i]}{\lambda_i}E_{0i}-\frac{E_{0i}^2}{P_{1,i}}+\frac{E_{0i}\cdot E_{1i}}{P_{1,i}}\right\}\end{array}\right\}\right\}}} \tag{38}$$

where:

$$\pi_0 = \left(\begin{array}{l}\frac{h_{10}\lambda_0^2}{2}\left(E_{00}\right)^2\left(\frac{1}{P_{10}}\right)+h_{40}\lambda_0 E_{10}+h_{10}\sum_{i=1}^{L}\left(\frac{\lambda_i E_{0i}}{P_{1i}}\sum_{i=1}^{L}\left(\lambda_i E_{0i}\right)\right)+ \\ +h_{10}\sum_{i=1}^{L}\left(-\frac{\lambda_i}{P_{1i}}E_{0i}\sum_{j=1}^{i}\left(\lambda_j E_{0j}\right)\right)\end{array}\right)$$

In a real-life situation, the number of shipments n takes on integer values only. However Eq. (38) results in a real number. In order to determine the integer value of n^* that minimizes the cost function $E[TCU(T, n)]$, two adjacent integers to n must be examined respectively. Let n^+ denote the smallest integer greater than or equal to n (as derived from Eq.(38)) and n^- denote the largest integer less than or equal to n – we perform the substitution of n^+ and n^- respectively in Eq. (37), apply the resulting (T, n^+) and (T, n^-) in Eq. (28), respectively, then select the one that gives the minimum value of $E[TCU(T, n)]$ as the optimal production-shipment policy (T^*, n^*).

4. NUMERICAL EXAMPLE

In this section, we provide a numerical example to exhibit practical usage of aforementioned results. Suppose a producer needs to fabricate five products to meet annual customer demands λ_i of *3000*, *3200*, *3400*, *3600*, and *3800* units, respectively.

On a single stage production basis [31], the annual production rates P_i for these five products are *58000*, *59000*, *60000*, *61000*, and *62000* units, respectively. The random defective rates x_i follow the uniform distribution over the intervals of [*0*, *0.05*], [*0*, *0.10*], [*0*, *0.15*], [*0*, *0.20*], and [*0*, *0.25*], respectively. Unit scrap costs $C_{S,i}$ are *$20*, *$25*, *$30*, *$35*, and *$40*, respectively. Unit production costs C_i are *$80*, *$90*, *$100*, *$110*, and *$120*, respectively. Unit holding costs $h_{1,i}$ are *$10*, *$15*, *$20*, *$25*, and *$30* respectively. Setup costs K_i are *$17000*, *$17500*, *$18000*, *$18500*, and *$19000*, respectively. The fixed delivery costs per shipment K_{1i} are *$1800*, *$1900*, *$2000*, *$2100*, and *$2200*. Unit holding costs $h_{3,i}$ in the customers' end are *$70*, *$75*, *$80*, *$85*, and *$90* respectively. Unit holding costs for safety stocks $h_{4,i}$ are *$10*, *$15*, *$20*, *$25*, and *$30* respectively. Unit transportation costs $C_{T,i}$ are *$0.1*, *$0.2*, *$0.3*, *$0.4*, and *$0.5* respectively.

It is further assumed that they share a *common intermediate part*, which is about *50%* completion as compared to the finished products (i.e., $\alpha = 0.5$), and the common part can be produced at a faster annual rate of $P_{1,0} = 120000$ units. We further assume that the production of common intermediate parts has the unit manufacturing cost $C_0 = \$40$, setup cost $K_0 = \$8500$, unit scrap cost $C_{S,0} = \$20$, unit holding cost $h_{1,0} = \$5$, and defective rate = [*0*, *0.04*]. Suppose the producer adopts the two-stage single-machine common production cycle time policy instead of single-stage production policy, then production rates and cost related parameters in the second stage (fabricating the finished products) will be in (linear) proportion to α. Therefore, annual production rates $P_{1,i}$ for the finished products in the second stage become *112258*, *116066*, *124068*, *128276*, and *600667* units, respectively. Setup costs K_i become *$8500*, *$9000*, *$9500*, *$10000*, and *$10500*, respectively. Unit production costs C_i are *$40*, *$50*, *$60*, *$70*, and *$80*, respectively. The random defective rates x_i follow the uniform distribution over the intervals of [*0*, *0.01*], [*0, 0.06*], [*0*, *0.11*], [*0*, *0.16*], and [*0*, *0.21*], respectively. Unit scrap costs $C_{S,i}$ are *$10*, *$15*, *$20*, *$25*, and *$30*, respectively.

From equations (9) and (10), we have $\lambda_0 = 19961$. Applying Eqs. (37), (38), and (28), we obtain the optimal number of delivery $n^* = 3$, the optimal common production cycle time $T^* = 0.4588$ (years), and the long-run average production-inventory-delivery costs per unit time for L products in the proposed system $E[TCU(T^*, n^*)] = \$2{,}278{,}391$.

The behaviour of $E[TCU(T, n)]$ with respect to the number of shipments n per cycle is illustrated in Figure 5.

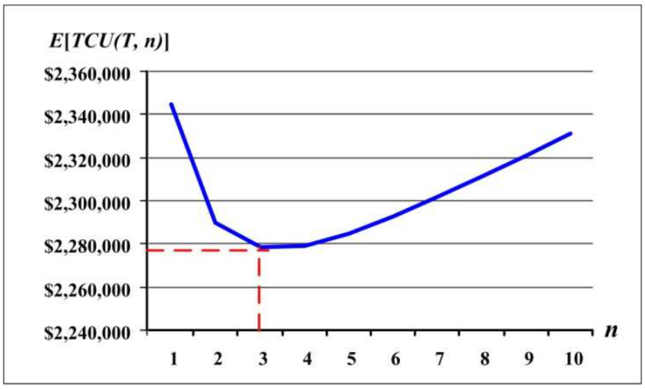

Fig. 5 *The behaviour of E[TCU(T, n)] with respect to number of shipments n per cycle*

Variation of the completion rate (α) of the common intermediate part effects on $E[TCU(T, n)]$ is depicted in Figure 6. It is noted that, as the completion rate (α) increases, the expected system costs $E[TCU(T, n)]$ decrease significantly.

From Figure 6 one also notices that the proposed two-stage delayed differentiation multi-product FPR model has a significantly lower system cost than that obtained from the single-stage multi-product FPR model [31].

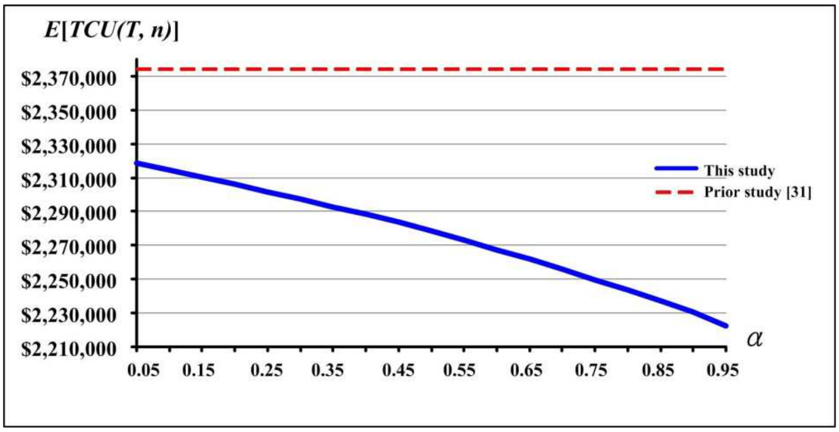

Fig. 6 *The behaviour of E[TCU(T, n)] with respect to the completion rate (α) of the common intermediate part*

Variation of the completion rate (α) of the common intermediate part effects on the optimal production cycle time T^* is illustrated in Figure 7. It is noted that, as the completion rate (α) increases, T^* decreases significantly. One also notices that the proposed two-stage delayed differentiation multi-product FPR model has significantly shorter production cycle time than that obtained from the single-stage multi-product FPR model [31].

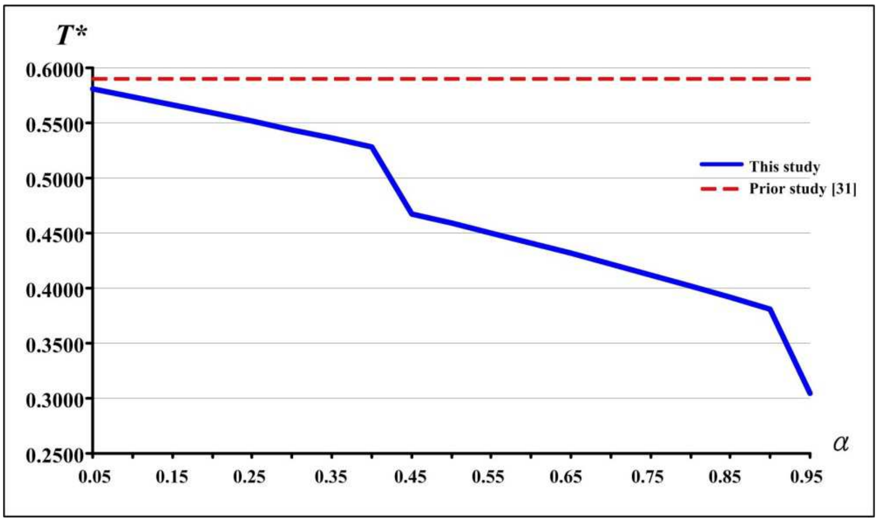

Fig. 7 *The behaviour of optimal production cycle time T* with respect to the completion rate (α) of the common intermediate part*

5. CONCLUDING REMARKS

In real-life production-shipment integrated systems, *w*hen a family of multiple products shares a *common intermediate part*, with the aim of reducing total production-inventory delivery costs as well as shortening the production cycle time, it is ordinary to consider delayed differentiation strategy for such a multi-product FPR system. This study uses mathematical modelling to explore a multi-product two-stage imperfect FPR system using a single-machine production scheme.

As a result, we derive a closed-form optimal replenishment cycle time and delivery policy that minimizes the long-run average cost per unit time for the proposed model. A numerical example is provided to demonstrate its practical usage and compare our result to that obtained from the traditional single-stage multi-product FPR model. It is noted that the research results enable management to better understand, plan, and control such a real multi-product FPR system with delayed differentiation. For future study, one interesting topic is to consider the effect of dual machines (i.e., one additional machine for fabricating specifically the common intermediate parts) on the optimal operating policy.

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