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On the mass modelling in vibration analysis of thin-walled structures using the finite element method

Ivo Senjanović, **Nikola Vladimir** and **Neven Hadžić**

University of Zagreb, Faculty of Mechanical Engineering and Naval Architecture,
I. Lučića 5, HR-10000 Zagreb, CROATIA
e-mail: ivo.senjanovic@fsb.hr

SUMMARY

Modelling of mass in vibration analysis of thin-walled structures using the finite element method has been considered, i.e. consistent and lumped mass matrices have been compared to a newly introduced, simplified mass matrix. The matrices have been specified for a two-noded beam element, three-noded triangular plate element and four-noded rectangular plate element. The simplified mass matrix has been derived on the basis of bar and membrane shape functions instead of the bending ones. As a result, the distributed mass has been considered only to the deflectional degree of freedom. Accuracy and advantage of such a mass matrix formulation is illustrated for the cases of beam and square plate vibrations.

Key words: *thin-walled structures, vibrations, mass modelling, finite element method.*

1. INTRODUCTION

The finite element method presents a very efficient tool for structural analysis of engineering structures [1, 2]. Vibration of thin-walled structures involves dealing with stiffness matrix, mass matrix and damping matrix. In order to couple in-plane and flexural vibrations, the shell finite elements comprising the membrane and plate properties have been used. The plate stiffness is based on the shape functions related to both deflection and rotations of the element nodes. The same shape functions are used for determining the consistent structural mass matrix. Non-structural mass (for instance, equipment and cargo mass in ships) is mainly considered as lumped mass that results in a diagonal mass matrix. Sometimes, due to simplicity, the structural mass is also discretized.

Error of structural frequencies of axial rod vibrations determined by consistent and lumped mass matrices are approximately equal and opposite. As a result, the mass matrix computed from the average of these two matrices reduces the error, as elaborated in Ref. [3]. A superior behaviour of the averaged consistent and lumped mass matrix is also found for natural vibrations of simply supported beam [3, 4]. However, in the case of a free beam, the error due to application of the lumped mass matrix is much higher than that of the consistent mass matrix. Therefore, the concept of the averaged mass matrix cannot be used as a rule, since the error depends on the boundary conditions. In order to increase accuracy, special lumping technique has been elaborated in Ref. [5], where total mass has been lumped to the translational degree of freedom (d.o.f.) proportionally to the diagonal entries of the consistent mass.

Inertia forces and moments of a dynamic system are actually internal loads. Kinetic energy of the forces is dominant, compared to that of the moments. Therefore, as the third physically based option, the mass matrix for flexural vibrations can be determined by employing the shape functions of the in-plane displacements for the plate deflection, while the mass moment of inertia can be distributed per rotational d.o.f. as lumped quantities. The simplified mass matrix based on this approach has been derived for ordinary beam and plate finite elements.

2. BEAM FINITE ELEMENT

Natural vibrations of a dynamic system are obtained by solving the eigenvalue problem:

$$\left(\boldsymbol{K}-\omega^2\boldsymbol{M}\right)\boldsymbol{\delta}=0\,, \tag{1}$$

where $\boldsymbol{K}$ is the global stiffness matrix, $\boldsymbol{M}$ is the global mass matrix, $\boldsymbol{\delta}$ represents the nodal displacement vector and ω is the natural frequency.

The finite element stiffness matrix of the beam and the consistent mass matrix are derived with the shape functions in the form of the third order (Hermitian) polynomials [6]:

$$\left\langle N\right\rangle=\left\langle 1-\xi^2\left(3-2\xi\right),\ l\xi\left(1-\xi\right)^2,\right.$$
$$\left.\xi^2\left(3-2\xi\right),\ -l\xi^2\left(1-\xi\right)\right\rangle \tag{2}$$

where $\xi=\xi/l$ and l is the element length. The stiffness matrix reads:

$$\boldsymbol{K}=\frac{2EI}{l^3}\begin{bmatrix}6 & 3l & -6 & 3l\\ & 2l^2 & -3l & l^2\\ & & 6 & -3l\\ Sym. & & & 2l^2\end{bmatrix} \tag{3}$$

where EI is the bending stiffness.

The consistent mass matrix is obtained according to the definition:

$$\boldsymbol{m}_1=m\int\limits_0^l\left\{N\right\}\left\langle N\right\rangle\mathrm{d}x=\frac{ml}{420}\begin{bmatrix}156 & 22l & 54 & -13l\\ & 4l^2 & 13l & -3l^2\\ & & 156 & -22l\\ Sym. & & & 4l^2\end{bmatrix} \tag{4}$$

where m is mass per unit length.

The lumped mass matrix reads:

$$\boldsymbol{m}_2=\frac{ml}{2}\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}. \tag{5}$$

For diagonal angular terms in Eq. (5), very small positive values have to be assumed in order to ensure a positive definite matrix as a prerogative for successful computing.

The simplified mass matrix for deflection d.o.f. is derived with the linear shape functions used for bar tension:

$$\left\langle N\right\rangle=\left\langle 1-\xi,\ 0,\ \xi,\ 0\right\rangle. \tag{6}$$

Hence, one finds:

$$\boldsymbol{m}_3^d=\frac{ml}{6}\begin{bmatrix}2 & 0 & 1 & 0\\ 0 & 0 & 0 & 0\\ 1 & 0 & 2 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}. \tag{7}$$

The linear shape functions imply rigid body motion with constant rotational angle. The element mass moment of inertia reads $J=ml^3/12$ and one half is assigned to each rotational d.o.f. In that way $\boldsymbol{m}_3^d$ is extended to:

$$\boldsymbol{m}_3=\frac{ml}{420}\begin{bmatrix}140 & 0 & 70 & 0\\ & 17.5l^2 & 0 & 0\\ & & 140 & 0\\ Sym. & & & 17.5l^2\end{bmatrix}. \tag{8}$$

Matrix $\boldsymbol{m}_3$ can be compared with $\boldsymbol{m}_1$, Eqs. (8) and (4), respectively.

3. TRIANGULAR PLATE ELEMENT

Triangular plate element with 9 d.o.f. is considered in the Cartesian coordinate system, Figure 1. The shape functions are the third order polynomials of a quite complex form. As a result, the stiffness and mass matrices are of very complex form, too, [7].

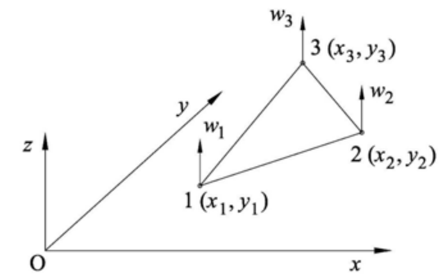

Fig. 1 Triangular plate element

In order to simplify the mass matrix, the shape functions of membrane displacements are employed for deflection [1]:

$$\begin{aligned}N_1&=\frac{1}{2A}\left(\alpha_1+y_{23}x+x_{32}y\right)\\ N_2&=\frac{1}{2A}\left(\alpha_2+y_{31}x+x_{13}y\right)\\ N_3&=\frac{1}{2A}\left(\alpha_3+y_{12}x+x_{21}y\right)\end{aligned} \tag{9}$$

where A is the element area and:

$$\begin{aligned}&\alpha_1=x_2y_3-x_3y_2,\ \alpha_2=x_3y_1-x_1y_3,\ \alpha_3=x_1y_2-x_2y_1\\ &x_{ij}=x_i-x_j,\quad y_{ij}=y_i-y_j,\quad i,j=1,2,3\end{aligned} \tag{10}$$

According to definition, the simplified mass matrix for deflection takes the following form:

$$\boldsymbol{m}_3^d = m\int_A \{N\}\langle N\rangle \mathrm{d}A = m\int_A \begin{bmatrix} N_1^2 & N_1N_2 & N_1N_3 \\ & N_2^2 & N_2N_3 \\ Sym. & & N_3^2 \end{bmatrix} \mathrm{d}A \tag{11}$$

where m is mass per unit area. In order to make analytical integration in Eq. (11) possible, the Cartesian coordinates are expressed with triangular ones [8], see Figure 2:

$$\begin{aligned} x &= x_1 + x_{21}\xi + x_{32}\xi\eta \\ y &= y_1 + y_{21}\xi + y_{32}\xi\eta \end{aligned} \tag{12}$$

where $0 \le \xi \le 1$ and $0 \le \eta \le 1$. In this case the shape functions (9) read:

$$N_1 = 1 - \xi,\ N_2 = \xi(1-\eta),\ N_3 = \xi\eta\ . \tag{13}$$

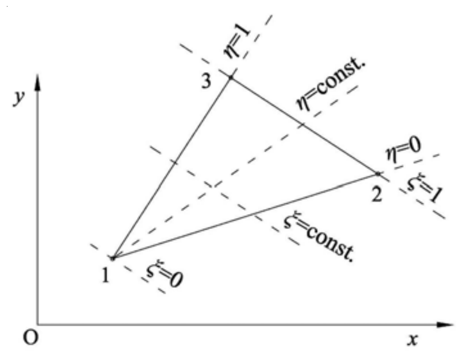

Fig. 2 Triangular coordinates

Furthermore:

$$dA = Jd\xi d\eta\ , \tag{14}$$

where:

$$J = \begin{vmatrix} \dfrac{\partial x}{\partial \xi} & \dfrac{\partial x}{\partial \eta} \\ \dfrac{\partial y}{\partial \xi} & \dfrac{\partial y}{\partial \eta} \end{vmatrix} = 2A\xi \tag{15}$$

is Jacobian. Hence, the simplified mass matrix for deflection, Eq. (11), yields:

$$\boldsymbol{m}_3^d = \frac{mA}{12}\begin{bmatrix} 2 & 1 & 1 \\ & 2 & 1 \\ Sym. & & 2 \end{bmatrix} \tag{16}$$

The mass moments of inertia of the complete finite element around x and y axis of the local coordinate system, with origin located at the centroid of the triangle, read:

$$I_x = m\int y^{*2} dA, \quad I_y = m\int x^{*2} dA\ . \tag{17}$$

For solving the above integrals it is convenient to use the area coordinates ξ_1, ξ_2 and ξ_3 [1]. By employing the transformation relationship:

$$\begin{Bmatrix} x^* \\ y^* \\ 1 \end{Bmatrix} = \begin{bmatrix} x_1^* & x_2^* & x_3^* \\ y_1^* & y_2^* & y_3^* \\ 1 & 1 & 1 \end{bmatrix} \begin{Bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{Bmatrix}, \tag{18}$$

the integrand functions in I_x and I_y, Eqs. (17) and (18), consist of terms which can be integrated by the general formula:

$$\int \xi_1^m \xi_2^n \xi_3^p \mathrm{d}A = 2A\frac{m!n!p!}{(m+n+p+2)!} \tag{19}$$

that leads to:

$$I_{rs} = \int_A x^{*r} y^{*s} dA = \frac{A}{12}\left(x_1^{*r} y_1^{*s} + x_2^{*r} y_2^{*s} + x_3^{*r} y_3^{*s}\right) \tag{20}$$

if $r+s=2$. Thus:

$$\begin{aligned} I_x &= \frac{mA}{12}\left(y_1^{*2} + y_2^{*2} + y_3^{*2}\right), \\ I_y &= \frac{mA}{12}\left(x_1^{*2} + x_2^{*2} + x_3^{*2}\right) \end{aligned} \tag{21}$$

The mass moments of inertia can be spread to the nodes in an approximate way, proportionally to the square of node coordinates. Finally, the complete simplified mass matrix is obtained by extending $\boldsymbol{m}_3^d$, Eq. (16), to all d.o.f.:

$$\boldsymbol{m}_3 = \frac{mA}{12}\begin{bmatrix} 2 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ & y_1^{*2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & & x_1^{*2} & 0 & 0 & 0 & 0 & 0 & 0 \\ & & & 2 & 0 & 0 & 1 & 0 & 0 \\ & & & & y_2^{*2} & 0 & 0 & 0 & 0 \\ & & & & & x_2^{*2} & 0 & 0 & 0 \\ & & & & & & 2 & 0 & 0 \\ & & & & & & & y_3^{*2} & 0 \\ Sym. & & & & & & & & x_3^{*2} \end{bmatrix} \tag{22}$$

The lumped mass matrix is diagonal and quite simple:

$$\boldsymbol{m}_2 = \frac{mA}{3}\begin{bmatrix} 1 & & & & & & & & \\ & 0 & & & & & & & \\ & & 0 & & & & & & \\ & & & 1 & & & & & \\ & & & & 0 & & & & \\ & & & & & 0 & & & \\ & & & & & & 1 & & \\ & & & & & & & 0 & \\ & & & & & & & & 0 \end{bmatrix} \tag{23}$$

4. RECTANGULAR PLATE ELEMENT

Instead of consistent shape functions for plate bending, those for membrane in-plane displacements can be used for deflection [1]:

$$N_i = \frac{1}{4}\left(1+\xi_i\xi\right)\left(1+\eta_i\eta\right),\ i=1,2,3,4\ , \tag{24}$$

where $\xi = \frac{x}{a}$ and $\eta = \frac{y}{b}$, Figure 3. For integrals of the shape functions in the mass matrix one finds:

$$m\int_A N_i N_j dA = \frac{mA}{16}\left(1+\frac{1}{3}\xi_i\xi_j\right)\left(1+\frac{1}{3}\eta_i\eta_j\right). \tag{25}$$

As a result, the simplified mass matrix for deflection yields:

$$\boldsymbol{m}_3^d = \frac{mA}{36}\begin{bmatrix} 4 & 2 & 1 & 2 \\ & 4 & 2 & 1 \\ & & 4 & 2 \\ Sym. & & & 4 \end{bmatrix}. \tag{26}$$

The mass moments of inertia of the complete element around x and y axes read:

$$I_x = \frac{mA}{3}b^2, \quad I_y = \frac{mA}{3}a^2 \tag{27}$$

and they are lumped to four nodes. Hence, the complete simplified mass matrix takes form:

$$\boldsymbol{m}_3 = \frac{mA}{36}\begin{bmatrix} 4 & 0 & 0 & 2 & 0 & 0 & 1 & 0 & 0 & 2 & 0 & 0 \\ & 3b^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & & 3a^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & & & 4 & 0 & 0 & 2 & 0 & 0 & 1 & 0 & 0 \\ & & & & 3b^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & & & & & 3a^2 & 0 & 0 & 0 & 0 & 0 & 0 \\ & & & & & & 4 & 0 & 0 & 2 & 0 & 0 \\ & & & & & & & 3b^2 & 0 & 0 & 0 & 0 \\ & & & & & & & & 3a^2 & 0 & 0 & 0 \\ & & & & & & & & & 4 & 0 & 0 \\ & & & & & & & & & & 3b^2 & 0 \\ Sym. & & & & & & & & & & & 3a^2 \end{bmatrix} \tag{28}$$

On the other side the lumped mass matrix reads:

$$\boldsymbol{m}_2 = \frac{mA}{4}\begin{bmatrix} 1 \\ & 0 \\ & & 0 \\ & & & 1 \\ & & & & 0 \\ & & & & & 0 \\ & & & & & & 1 \\ & & & & & & & 0 \\ & & & & & & & & 0 \\ & & & & & & & & & 1 \\ & & & & & & & & & & 0 \\ & & & & & & & & & & & 0 \end{bmatrix} \tag{29}$$

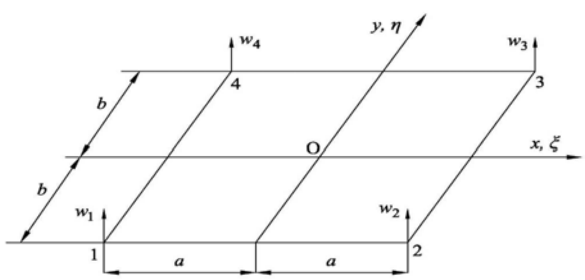

Fig. 3 Rectangular plate element

5. SHELL FINITE ELEMENTS

Simple shell element is constituted of membrane and plate elements. Consistent mass matrix for triangular membrane element is determined with linear shape functions (9) and reads:

$$\boldsymbol{m}_m = \frac{mA}{12}\begin{bmatrix} 2 & 0 & 1 & 0 & 1 & 0 \\ & 2 & 0 & 1 & 0 & 1 \\ & & 2 & 0 & 1 & 0 \\ & & & 2 & 0 & 1 \\ & & & & 2 & 0 \\ Sym. & & & & & 2 \end{bmatrix}. \tag{30}$$

The node displacement vector of triangular shell element is:

$$\boldsymbol{\delta} = \begin{Bmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \end{Bmatrix}, \qquad \boldsymbol{\delta}_i = \begin{Bmatrix} u_i \\ v_i \\ w_i \\ \varphi_{xi} \\ \varphi_{yi} \\ \varphi_{zi} \end{Bmatrix}, \tag{31}$$

where φ_{zi} is so called dummy d.o.f. without stiffness. However, it has mass moment of inertia $\frac{mA}{12} r_i^{*2}$, $r_i^{*2} = x_i^{*2} + y_i^{*2}$. The shell mass matrix is comprised of the rearranged terms of matrices $\boldsymbol{m}_3$ and $\boldsymbol{m}_m$, Eqs. (22) and (30), according to the displacement vector $\boldsymbol{\delta}$:

$$\boldsymbol{M} = \frac{mA}{12}\left[\begin{array}{cccccccccccccccccc}
2 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
 & 2 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
 & & 2 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
 & & & y_1^{*2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & & & & x_1^{*2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & & & & & r_1^{*2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & & & & & & 2 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
 & & & & & & & 2 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
 & & & & & & & & 2 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
 & & & & & & & & & y_2^{*2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & & & & & & & & & & x_2^{*2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & & & & & & & & & & & r_2^{*2} & 0 & 0 & 0 & 0 & 0 & 0 \\
 & & & & & & & & & & & & 2 & 0 & 0 & 0 & 0 & 0 \\
 & & & & & & & & & & & & & 2 & 0 & 0 & 0 & 0 \\
 & & & & & & & & & & & & & & 2 & 0 & 0 & 0 \\
 & & & & & & & & & & & & & & & y_3^{*2} & 0 & 0 \\
 & & & & & & & & & & & & & & & & x_3^{*2} & 0 \\
Sym. & & & & & & & & & & & & & & & & & r_3^{*2}
\end{array}\right]. \tag{32}$$

In similar way, mass matrix of the rectangular shell element can be constructed by comprising the membrane mass matrix for displacements u_i and v_i [6]:

$$\boldsymbol{m}_m = \frac{mA}{36}\begin{bmatrix} 4 & 0 & 2 & 0 & 1 & 0 & 2 & 0 \\ & 4 & 0 & 2 & 0 & 1 & 0 & 2 \\ & & 4 & 0 & 2 & 0 & 1 & 0 \\ & & & 4 & 0 & 2 & 0 & 1 \\ & & & & 4 & 0 & 2 & 0 \\ & & & & & 4 & 0 & 2 \\ & & & & & & 4 & 0 \\ \text{Sym.} & & & & & & & 4 \end{bmatrix}, \tag{33}$$

and the plate mass matrix $\boldsymbol{m}_3$, Eq. (28), for displacements w_i, φ_{xi} and φ_{yi}, extended to φ_{zi}. In that way the complete simplified mass matrix yields:

$$\boldsymbol{M} = \frac{mA}{36}\left[\begin{array}{cccccccccccccccccccccccc}
4&0&0&0&0&0&2&0&0&0&0&0&1&0&0&0&0&0&2&0&0&0&0&0 \\
&4&0&0&0&0&0&2&0&0&0&0&0&1&0&0&0&0&0&2&0&0&0&0 \\
&&4&0&0&0&0&0&2&0&0&0&0&0&1&0&0&0&0&0&2&0&0&0 \\
&&&3b^2&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
&&&&3a^2&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
&&&&&3(a+b)^2&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
&&&&&&4&0&0&0&0&0&2&0&0&0&0&0&1&0&0&0&0&0 \\
&&&&&&&4&0&0&0&0&0&2&0&0&0&0&0&1&0&0&0&0 \\
&&&&&&&&4&0&0&0&0&0&2&0&0&0&0&0&1&0&0&0 \\
&&&&&&&&&3b^2&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
&&&&&&&&&&3a^2&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
&&&&&&&&&&&3(a+b)^2&0&0&0&0&0&0&0&0&0&0&0&0 \\
&&&&&&&&&&&&4&0&0&0&0&0&2&0&0&0&0&0 \\
&&&&&&&&&&&&&4&0&0&0&0&0&2&0&0&0&0 \\
&&&&&&&&&&&&&&4&0&0&0&0&0&2&0&0&0 \\
&&&&&&&&&&&&&&&3b^2&0&0&0&0&0&0&0&0 \\
&&&&&&&&&&&&&&&&3a^2&0&0&0&0&0&0&0 \\
&&&&&&&&&&&&&&&&&3(a+b)^2&0&0&0&0&0&0 \\
&&&&&&&&&&&&&&&&&&4&0&0&0&0&0 \\
&&&&&\text{Sym.}&&&&&&&&&&&&&&4&0&0&0&0 \\
&&&&&&&&&&&&&&&&&&&&4&0&0&0 \\
&&&&&&&&&&&&&&&&&&&&&3b^2&0&0 \\
&&&&&&&&&&&&&&&&&&&&&&3a^2&0 \\
&&&&&&&&&&&&&&&&&&&&&&&3(a+b)^2
\end{array}\right] \tag{34}$$

6. ILLUSTRATIVE EXAMPLES

6.1. Beam vibrations

The application of different mass modelling and the resulting accuracy has been illustrated for the case of natural beam vibrations with the following parameters:

Length	$L = 40\ m$
Breadth	$B = 2\ m$
Height	$H = 1\ m$
Cross-section area	$A = 2\ m^2$
Moment of inertia of cross-section	$I = 0.1667\ m^4$
Mass	$M = 6.28\times10^5\ kg$
Young's modulus	$E = 2.1\times10^{11}\ N/m^2$

The analytical values of natural frequencies for a free beam are determined by the following formula:

$$\omega_n = \frac{(\beta_n l/2)^2}{(l/2)^2}\sqrt{\frac{EI}{m}}, \tag{35}$$

where the roots for the symmetric and antisymmetric elastic modes with odd and even indeces respectively, read $\beta_1 l/2 = 2.365$, $\beta_2 l/2 = 3.925$, $\beta_3 l/2 = 5.497$,

$\beta_4 l/2 = 7.068$. They are obtained from the corresponding frequency equations:

$$cosh\left(\frac{\beta_n l}{2}\right)sin\left(\frac{\beta_n l}{2}\right) \pm sinh\left(\frac{\beta_n l}{2}\right)cos\left(\frac{\beta_n l}{2}\right) = 0 \quad (36)$$

The problem has been analyzed in the Cartesian coordinate system with origin in the middle of beam.

Natural modes of simply supported beam are sinusoidal and natural frequencies read:

$$\omega_n = \left(\frac{n\pi}{l}\right)^2 \sqrt[4]{\frac{EI}{m}}, \qquad n = 1,2... \qquad (37)$$

It is interesting that natural frequencies for the clamped beam are the same as those for the free one, Eq. (34), but the mode shapes are different.

Numerical calculation of flexural beam vibrations has been performed for different boundary conditions, i.e. for the cases of free, simply supported and clamped beams. The beam has been divided into 8 and 16 finite elements in order to check the convergence of the results. The obtained natural frequencies for the consistent, lumped and simplified mass matrices are listed in Tables 1 to 6 and compared with analytical solutions. As expected, the consistent mass matrix gives the best results in all considered cases. The lumped mass matrix induces very large discrepancies for the free beam, while for the simply supported and clamped beam the results are, surprisingly, as good as those obtained with the consistent mass matrix. Discrepancies due to the application of the simplified mass matrix are medium and stable in all cases.

Table 1 Natural frequencies of free beam ω_i [Hz], 8 finite elements

Mode no.	Analytical	Consistent mass m_1	Lumped mass m_2	Simplified mass m_3	Discrepancy δ (%)		
					δ_1	δ_2	δ_3
1	3.323	3.323	3.171	3.267	0.00	-4.79	-1.71
2	9.151	9.165	8.481	8.996	0.15	-7.90	-1.72
3	17.951	17.994	16.180	17.615	0.24	-10.95	-1.91
4	29.678	29.841	26.079	28.779	0.55	-13.80	-3.12

Table 2 Natural frequencies of free beam ω_i [Hz], 16 finite elements

Mode no.	Analytical	Consistent mass m_1	Lumped mass m_2	Simplified mass m_3	Discrepancy δ (%)		
					δ_1	δ_2	δ_3
1	3.323	3.323	3.283	3.309	0.00	-1.22	-0.42
2	9.151	9.160	8.977	9.130	0.10	-1.71	-0.23
3	17.951	17.959	17.459	17.957	0.04	-2.82	0.03
4	29.678	29.695	28.634	29.829	0.06	-3.65	0.51

Table 3 Natural frequencies of simply supported beam ω_i [Hz], 8 finite elements

Mode no.	Analytical	Consistent mass m_1	Lumped mass m_2	Simplified mass m_3	Discrepancy δ (%)		
					δ_1	δ_2	δ_3
1	1.464	1.466	1.466	1.475	0.136	0.136	0.75
2	5.857	5.865	5.862	6.002	0.136	0.085	2.42
3	13.179	13.209	13.168	13.811	0.227	-0.084	4.58
4	23.430	23.546	23.283	24.981	0.493	-0.631	6.21

Table 4 Natural frequencies of simply supported beam ω_i [Hz], 16 finite elements

Mode no.	Analytical	Consistent mass m_1	Lumped mass m_2	Simplified mass m_3	Discrepancy δ (%)		
					δ_1	δ_2	δ_3
1	1.464	1.466	1.466	1.468	0.136	0.136	0.27
2	5.857	5.863	5.863	5.900	0.102	0.102	0.73
3	13.179	13.194	13.191	13.375	0.114	0.091	1.47
4	23.430	23.459	23.446	24.009	0.124	0.068	2.41

Table 5 Natural frequencies of clamped beam ω_i [Hz], 8 finite elements

Mode no.	Analytical	Consistent mass m_1	Lumped mass m_2	Simplified mass m_3	Discrepancy δ (%)		
					δ_1	δ_2	δ_3
1	3.323	3.323	3.323	3.347	0.00	0.00	0.72
2	9.151	9.165	9.143	9.383	0.15	-0.09	2.47
3	17.951	17.999	17.863	18.736	0.27	-0.49	4.19
4	29.678	29.868	29.142	31.295	0.64	-1.84	5.17

Table 6 Natural frequencies of clamped beam ω_i [Hz], 16 finite elements

Mode no.	Analytical	Consistent mass m_1	Lumped mass m_2	Simplified mass m_3	Discrepancy δ (%)		
					δ_1	δ_2	δ_3
1	3.323	3.323	3.323	3.329	0.00	0.00	0.18
2	9.151	9.160	9.159	9.225	0.10	0.09	0.80
3	17.951	17.959	17.953	18.222	0.04	0.01	1.49
4	29.678	29.695	29.666	30.407	0.06	-0.04	2.40

6.2. Plate vibrations

Let us consider vibrations of a simply supported rectangular plate, since this case of boundary conditions usually appears in practise and there is a relatively simple analytical solution for natural frequencies [9]:

$$\omega_{kl} = \pi^2 \sqrt{\frac{D}{m}} \left[\left(\frac{k}{a} \right)^2 + \left(\frac{l}{b} \right)^2 \right], \tag{38}$$

where:

$$D = \frac{Et^3}{12\left(1-\nu^2\right)} \tag{39}$$

is the plate stiffness, a and b are the plate length and width respectively, m is the mass per unit area, and t is the plate thickness.

Natural frequencies for square plate of the following parameters have been analyzed: $a = b = 2$ m, $t = 0.01$ m, and $m = 78.5$ kg/m^2. The plate has been modelled with $8\times8=64$ rectangular elements with four corner nodes with compatible deflections [9]. The natural frequencies determined with consistent, lumped and simplified mass matrices are listed in Table 7. The eigenvalue problem has been solved using the standard routine for eigenvalues in MATLAB software [10].

It is obvious that all numerical results are underestimated. Frequencies determined with the simplified mass matrix are closer to the analytical solution than those determined using the lumped mass matrix. As a result of the applied numerical method, in all numerical solutions there has been a small difference between natural frequencies ω_{21} and ω_{12}.

In commercial computer programs, more sophisticated finite elements containing consistent, lumped and coupled mass matrices are used. For illustration, natural frequencies of the considered square plate with the same mesh density are determined with SESAM [11] and NASTRAN [12]. The obtained results are presented in Tables 8 and 9, respectively. Discrepancies for SESAM solution using the consistent mass matrix are low and uniformly increased with modes. That is not the case for the lumped mass matrix. NASTRAN results are overestimated and underestimated for the coupled and lumped mass matrices, respectively.

Table 7 Natural frequencies of simply supported square plate, ω_{kl} [Hz], 64 elements

Mode no.	Analytical	Consistent mass m_1	Lumped mass m_2	Simplified mass m_3	Discrepancy δ (%)		
					δ_1	δ_2	δ_3
1,1	12.287	11.994	10.896	11.366	-2.38	-11.32	-7.49
1,2	30.717	28.913	27.870	29.937	-5.87	-9.26	-2.54
2,1	30.717	29.727	28.920	31.160	-3.22	-5.85	1.44
2,2	49.147	45.780	42.436	44.715	-6.85	-13.65	-9.02

Table 8 Natural frequencies of simply supported square plate, SESAM, ω_{kl} [Hz], 64 elements

Mode no.	Consistent mass	Lumped mass	Discrepancy δ (%)	
			δ_1	δ_2
1,1	12.122	12.155	-1.34	-1.07
1,2	30.088	28.348	-2.04	-7.71
2,1	30.088	31.099	-2.04	1.24
2,2	46.705	42.929	-4.97	-12.65

Table 9 Natural frequencies of simply supported square plate, NASTRAN, ω_{kl} [Hz], 64 elements

Mode no.	Consistent mass	Lumped mass	Discrepancy δ (%)	
			δ_1	δ_2
1,1	12.374	12.060	0.71	2.55
1,2	31.970	29.982	4.08	-2.39
2,1	31.970	29.982	4.08	-2.39
2,2	51.377	46.361	4.54	-5.67

7. CONCLUSION

In order to simplify the mass matrix formulation and reduce computing time, simplified mass matrices for beam and rectangular finite elements, applied in the modelling of thin-walled structures, have been derived. Shape functions for bar and membrane have been used for the deflection of beam and plate so that the distributed mass is reduced to the deflectional degree of freedom. In this way, the mass matrix for shell elements has been obtained in quite a simple form with the same type of elements for all translational degrees of freedom. The numerical examples of beam and square plate vibrations have shown satisfactory accuracy.

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MODELIRANJE MASE METODOM KONAČNIH ELEMENATA U ANALIZI VIBRACIJA TANKOSTIJENIH KONSTRUKCIJA

SAŽETAK

U ovome se radu razmatra modeliranje mase pomoću metode konačnih elemenata u analizi vibracija tankostijenih konstrukcija, odnosno, uspoređuju se prednosti i nedostaci korištenja konzistentne i koncentrirane matrice mase u odnosu na novopredloženu, pojednostavljenu matricu mase. Matrice su određene za dvočvorni gredni element, tročvorni trokutasti pločasti element i četveročvorni pravokutni pločasti element. Pojednostavljena matrica mase dobivena je na temelju oblikovnih funkcija za štap i membranu umjesto na temelju oblikovnih funkcija za savijanje. Stoga su raspodijeljene mase razmatrane samo za slučaj stupnjeva slobode pomaka. Točnost i prednosti takve vrste formulacije matrice mase su pokazane na slučajevima vibracije greda i kvadratnih ploča.

Ključne riječi: *tankostijene konstrukcije, vibracije, modeliranje mase, metoda konačnih elemenata.*