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A two-phase loading model of the cable structures

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SUMMARY

A method for calculating displacements and stresses of prestressed tensile cable structures is presented. The developed numerical model takes into account the material and geometric nonlinearities. The behavior of the structure under an increasing load, from zero up to the final value is described. The load is usually applied in two phases: the first phase is prestressing, and the second phase is loading by dead and live gravity loads. An incremental approach with the successive application of the Total Lagrange formulation with small displacements is used to solve the problem of large displacements. The spatial discretization of the system is performed by two-noded beam elements. The fiber discretization of the cross-section is performed by triangular elements, where the mechanical properties of each fiber are presented by the stress-strain curve. The described model represents a practical way of implementing the large displacements theory for finding the appropriate form and stresses in prestressed cable structures.

Key words: *cables, tension structures, nonlinear analysis, space structures, form-finding.*

1. INTRODUCTION

Many models for the computation of form-finding in cable structures are developed from its beginning [1]. Nowadays, many system analyses are aimed at structural optimization. As a result, lightweight flexible structures, especially cable ones, are being intensively developed in the last decades [2-4]. Generally, they are used in roof structures of stadiums, sports halls, exhibition halls and bridges. The development of structural theories, computers, as well as new materials with good performances, provides fast and accurate analysis with graphical representation of the shape and response of the structures.

In the analysis of flexible cable structures, it is necessary to take into account the geometric nonlinearity (due to large displacements) and very often the material nonlinearity. The daily practice is that, after complex analyses and comparison of many versions with many drafts, the final shape of such structures is selected.

It is necessary to know the behavior of a structure under various load conditions. Generally, flexible cable structures may exhibit very different shapes of failure, depending on the load types.

A structural analysis usually starts with a known finite geometry of the structure but in case of cable structures the appropriate shape of structures has to be found. Particularly, the shape of flexible cable structures can not be predefined. Instead, it has to be obtained from the equilibrium condition with respect to the boundary conditions, structural topology and internal force distribution.

For the static and dynamic analyses of cable structures, there already exist several techniques. The purpose of this paper is to illustrate the relative complexity found by previous researchers of this problem.

The dynamic relaxation method presents an approach to the design of prestressed cable networks through the use of the D'Alembert principle [5, 6]. The stiffness matrix approach is solved iteratively by Newton-Raphson method [7], and by incremental loading for situations where error accumulation is too large. The theoretical approach of minimizing the total potential energy has been described [8-10] with the same theoretical approach but with a different choice of the minimization algorithm. An approximate linear approach to geometric nonlinearity can be used by observing that a general load on a cable net can be decomposed into two parts (one part with extensional displacements, and the other without them) [11, 12]. Some researchers have applied a more general finite element method approach, for example, curved member for shallow cable nets [13], hyper cable (a cable connected to intermediate pulleys along its length [14]), and two-link structure [15].

The starting point of the procedure, in presented numerical model of cable structures, assumes a known initial geometry (the initial coordinates of points and initial positions of elements).

The analysis is than performed in two phases. The first phase is with pretension, if applicable, leading to a new geometry. In the second phase, other loads (such as the gravity load) are added on to the new geometry.

The finite geometry and finite internal and external forces in the system are obtained by applying the large displacements theory.

2. LARGE DISPLACEMENTS THEORY

Large displacements are obtained by the incremental application of the small displacements theory. The small displacements theory follows the Total Lagrange formulation. The updating of the geometric configuration from increment to increment is implemented by a special procedure called the Null Configuration Principle [16, 17]. The updating of the effects on the structure includes the updating of the basic and geometric stiffness matrices of small deformations, and then the updating of the loads and their influences on the internal forces.

Space cable structures are modeled by linear finite elements. The updating of large translational deformations in the incremental procedure follows from their vector summation from increment to increment. The condition of large translations, with the application of linear elements, is approximated such that each element, for the purpose of interpreting the geometry, is considered as a rigid body which accepts large rotations. Formally, a discontinuity in rotations is introduced. This assumption is acceptable if the length of the element is small enough, and the stress in the element is expressed within the small deformations of each increment.

The governing global equilibrium equation for a structure gives:

$$\left[\boldsymbol{K}_b\left(u\right)+\boldsymbol{K}_g\left(u\right)+\boldsymbol{K}_L\left(u\right)\right]\boldsymbol{u}=\boldsymbol{F}\left(u\right) \qquad (1)$$

where $\boldsymbol{K}_b(u)$ is the basic stiffness matrix, $\boldsymbol{K}_g(u)$ is the geometric stiffness matrix, $\boldsymbol{K}_L(u)$ is the geometric stiffness matrix due to the large displacements, $\boldsymbol{u}$ is the vector of unknown displacements and $\boldsymbol{F}(u)$ is the vector of nodal forces. Matrix $\boldsymbol{K}_L(u)$ may be defined as a change of the basic stiffness matrix under the new coordinates.

Considering only large translation displacements, the procedure of successive application of the Total Lagrange formulation is applied. For practical purposes, an incremental-iterative procedure is developed into computer program [18]. The changes in configuration and structural stiffness are included in each increment.

Under arbitrary load level, as shown in Figure 1, in the secant stiffness method relationship between displacements and loading is:

$$\tilde{\boldsymbol{u}}=\boldsymbol{K}^{-1}\boldsymbol{F} \qquad (2)$$

where $\boldsymbol{K}$ is the total stiffness of structure.

Variation of displacement on current load level is given by:

$$\delta\tilde{\boldsymbol{u}}=\delta\boldsymbol{K}^{-1}\boldsymbol{F}+\boldsymbol{K}^{-1}\delta\boldsymbol{F} \qquad (3)$$

The first member has an impact on variation due to changing the flexible matrix (inversion of the stiffness matrix) and the second member has an impact under changes in load. In the case of large displacement geometric nonlinearity, the impact of the first member cannot be omitted.

It represents the impact of the previously applied load on displacement, due to changes of position. That position is caused by changes in geometry and presents the actual position of total load.

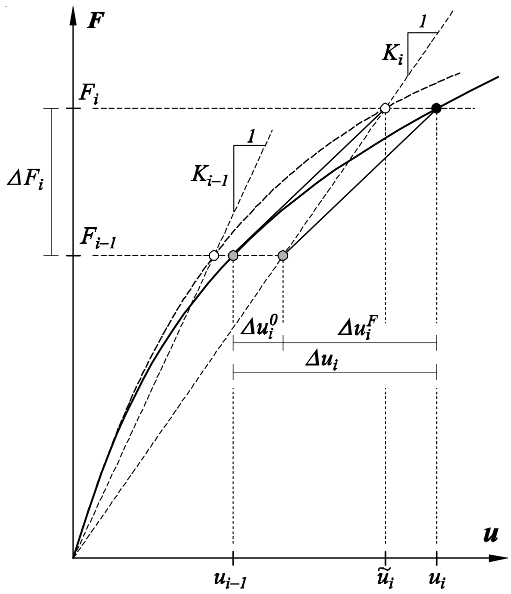


Fig. 1 Symbolic description of large displacements of single degree-of-freedom system

In the case of finite increments, Eq. (2) is given by:

$$\Delta \boldsymbol{u} = \Delta \boldsymbol{K}^{-1} \boldsymbol{F} + \boldsymbol{K}^{-1} \Delta \boldsymbol{F} \tag{4}$$

hence:

$$\Delta \boldsymbol{u}_i = \left(\boldsymbol{K}_{i-1}^{-1} - \boldsymbol{K}_i^{-1} \right) \boldsymbol{F}_{i-1} + \boldsymbol{K}_i^{-1} \Delta \boldsymbol{F}_i \tag{5}$$

what can be written as:

$$\Delta \boldsymbol{u}_i = \Delta \boldsymbol{u}_i^0 + \Delta \boldsymbol{u}_i^F \tag{6}$$

Subsequently:

$$\boldsymbol{u}_i = \boldsymbol{u}_{i-1} + \Delta \tilde{\boldsymbol{u}}_i \tag{7}$$

Thereafter, in the incremental procedure, an update of the internal forces and stresses within the structural model based on the achieved displacements, is required:

$$\boldsymbol{S}_i = \boldsymbol{S}_i \left(\boldsymbol{u}_i \right) \tag{8}$$

Afterwards, the next increment step may be applied.

The solution of Eq. (1) is performed by the incremental-iterative procedure which is implemented in a developed computer program [18].

3. NUMERICAL MODEL OF SPACE CABLE STRUCTURES

3.1 Basic hypotheses

For practical usage of this numerical model it is necessary to adopt some hypotheses:

- The analysis of space structures is assumed.
- The structural dimensions, cross-sections, and types of materials are predefined, and the necessary discretization is implemented.
- This model includes analyses by implementing the large displacements theory and the assumption of the follower load type, taking into account the geometric nonlinearity.
- This model takes into account the material nonlinearity which is assumed by an uni-axial stress-strain relationship that is given numerically.
- Load combinations have to be calculated because the geometric and material nonlinearities are taken into account.
- Certain initial imperfection is taken into account by the initial geometry.

3.2 Structural discretization

In the model of form-finding for cable structures, as presented in this paper, linear, ideal straight two-noded finite elements are used (Figure 2). All the nodes have six degrees of freedom (three translations and three rotations). It is necessary to pay attention to the size of each individual element, because large differences may cause significant numerical errors.

Finite elements are connected at nodes, and every node is necessary for the finite element mesh.

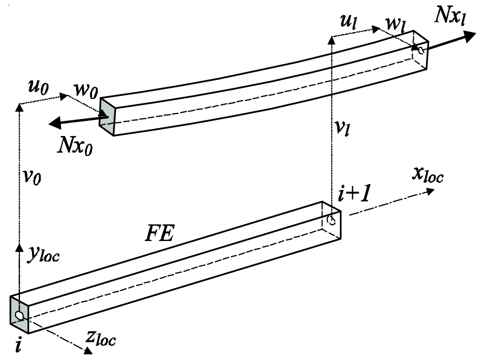


Fig. 2 Tensile two-noded space finite element

3.3 Modeling of cross-section

A cross-section of the element is discretized by quadrilateral and/or triangular elements (Figure 3a), which is represented by filament. Such filament discretization enables the monitoring of the normal stresses in the element and hence the state of the cross-section under the general action of the longitudinal forces.

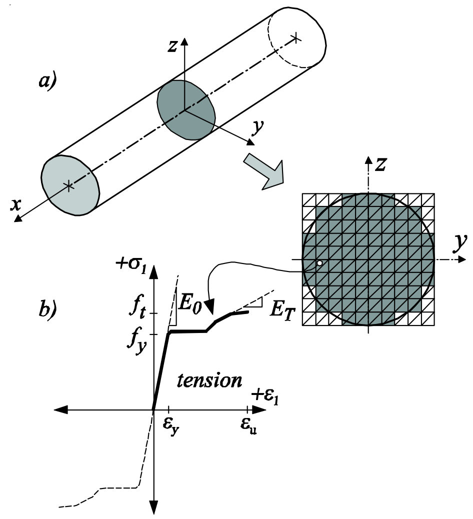


Figure 3 Modeling of cross-section:
a) Cross-section discretization;
b) Stress-strain relationship for steel

The material properties for uniaxial stress-strain relationship are given numerically in a polygonal form (Figure 3b). The equilibrium of the cross-section is obtained by an iterative procedure.

3.4 Two-phase loading model

The model presented in this paper is solved in two phases. The first phase includes self-weight and pretension load (Figure 4a). In most cases, the influence of self-weight is negligible as compared to the pretension forces in cables between two nodes. Afterwards, the cable structure, under the load from the first phase, obtains a new geometry. In the second phase, the gravity load is applied as point or concentrated loads (Figure 4b). In both phases the load can be a point load on a node or a distributed load on an element. The load increases from zero up to the failure level.

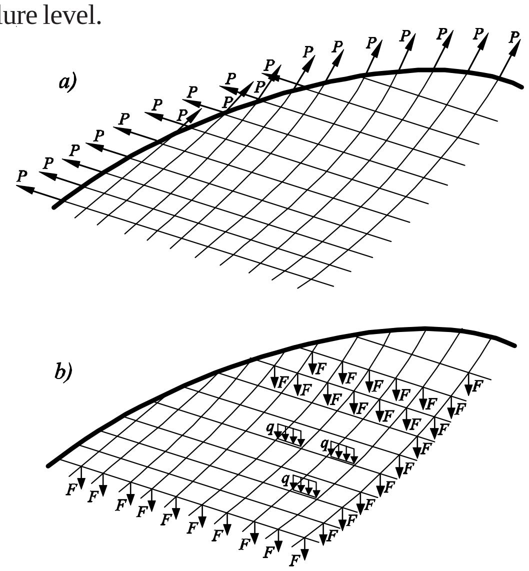


Fig. 4 The load phases:
a) self-weight and pretension load;
b) gravity load

4. NUMERICAL SOLUTION ACCURACY

Generally, the equilibrium equation (1) is solved by an incremental-iterative method. The accuracy of the numerical solution depends on: (i) The discretization of the model by linear elements and the choice for the basis functions, (ii) The cross-section discretization; (iii) The equilibrium of the cross-section; (iv) The nonlinear material properties; (v) The number of iterations per increment; (vi) The increment size.

A study of the numerical error is presented using a specific example from the literature [19]. The geometric properties are shown in Figure 5. Area of the cable is $A=500\ mm^2$, the modulus of elasticity $E=105\ kN/mm^2$ and the initial unstrained lengths of cable is *15.00 m*. The first phase is the gravity loading of *0.01 kN* applied at each node from *2* to *5*, and the second phase is the gravity loading of *1.0 kN* also applied at each node from *2* to *5*. A study of the behavior of this structure by different discretizations and load increments is presented below.

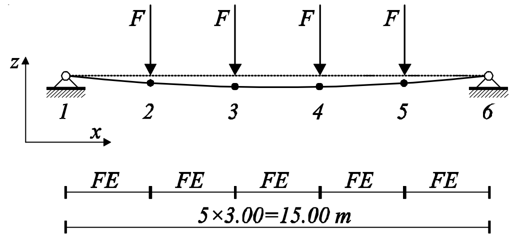


Fig. 5 Structure discretized by 5 elements

The displacements for 5 elements discretization and for different load increments are presented in Table 1.

The displacements for 10 elements discretization (Figure 6) and for different load increments are presented in Table 2.

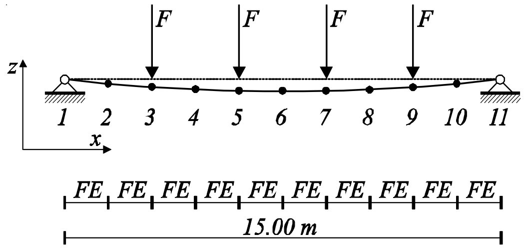


Fig. 6 Structure discretized by 10 elements

The results obtained by this example with different structural discretizations and load increments are quite satisfactory.

For studying the convergence, the same example is selected, except that in the second phase the gravity load of *2.0 kN* is also applied to nodes from *2* to *5*.

A study of numerical errors conditioned by large displacements, which are implemented as a series of small displacements, through the effects of the increment size and the density of discretization, is presented. At any increment, one equilibrium iterative step is applied, and the equilibrium criterion per force norm is *0.05*. For studying the effect of the increment size, the load increment for both discretizations is accepted between *0.5%* and *9.5%* of the specific load.

The results are presented in an accession curve diagram (Figure 7), which represents the numerically obtained characteristic load factor f with respect to the relative increment size δf. The relative increment size δf is the ratio between the incremental load ΔF_{inc} and the total applied load (i.e., the load corresponding to unit characteristic load factor $f=1.00$):

$$\delta f = \frac{\Delta F_{inc}}{F_{(f=1.00)}} \tag{9}$$

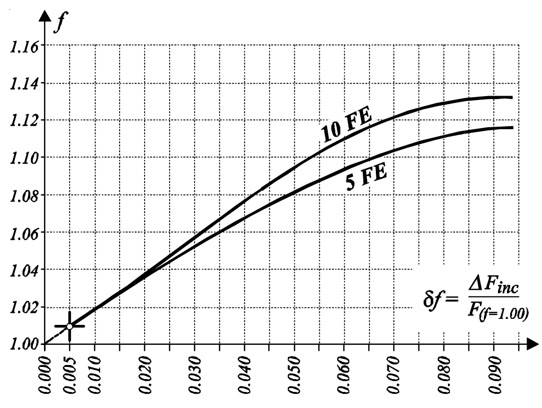


Fig. 7 Accession curves

The accession curves are approaching with decreased increment size, especially under *3%* of the specific load, in the numerical convergence zone. The increment size of *0.5%* is a limit, because in this case the solution is diverging. The minimal increment size of *0.5%* to the specific load reaches an accuracy level of *1%*.

The presented model is applied to a few more examples, where a faster convergence is observed when the initial load is sufficiently low as compared to the breakdown load. Hence, it is more sensitive to define the time and place of the breakdown when the load increment is lower.

Table 1 Nodal displacements for structure discretized by 5 elements

Node	Horizontal displacement (mm)			Vertical displacement (mm)		
	Millar and Barghian [19]	This paper		Millar and Barghian [19]	This paper	
		10 load increment	100 load increment		10 load increment	100 load increment
2	-0.0200	-0.023	-0.0225	-8.4597	-8.1835	-7.9899
3	-0.0099	-0.012	-0.0121	-12.5320	-12.7480	-12.4430
4	0.0099	0.012	0.0121	-12.5320	-12.7480	-12.4430
5	0.0200	0.023	0.0225	-8.4597	-8.1835	-7.9899

Table 2 Nodal displacements for structure discretized by 10 elements

Node	Horizontal displacement (mm)			Vertical displacement (mm)		
	Millar and Barghian [19]	This paper		Millar and Barghian [19]	This paper	
		10 load increment	100 load increment		10 load increment	100 load increment
2	-0.0200	-0.0225	-0.0221	-8.4597	-8.1201	-7.9295
3	-0.0099	-0.0121	-0.0110	-12.5320	-12.5590	-12.3580
4	0.0099	0.0121	0.0110	-12.5320	-12.5590	-12.3580
5	0.0200	0.0225	0.0221	-8.4597	-8.1201	-7.9295

5. NUMERICAL EXAMPLES

The model presented in this paper is verified with results from the literature. The experimental results [5] and the results of the numerical model with geometric nonlinearity [15] are attached.

5.1. Flat net

The flat net example consists of a cable net lying on a 3×3 square grid with a cell side length of $0.4\ m$, as shown in Figure 8. The cables have cross-sectional stiffness $EA=97.97\ kN$.

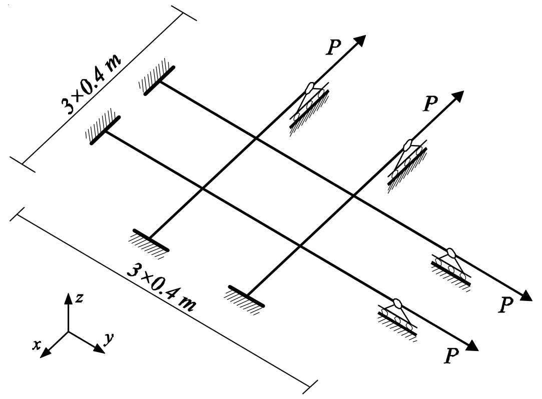


Fig. 8 Flat net (3 ×3) in three-dimensional space (The first phase)

In the first phase, all the cables have a pretension of $200\ N$, and the analysis is achieved within 100 incremental steps. Then in the second phase, all the cables are anchored at their ends and weighted by a vertical force of $150\ N$ as is shown in Figure 9.

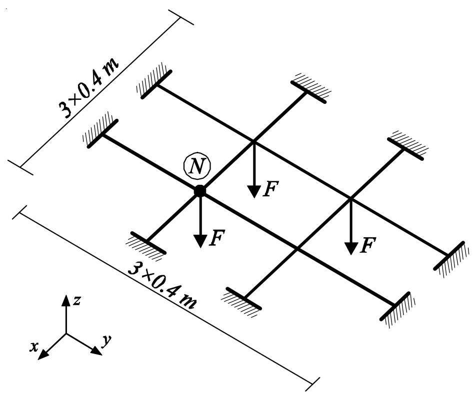


Fig. 9 Flat net (3 ×3) in three-dimensional space (The second phase)

The equilibrium path for the second phase loading for the marked node in Figure 9 is presented in Figure 10, which clearly shows the effect of geometric nonlinearity due to large displacements (F is the current load and F_0 is the ultimate load). The structure is hardening as the load increases.

The intensities of the cable internal forces and displacements are *12.5 %* of the second phase load, and are shown in Figure 11.

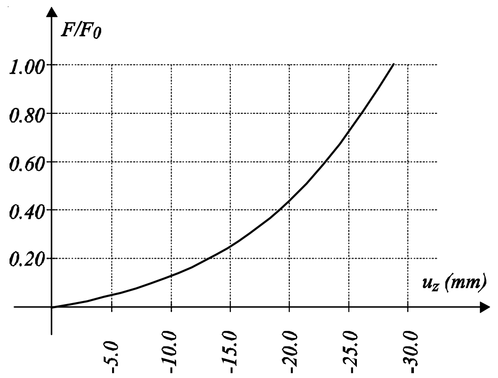
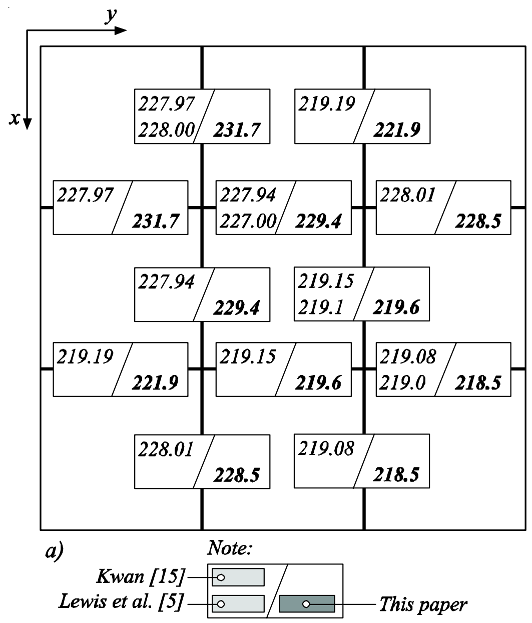


Fig. 10 Equilibrium path for node N (marked in Figure 9)



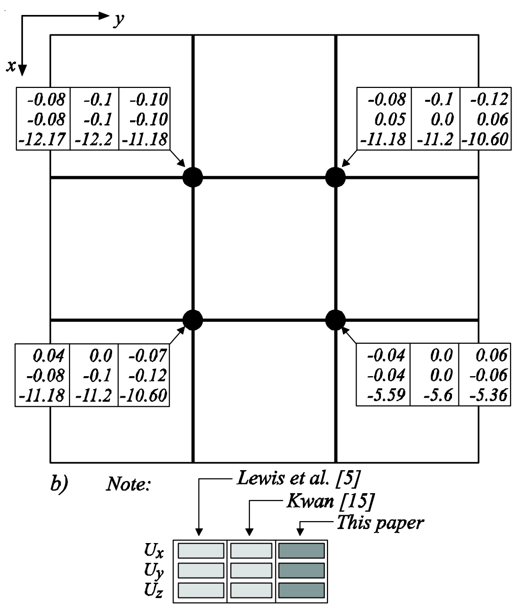


Fig. 11 Comparison with results from the literature:
a) Tension cable forces (N);
b) Nodal displacements in x, y, z directions (mm)

5.2 Hyperbolic paraboloid network

The model presented in this paper is verified on the hyperbolic paraboloid network shown in Figure 12a. The characteristics of the geometry and material parameters are given in Ref. [15].

The system is discretized by 31 elements (with *EA=100.72 kN*). In the first phase all the cables have a pretension force of *200 N*, and the analysis is achieved within 100 incremental steps. Then in the second phase, all the cables are anchored at their ends and weighted by a vertical force of *15.7 N* as is shown in Figure 12b. The second phase is completed within 10 incremental steps.

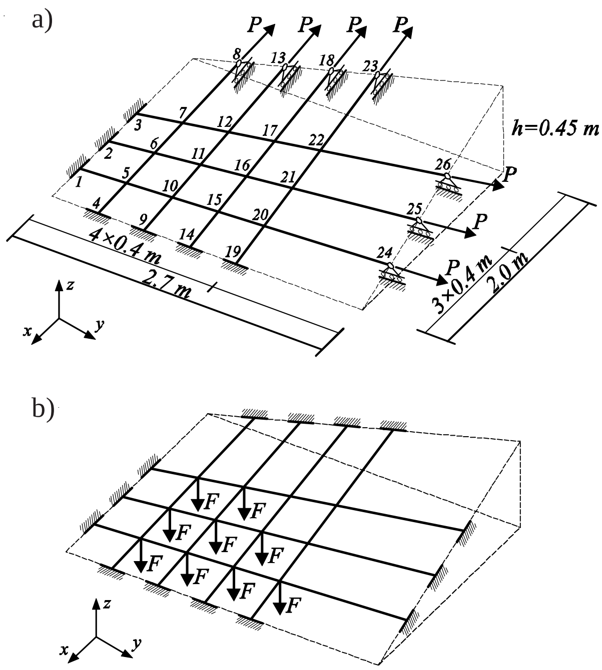


Fig. 12 Hyperbolic paraboloid network:
a) Geometry of network and first phase loading;
b) Second phase loading

Table 3 shows the results given by different authors [5, 7, 9, 15, 20] and the results obtained by the application of the large displacements theory. The results shown in Table 3 reveal an analogy of the numerical model presented in this paper and the results given by Ref. [15].

5.3 Saddle net

The saddle net presented in Figures 13 and 14 consist of 201 elements with 132 nodes.

The cross-section is circular and discretized by 10×10 quadrilateral elements with area $A = 19.63\ cm^2$. In this example a linear-elastic material is used with Young's modulus $E = 2.1 \times 10^5\ MPa$.

In the first phase some cables have a pretension force of $P = 30\ kN$ as shown in Figures 13 and 14. In the second phase all the cables are anchored, and the free nodes are weighted by a vertical force $F = 90\ kN$. Figure 14 shows the z - deflections of each node after

Table 3 Comparison of experimental and theoretical results for the hyperbolic net

		Deflection (mm)						
Node	Load (kN)	Experiment Lewis [5]	Numerical Krishna [7]	Numerical Sufian [9]	Numerical Lewis [20]	Numerical Kwan [15]	Numerical Kwan [15]	Numerical (this paper)
5	0.0157	19.5	19.6	19.3	19.3	19.38	19.52	19.00
6	0.0157	25.3	25.9	25.5	25.3	25.62	25.35	25.30
7	0.0157	22.8	23.7	23.1	23.0	22.95	23.31	22.80
10	0.0157	25.4	25.3	25.8	25.9	25.57	25.86	25.70
11	0.0157	33.6	33.0	34.0	33.8	33.79	34.05	34.40
12	0.0157	28.8	28.2	29.4	29.4	29.32	29.49	29.50
15	0.0157	25.2	25.8	25.7	26.4	25.43	25.79	25.50
16	0.0157	30.6	31.3	31.2	31.7	31.11	31.31	31.50
17	-	21.0	21.4	21.1	21.9	21.28	21.42	21.50
20	0.0157	21.0	22.0	21.1	21.9	21.16	21.48	20.80
21	-	19.8	21.1	19.9	20.5	19.79	20.00	19.90
22	-	14.2	15.7	14.3	14.8	14.29	14.40	14.10

the ultimate load. The first phase is completed within 500 incremental steps, and the second phase within 435 incremental steps.

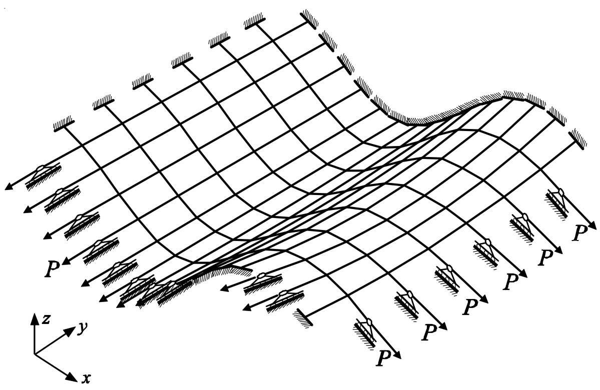


Fig. 13 Saddle net - isometric view with first phase loading

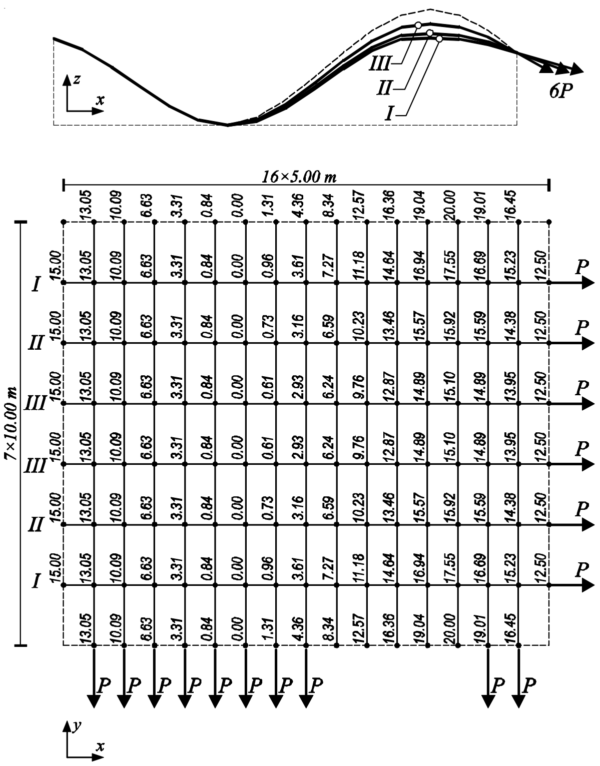


Fig. 14 Saddle net: startup geometry and first phase loading

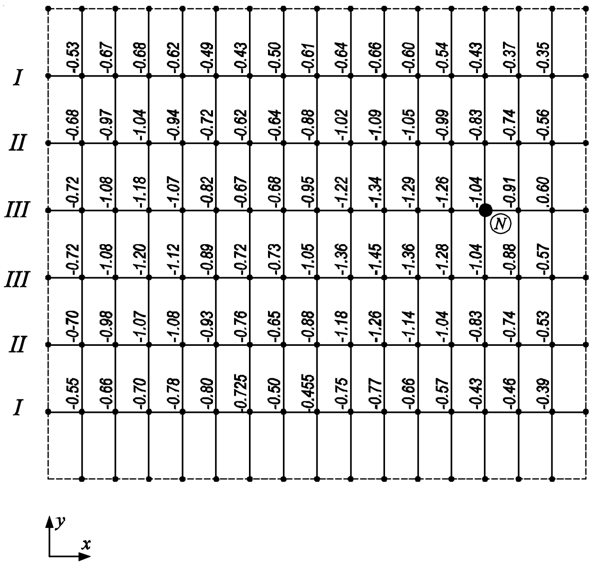


Fig. 15 Saddle net: values of displacement in global z direction (m) for the ultimate load

The equilibrium path for the second phase loading for the node marked in Figure 15 is shown in Figure 16, which also clearly shows the geometric nonlinearity effect due to large displacements.

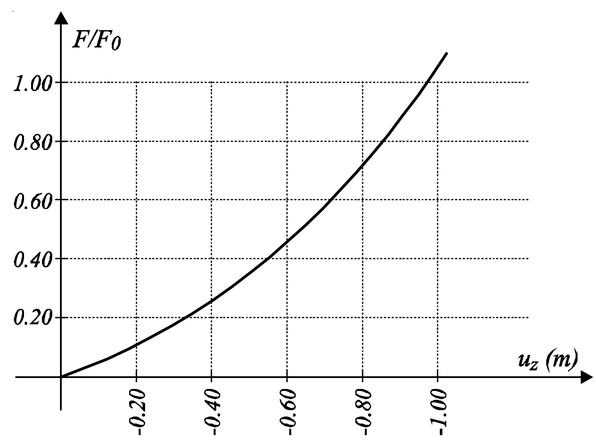


Fig. 16 Equilibrium path for node N (marked in Figure 15)

6. CONCLUSIONS

The presented numerical large displacements model can solve the problem of form-finding for cable structures with known initial geometry and total gravity load.

This model includes two phases of loading. It is possible to introduce the pretension and self-weight in the model in the first phase, while in the second phase point and distributed gravity loads are introduced. By this model, the analysis up to the failure or stability loss point is performed. The presented examples show a good convergence of the incremental-iterative method, and a good agreement with their experimental results.

The equilibrium paths for some nodes clearly show the effect of geometric nonlinearity due to large displacements. The structure is hardening as the load increases, which is typical for cable structures.

7. ACKNOWLEDGMENT

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8. REFERENCES

[1] H.-J. Schek, The force density method for form finding and computation of general networks, *Computer Methods in Applied Mechanics and Engineering*, Vol. 3, No. 1, pp. 115-134, 1974.

[2] W.J. Lewis, *Tension Structures: Form and Behaviour*, Thomas Telford Ltd, London, 2003.

[3] K.-U. Bletzinger, R. Wüchner, F. Daoud and N. Camprubí, Computational methods for form finding and optimization of shells and membranes, *Computer Methods in Applied Mechanics and Engineering*, Vol. 194, Issue 30-33, pp. 3438-3452, 2005.

[4] J. Dvornik and D. Lazarević, Prestressed flexible cable and fabric structures, *Građevinar*, Vol. 47, No. 4, pp. 185–199, 1995. (in Croatian)

[5] W.J. Lewis, M.S. Jones and K.R. Rushton, Dynamic relaxation analysis of the non-linear static response of pretensioned cable roofs, *Computers & Structures*, Vol. 18, No. 6, pp. 989-997, 1984.

[6] M.R. Barnes, Form-finding and analysis of prestressed nets and membranes, *Computers & Structures*, Vol. 30, No. 3, pp. 685- 695, 1988.

[7] P. Krishna, *Cable-suspended roofs*, McGraw-Hill Inc., 1978.

[8] J.P. Coyette and P. Guisset, Cable network analysis by a nonlinear programming technique, *Engineering Structures*, Vol. 10, No. 4, pp. 41-46, 1988.

[9] F.M.A. Sufian and A.B. Templeman, On the non-linear analysis of pretensioned cable net structures, *Structural Engineering*, Vol. 4, No. 2, pp. 147-158, 1992.

[10] G.D. Stefanou, E. Moossavi, S. Bishop and P. Koliopoulos, Conjugate gradients method for calculating the response of large cable nets to static loads, *Computers & Structures*, Vol. 49, No. 5, pp. 843-848, 1993.

[11] S. Pelegrino, A class of tensegrity domes, *International Journal of Space Structures*, Vol. 7, No. 2, pp. 127-142, 1992.

[12] O. Vilnay and P. Rogers, Statical and dynamical response of cable nets, *International Journal of Solids and Structures*, Vol. 26, No. 3, pp. 299-312, 1990.

[13] M.L. Gambhir and B.V. de Batchelor, A finite element for 3-D prestressed cablenets, *International Journal of Numerical Methods in Engineering*, Vol. 11, No. 4, pp. 1699-1718, 1977.

[14] J. Mitsugi, Static analysis of cable networks and their supporting structures, *Computers & Structures*, Vol. 51, No. 1, pp. 47-56, 1994.

[15] A.S.K. Kwan, A new approach to geometric nonlinearity of cable structures, *Computers & Structures*, Vol. 67, No. 4, pp. 243-252, 1998.

[16] B. Trogrlić, A. Harapin and A. Mihanović, The null configuration model in limit load analysis of steel space frames, *Materialwissenschaft und Werkstofftechnik*, Vol. 42, No. 5, pp. 417-428, 2011.

[17] B. Trogrlić and A. Mihanović, The comparative body model in material and geometric nonlinear analysis of space R/C frames, *Engineering Computations*, Vol. 25, No. 2, pp. 155-171, 2008.

[18] A. Mihanović and B. Trogrlić, PRONEL - Computer program for nonlinear analysis of 3D frame structures, University of Split, Faculty of Civil Engineering and Architecture, Split, 2003.

[19] M.A. Millar and M. Barghian, Snap-through behaviour of cables in flexible structures, *Computers & Structures*, Vol. 77, No. 4, pp. 361-369, 2000.

[20] W.J. Lewis, A comparative study of numerical methods for the solution of pretensioned cable networks, Proc. of the Int. Conf. on Design and Construction of Non-Conventional Structures, Edinburgh, Ed. B.H.V. Topping, Civil-Comp Press, Edinburgh, Vol. 2, pp. 27-33, 1987.

MODEL DVOFAZNOG OPTEREĆENJA ZA PRORAČUN KONSTRUKCIJA OD UŽADI

SAŽETAK

Prikazana je metoda proračuna pomaka i naprezanja konstrukcije od prednapregnutih kablova. Razvijeni numerički model uključuje materijalnu i geometrijsku nelinearnost. Prati se ponašanje konstrukcije pri povećanju opterećenja od nultog do slomnog. Opterećenje se nanosi u dvije faze: prva faza je prednaprezanje, a druga faza je opterećenje vlastitom težinom i dodatnim opterećenjem gravitacijskog tipa. Rabi se inkrementalni postupak sukcesivne Lagrange-ove obnovljive metode u rješenju problema velikih pomaka. Primijenjena je prostorna diskretizacija sustava dvočvornim konačnim elementima. Poprečni presjek se diskretizira trokutnim elementima, a mehaničko ponašanje vlakana u težištu elemenata se zadaje dijagramom naprezanje-deformacija. Prikazani model predstavlja praktičan put primjene teorije velikih pomaka u traženju oblika i proračunu naprezanja konstrukcija od užadi.

Ključne riječi: *užadi, vlačne konstrukcije, nelinearna analiza, prostorne konstrukcije, traženje oblika.*