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Potential theory of in-plane vibrations of rectangular and circular plates

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SUMMARY

Basic equations of in-plane plate vibrations are specified. Governing differential equations of motion are solved analytically introducing displacement potentials and employing the method of separation of variables. Frequency equations for a rectangular plate with two opposite simply supported edges and two remaining edges clamped, free and combined clamped-free are derived. Three types of solutions are possible, depending on values of involved function arguments. For the purpose of vibration analysis of a circular plate, differential equations of motion of a rectangular plate are transformed. Concerning a circular plate, circumferential variation of displacement potentials is assumed in the form of trigonometric series, while variation in radial direction is obtained by solving Bessel's differential equations. Frequency equations for clamped and free plate edges are given, and the same procedure is applied for the annular plate. Application of the developed theory is illustrated in cases of a rectangular plate simply supported at two opposite edges, clamped, free and combined clamped-free at two remained edges. Vibrations of clamped and free circular plates are also analyzed as well as of a clamped-free annular plate. In all the considered cases, analytical values are compared with FEM results.

***Key words**: rectangular plate, circular plate, in-plane vibration, potential theory, analytical solution, FEM.*

1. INTRODUCTION

Flexural vibrations of thin and thick rectangular and circular plates are of great practical importance since their transverse motion can be easily excited by external sources. Hence, there is an extensive body of literature related to free flexural vibrations of these structural elements. On the other side, relevant literature on in-plane plate vibrations is rather scarce since natural frequencies take much higher values than the ordinary excitation frequencies so that resonant response is realized quite rarely. However, in-plane vibrations can be, for instance, excited in a ship's hull plating, reducing thus the stability of plates exposed to in-plane load due to bending of the ship's hull in waves. Another practical example is in-plane vibration of a rotating disk in mechanical systems, which can be caused by imperfections in shaft alignment, etc.

One of the first works on this subject was presented by Lord Rayleigh [1] for cases of simply supported plates. A valuable survey of the relevant literature until 1996 is given in Ref. [2], where in-plane natural frequencies for rectangular plates are calculated using the Rayleigh–Ritz method. The superposition method, in order to obtain an analytical solution for rectangular plate vibrations, for free boundaries, clamped boundaries and elastically supported edges, is used in Refs. [3] and [4]. Problem regarding elastically restrained edges is also analyzed in Ref. [5] by employing double Fourier series and four complementary functions. The exact solution for in-plane vibrations of a rectangular plate with two simply supported opposite edges and two remaining edges being both clamped or both free is presented in Ref. [6].

Recent research of in-plane vibrations of rectangular plates, where method of separation of variables is applied, is presented in Refs. [7] and [8].

Using such an approach the exact solutions for simply supported boundaries at two opposite edges have been achieved. It has been shown that two types of simply supported boundaries are possible. In-plane plate vibrations deal with two orthogonal displacements, two normal stresses and two equal shear stresses. At boundaries of a rectangular plate 8 conditions can be satisfied, i.e. two per each edge. However, the solution in Ref. [7] is expressed with two x and two y functions, each with four homogenous solutions, resulting in total of 16 integration constants. In order to enable satisfaction of the boundary conditions, their number is further reduced to 8.

A short review of the in-plane plate vibrations is presented in Ref. [9], starting from Love's book [10] as an initial approach to the current problems. In-plane vibrations of circular and annular plates with free boundaries are analyzed in Ref. [11]. Natural frequencies for the in-plane vibrations of annular plates with four combinations of free and clamped boundary conditions at the inner and outer edges are calculated in Ref. [12] employing a transfer matrix procedure. Natural in-plane vibrations of circular plates assuming mode shapes in circumferential and radial direction are analyzed in Ref. [13] by trigonometric and Bessel functions, respectively. Since no paper dealing with an exact frequency equation for in-plane plate vibration of a clamped plate had been published at the time, that problem has recently been solved in Ref. [9], with the introduction of displacement potential functions.

Based on the aforementioned circumstances, potential function approach is generalized in this paper and used for both rectangular and circular plate vibration analysis. A method of the separation of variables is applied and a general solution of governing differential equations of motion with 8 integration constants, as number of possible boundary conditions for a rectangular plate, is directly obtained. Analytical solutions for plates with two simply supported opposite edges and any combination of boundary conditions at the remaining two edges are proposed. The same approach is used for in-plane vibrations of circular plates. Differential equations of motion derived for rectangular plates are transformed from orthogonal to polar coordinate system. Potential variation in circumferential direction is assumed by trigonometric functions and variation in radial direction is obtained in the form of the Bessel functions. The solution enables determination of frequency equation for any boundary value problem specified for an inner and outer edge. The application of the proposed procedure is illustrated for clamped, free and combined boundary conditions for rectangular and circular plates. The results are validated by a FEM analysis.

2. BASIC EQUATIONS OF IN-PLANE VIBRATIONS

A rectangular plate in the Cartesian coordinate system is considered with an aspect ratio a/b and a thickness h, see Figure 1. Longitudinal and transversal displacements are harmonic, i.e. $u(x,y,t) = U(x,y)\, \sin\omega t$ and $v(x,y,t) = V(x,y)\, \sin\omega t$, where ω is a natural frequency. Amplitudes of membrane stresses [14] read:

$$\begin{aligned}
\sigma_x &= \frac{E}{1-\nu^2}\left(\frac{\partial U}{\partial x} + \nu \frac{\partial V}{\partial y}\right), \\
\sigma_y &= \frac{E}{1-\nu^2}\left(\frac{\partial V}{\partial y} + \nu \frac{\partial U}{\partial x}\right), \\
\sigma_{xy} &= G\left(\frac{\partial U}{\partial y} + \frac{\partial V}{\partial x}\right),
\end{aligned} \tag{1}$$

where E is Young's modulus, G shear modulus and ν Poisson's ratio. Based on equilibrium of internal and inertia forces in x and y direction, i.e.:

$$\begin{aligned}
&\frac{\partial \sigma_x}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + \rho\omega^2 U = 0, \\
&\frac{\partial \sigma_y}{\partial y} + \frac{\partial \sigma_{xy}}{\partial x} + \rho\omega^2 V = 0,
\end{aligned} \tag{2}$$

system of two partial differential equations, [7], is obtained:

$$\begin{aligned}
&\frac{\partial^2 U}{\partial x^2} + \frac{1}{2}(1-\nu)\frac{\partial^2 U}{\partial y^2} + \frac{1}{2}(1+\nu)\frac{\partial^2 V}{\partial x \partial y} + \left(1-\nu^2\right)\frac{\rho}{E}\omega^2 U = 0, \\
&\frac{\partial^2 V}{\partial y^2} + \frac{1}{2}(1-\nu)\frac{\partial^2 V}{\partial x^2} + \frac{1}{2}(1+\nu)\frac{\partial^2 U}{\partial x \partial y} + \left(1-\nu^2\right)\frac{\rho}{E}\omega^2 V = 0.
\end{aligned} \tag{3}$$

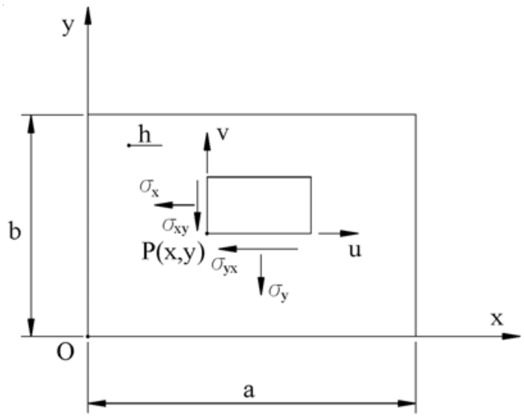

Fig. 1 Rectangular plate for in-plane vibrations

3. POTENTIAL THEORY OF PLATE VIBRATIONS

Direct solution to Eqs. (3) is rather complex and results in twice the number of integration constants, compared to available number of boundary conditions [7]. Therefore, displacement potential functions $\Phi(x,y)$ and $\Psi(x,y)$ are introduced according to Helmholtz, [15] and [16]:

$$U = \frac{\partial \Phi}{\partial x} + \frac{\partial \Psi}{\partial y}, \quad V = \frac{\partial \Phi}{\partial y} - \frac{\partial \Psi}{\partial x}. \tag{4}$$

Substitution of Eq. (4) into Eq. (3) yields:

$$\begin{aligned} &E^*\left(\frac{\partial^3 \Phi}{\partial x^3} + \frac{\partial^3 \Psi}{\partial x \partial y^2}\right) + \rho\omega^2 \frac{\partial \Phi}{\partial x} + \\ &+G\left(\frac{\partial^3 \Psi}{\partial x^2 \partial y} + \frac{\partial^3 \Psi}{\partial y^3}\right) + \rho\omega^2 \frac{\partial \Psi}{\partial y} = 0, \\ &E^*\left(\frac{\partial^3 \Phi}{\partial x^2 \partial y} + \frac{\partial^3 \Psi}{\partial y^3}\right) + \rho\omega^2 \frac{\partial \Phi}{\partial y} - \\ &-G\left(\frac{\partial^3 \Psi}{\partial x^3} + \frac{\partial^3 \Psi}{\partial x \partial y^2}\right) - \rho\omega^2 \frac{\partial \Psi}{\partial x} = 0, \end{aligned} \tag{5}$$

where $E^* = E/\left(1-\nu^2\right)$. If Eqs. (5) are derivated once per x and y respectively, summed up, and derivated once again per y and x, respectively, and subtracted, one arrives at:

$$E^*\Delta\Phi + \rho\omega^2\Phi = 0, \quad G\Delta\Psi + \rho\omega^2\Psi = 0. \tag{6}$$

In that way coupled Eqs. (3) are decomposed into two independent equations.

The variable separation method can be used to solve Eqs. (6), i.e. $\Phi(x,y) = X(x) \cdot Y(y)$ and $\Psi(x,y) = Z(x) \cdot W(y)$. Insertion of potential functions into Eq. (6) yields:

$$\begin{aligned} &\frac{1}{X}\frac{d^2X}{dx^2} + \frac{1}{Y}\frac{d^2Y}{dx^2} + \frac{\rho\omega^2}{E^*} = 0, \\ &\frac{1}{Z}\frac{d^2Z}{dx^2} + \frac{1}{W}\frac{d^2W}{dy^2} + \frac{\rho\omega^2}{G} = 0. \end{aligned} \tag{7}$$

Furthermore, the solutions of the unknown functions can be assumed in an exponential form, i.e. $X=Ae^{i\alpha x}$, $Y=Ce^{i\gamma y}$, $Z=Be^{i\beta x}$ and $W=De^{i\vartheta y}$. That leads to characteristic equations:

$$\alpha^2 + \gamma^2 - \omega^2\frac{\rho}{E^*} = 0, \quad \beta^2 + \vartheta^2 - \omega^2\frac{\rho}{G} = 0 \tag{8}$$

from which one can write:

$$\begin{aligned} &\alpha_{1,2} = \pm\sqrt{\omega^2\frac{\rho}{E^*} - \gamma^2}, \quad \gamma_{1,2} = \pm\sqrt{\omega^2\frac{\rho}{E^*} - \alpha^2}, \\ &\beta_{1,2} = \pm\sqrt{\omega^2\frac{\rho}{G} - \vartheta^2}, \quad \vartheta_{1,2} = \pm\sqrt{\omega^2\frac{\rho}{G} - \beta^2}. \end{aligned} \tag{9}$$

The assumed functions take the following form:

$$\begin{aligned} X(x) &= A_1\cos\alpha x + A_2\sin\alpha x, \\ Y(y) &= C_1\cos\gamma y + C_2\sin\gamma y, \\ Z(x) &= B_1\cos\beta x + B_2\sin\beta x, \\ W(y) &= D_1\cos\vartheta y + D_2\sin\vartheta y. \end{aligned} \tag{10}$$

Now, displacements, Eq. (4), read:

$$U = \frac{dX}{dx}Y + Z\frac{dW}{dy}, \quad V = X\frac{dY}{dy} - \frac{dZ}{dx}W. \tag{11}$$

Stresses, Eq. (1), can also be expressed with separated functions:

$$\begin{aligned} \sigma_x &= E^*\left(\frac{d^2X}{dx^2}Y + \nu X\frac{d^2Y}{dy^2} + (1-\nu)\frac{dZ}{dx}\frac{dW}{dy}\right), \\ \sigma_y &= E^*\left(X\frac{d^2Y}{dy^2} + \nu\frac{d^2X}{dx^2}Y - (1-\nu)\frac{dZ}{dx}\frac{dW}{dy}\right), \\ \sigma_{xy} &= G\left(2\frac{dX}{dx}\frac{dY}{dy} + Z\frac{d^2W}{dy^2} - \frac{d^2Z}{dx^2}W\right). \end{aligned} \tag{12}$$

A general solution of differential equations of motion, Eq. (10), includes 8 integration constants determined by satisfying 8 boundary conditions, i.e. two at each plate edge. Two displacements and three stress components are on disposal, so that a large number of boundary conditions can be specified. Each plate edge can be simply supported, clamped or free, with two possibilities in the case of a simply supported edge. A list of complete boundary conditions is shown in Table 1, [7].

Table 1. Boundary conditions for in-plane vibrations of a rectangular plate

	$x = 0$ or $x = a$	$y = 0$ or $y = b$
Simply supported, SS1	$V = 0,\ \sigma_x = 0$	$U = 0,\ \sigma_y = 0$
Simply supported, SS2	$U = 0,\ \sigma_{xy} = 0$	$V = 0,\ \sigma_{xy} = 0$
Clamped, C	$U = 0,\ V = 0$	$U = 0,\ V = 0$
Free, F	$\sigma_x = 0,\ \sigma_{xy} = 0$	$\sigma_y = 0,\ \sigma_{xy} = 0$

4. A RECTANGULAR PLATE WITH SIMPLY SUPPORTED EDGES y=0 AND y=b, SS

Let us consider a plate with simply supported longitudinal edges SS2, i.e. $V = 0$ and $\sigma_{xy} = 0$, as a more realistic case. That conditions are satisfied if separated functions are harmonic, i.e. $Y(y) = \cos\gamma y$ and $W(y) = \sin\vartheta y$, where $\gamma = \vartheta = n\pi/b$. Vibration parameters, Eq. (9), read:

$$\alpha = \sqrt{\omega^2 \frac{\rho}{E^*} - \gamma^2}, \quad \beta = \sqrt{\omega^2 \frac{\rho}{G} - \gamma^2}, \tag{13}$$

and they may be real or imaginary. Hence, functions $X(x)$ and $Z(x)$, Eqs. (10), can take one of the following types:

$$\begin{aligned}
\text{Type 1:} \quad & X(x) = A_1 \cos\alpha x + A_2 \sin\alpha x, \\
& Z(x) = B_1 \cos\beta x + B_2 \sin\beta x, \\
\text{Type 2:} \quad & X(x) = A_1 \cosh\bar{\alpha} x + A_2 \sinh\bar{\alpha} x, \\
& Z(x) = B_1 \cos\beta x + B_2 \sin\beta x, \\
\text{Type 3:} \quad & X(x) = A_1 \cosh\bar{\alpha} x + A_2 \sinh\bar{\alpha} x, \\
& Z(x) = B_1 \cosh\bar{\beta} x + B_2 \sinh\bar{\beta} x,
\end{aligned} \tag{14}$$

where:

$$\bar{\alpha} = \sqrt{\gamma^2 - \omega^2 \frac{\rho}{E^*}}, \quad \bar{\beta} = \sqrt{\gamma^2 - \omega^2 \frac{\rho}{G}}. \tag{15}$$

Depending on the type of solution displacements and stresses, Eqs. (11) and (12), take trigonometric, combined hyperbolic and trigonometric, or complete hyperbolic form:

Type 1:

$$\begin{aligned}
U &= \left[\alpha\left(-A_1 \sin\alpha x + A_2 \cos\alpha x\right) + \gamma\left(B_1 \cos\beta x + B_2 \sin\beta x\right)\right]\cos\gamma y, \\
V &= \left[-\gamma\left(A_1 \cos\alpha x + A_2 \sin\alpha x\right) + \beta\left(B_1 \sin\beta x - B_2 \cos\beta x\right)\right]\sin\gamma y, \\
\sigma_x &= E^*\left[-\left(\alpha^2 + \nu\gamma^2\right)\left(A_1 \cos\alpha x + A_2 \sin\alpha x\right) + (1-\nu)\beta\gamma\left(-B_1 \sin\beta x + B_2 \cos\beta x\right)\right]\cos\gamma y, \\
\sigma_y &= E^*\left[-\left(\nu\alpha^2 + \gamma^2\right)\left(A_1 \cos\alpha x + A_2 \sin\alpha x\right) + (1-\nu)\beta\gamma\left(B_1 \sin\beta x - B_2 \cos\beta x\right)\right]\cos\gamma y, \\
\sigma_{xy} &= G\left[2\alpha\gamma\left(A_1 \sin\alpha x - A_2 \cos\alpha x\right) + \left(\beta^2 - \gamma^2\right)\left(B_1 \cos\beta x + B_2 \sin\beta x\right)\right]\sin\gamma y,
\end{aligned} \tag{16}$$

Type 2:

$$\begin{aligned}
U &= \left[\bar{\alpha}\left(-A_1 \sinh\bar{\alpha} x + A_2 \cosh\bar{\alpha} x\right) + \gamma\left(B_1 \cos\beta x + B_2 \sin\beta x\right)\right]\cos\gamma y, \\
V &= \left[-\gamma\left(A_1 \cosh\bar{\alpha} x + A_2 \sinh\bar{\alpha} x\right) + \beta\left(B_1 \sin\beta x - B_2 \cos\beta x\right)\right]\sin\gamma y, \\
\sigma_x &= E^*\left[\left(\bar{\alpha}^2 - \nu\gamma^2\right)\left(A_1 \cosh\bar{\alpha} x + A_2 \sinh\bar{\alpha} x\right) + (1-\nu)\beta\gamma\left(-B_1 \sin\beta x + B_2 \cos\beta x\right)\right]\cos\gamma y, \\
\sigma_y &= E^*\left[\left(\nu\bar{\alpha}^2 - \gamma^2\right)\left(A_1 \cosh\bar{\alpha} x + A_2 \sinh\bar{\alpha} x\right) + (1-\nu)\beta\gamma\left(B_1 \sin\beta x + B_2 \cos\beta x\right)\right]\cos\gamma y, \\
\sigma_{xy} &= G\left[-2\bar{\alpha}\gamma\left(A_1 \sinh\bar{\alpha} x + A_2 \cosh\bar{\alpha} x\right) + \left(\beta^2 - \gamma^2\right)\left(B_1 \cos\beta x + B_2 \sin\beta x\right)\right]\sin\gamma y,
\end{aligned} \tag{17}$$

Type 3:

$$\begin{aligned}
U &= \left[\bar{\alpha}\left(A_1 \sinh\bar{\alpha} x + A_2 \cosh\bar{\alpha} x\right) + \gamma\left(B_1 \cosh\bar{\beta} x + B_2 \sinh\bar{\beta} x\right)\right]\cos\gamma y, \\
V &= \left[-\gamma\left(A_1 \cosh\bar{\alpha} x + A_2 \sinh\bar{\alpha} x\right) - \bar{\beta}\left(B_1 \sinh\bar{\beta} x + B_2 \cosh\bar{\beta} x\right)\right]\sin\gamma y, \\
\sigma_x &= E^*\left[\left(\bar{\alpha}^2 - \nu\gamma^2\right)\left(A_1 \cosh\bar{\alpha} x + A_2 \sinh\bar{\alpha} x\right) + (1-\nu)\bar{\beta}\gamma\left(B_1 \sinh\bar{\beta} x + B_2 \cosh\bar{\beta} x\right)\right]\cos\gamma y, \\
\sigma_y &= E^*\left[\left(\nu\bar{\alpha}^2 - \gamma^2\right)\left(A_1 \cosh\bar{\alpha} x + A_2 \sinh\bar{\alpha} x\right) - (1-\nu)\bar{\beta}\gamma\left(B_1 \sinh\bar{\beta} x + B_2 \cosh\bar{\beta} x\right)\right]\cos\gamma y, \\
\sigma_{xy} &= -G\left[2\bar{\alpha}\gamma\left(A_1 \sinh\bar{\alpha} x + A_2 \cosh\bar{\alpha} x\right) + \left(\bar{\beta}^2 + \gamma^2\right)\left(B_1 \cosh\bar{\beta} x + B_2 \sinh\bar{\beta} x\right)\right]\sin\gamma y.
\end{aligned} \tag{18}$$

Among the large number of combinations of boundary conditions, only three typical cases, i.e. $C-C$, $F-F$ and $C-F$ at edges $x=0$ and $x=a$ are analyzed. Employing expressions for displacements and stresses for three types of solutions, Eqs. (16), (17) and (18), the corresponding frequency equation specified in Table 2 is obtained, where:

$$
\begin{aligned}
&A=\alpha^2+\nu\gamma^2, \quad B=\beta^2-\gamma^2, \quad C=2\left(1-\nu\right)\alpha\beta\gamma^2,\\
&\bar{A}=\bar{\alpha}^2-\nu\gamma^2, \quad \bar{B}=\bar{\beta}^2+\gamma^2,\\
&C'=2\left(1-\nu\right)\bar{\alpha}\beta\gamma^2, \quad C''=2\left(1-\nu\right)\bar{\alpha}\bar{\beta}\gamma^2.
\end{aligned} \tag{19}
$$

Table 2. Frequency equations for a rectangular plate with simply supported edges, y=0 and y=b

C–SS2–C–SS2	$0\le x\le a$ Eq. (a)	$-a/2\le x\le a/2$ Eq. (b)	Eq. (c)
Type 1	$\frac{1-\cos\alpha a\cos\beta a}{\sin\alpha a\sin\beta a}=-\frac{1}{2}\left(\frac{\alpha\beta}{\gamma^2}+\frac{\gamma^2}{\alpha\beta}\right)$	$\frac{\tan\beta\frac{a}{2}}{\tan\alpha\frac{a}{2}}=-\frac{\alpha\beta}{\gamma^2}$,	$\frac{\tan\beta\frac{a}{2}}{\tan\alpha\frac{a}{2}}=-\frac{\gamma^2}{\alpha\beta}$
Type 2	$\frac{1-\cosh\bar{\alpha}a\cos\beta a}{\sinh\bar{\alpha}a\sin\beta a}=\frac{1}{2}\left(\frac{\bar{\alpha}\beta}{\gamma^2}-\frac{\gamma^2}{\bar{\alpha}\beta}\right)$	$\frac{\tan\beta\frac{a}{2}}{\tanh\bar{\alpha}\frac{a}{2}}=\frac{\bar{\alpha}\beta}{\gamma^2}$,	$\frac{\tan\beta\frac{a}{2}}{\tanh\bar{\alpha}\frac{a}{2}}=-\frac{\gamma^2}{\bar{\alpha}\beta}$
Type 3	$\frac{1-\cosh\bar{\alpha}a\cosh\bar{\beta}a}{\sinh\bar{\alpha}a\sinh\bar{\beta}a}=-\frac{1}{2}\left(\frac{\bar{\alpha}\bar{\beta}}{\gamma^2}+\frac{\gamma^2}{\bar{\alpha}\bar{\beta}}\right)$	$\frac{\tanh\bar{\beta}\frac{a}{2}}{\tanh\bar{\alpha}\frac{a}{2}}=\frac{\bar{\alpha}\bar{\beta}}{\gamma^2}$,	$\frac{\tanh\bar{\beta}\frac{a}{2}}{\tanh\bar{\alpha}\frac{a}{2}}=\frac{\gamma^2}{\bar{\alpha}\bar{\beta}}$
F–SS–F–SS2	$0\le x\le a$ Eq. (a)	$-a/2\le x\le a/2$ Eq. (b)	Eq. (c)
Type 1	$\frac{1-\cos\alpha a\cos\beta a}{\sin\alpha a\sin\beta a}=-\frac{1}{2}\left(\frac{AB}{C}+\frac{C}{AB}\right)$	$\frac{\tan\beta\frac{a}{2}}{\tan\alpha\frac{a}{2}}=-\frac{AB}{C}$,	$\frac{\tan\beta\frac{a}{2}}{\tan\alpha\frac{a}{2}}=-\frac{C}{AB}$
Type 2	$\frac{1-\cosh\bar{\alpha}a\cos\beta a}{\sinh\bar{\alpha}a\sin\beta a}=\frac{1}{2}\left(\frac{\bar{A}B}{C'}-\frac{C'}{\bar{A}B}\right)$	$\frac{\tan\beta\frac{a}{2}}{\tanh\bar{\alpha}\frac{a}{2}}=\frac{\bar{A}B}{C'}$,	$\frac{\tan\beta\frac{a}{2}}{\tanh\bar{\alpha}\frac{a}{2}}=-\frac{C'}{\bar{A}B}$
Type 3	$\frac{1-\cosh\bar{\alpha}a\cosh\bar{\beta}a}{\sinh\bar{\alpha}a\sinh\bar{\beta}a}=-\frac{1}{2}\left(\frac{\bar{A}\bar{B}}{C''}+\frac{C''}{\bar{A}\bar{B}}\right)$	$\frac{\tanh\bar{\beta}\frac{a}{2}}{\tanh\bar{\alpha}\frac{a}{2}}=\frac{\bar{A}\bar{B}}{C''}$,	$\frac{\tanh\bar{\beta}\frac{a}{2}}{\tanh\bar{\alpha}\frac{a}{2}}=\frac{C''}{\bar{A}\bar{B}}$
C–SS–F–SS2	$0\le x\le a$		
Type 1	$\left[2\left(1-\nu\right)\alpha\beta+\frac{AB}{\alpha\beta}\right]\sin\alpha a\sin\beta a-\left[2\left(1-\nu\right)\gamma^2+\frac{AB}{\gamma^2}\right]\cos\alpha a\cos\beta a=2A+\left(1-\nu\right)B$		
Type 2	$\left[2\left(1-\nu\right)\bar{\alpha}\beta+\frac{\bar{A}B}{\bar{\alpha}\beta}\right]\sinh\bar{\alpha}a\sin\beta a+\left[2\left(1-\nu\right)\gamma^2-\frac{\bar{A}B}{\gamma^2}\right]\cosh\bar{\alpha}a\cos\beta a=2\bar{A}-\left(1-\nu\right)B$		
Type 3	$\left[2\left(1-\nu\right)\bar{\alpha}\bar{\beta}+\frac{\bar{A}\bar{B}}{\bar{\alpha}\bar{\beta}}\right]\sinh\bar{\alpha}a\sinh\bar{\beta}a-\left[2\left(1-\nu\right)\gamma^2+\frac{\bar{A}\bar{B}}{\gamma^2}\right]\cosh\bar{\alpha}a\cosh\bar{\beta}a=-\left[2\bar{A}+\left(1-\nu\right)\bar{B}\right]$		

Boundary conditions $C-C$ and $F-F$ are symmetric and by taking edge coordinate $x=-a/2$ and $x=a/2$ two simpler frequency equations are obtained, Table 2. Derivation of frequency equation is illustrated in Appendix.

The above formulae cannot be used in case $n=0$, which is related to a bar longitudinal vibrations with restrained transverse contraction. The governing differential equation of motion is obtained from the first of Eqs. (3):

$$
\frac{d^2U}{dx^2}+\frac{\rho}{E^*}\omega^2U=0. \tag{20}
$$

Assuming the solution of Eq. (20) in the trigonometric form $U=A\sin\frac{k\pi x}{a}$, where k is a number of the mode half-waves, natural frequencies are obtained:

$$\omega_k = \frac{k\pi}{a}\sqrt{\frac{E^*}{\rho}}. \qquad (21)$$

Eq. (21) expressed in a non-dimensional form reads:

$$\Omega_k = \frac{\omega_k a}{\pi}\sqrt{\frac{\rho}{E^*}} = k, \qquad (22)$$

where $k = 1, 2, ...$ for clamped (C-C) and free (F-F) ends, and $k = 0.5, 1.5, ...$ for combined clamped-free ends (C-F). It is convenient to use expression (22) as a norm for the definition of frequency parameter for a general case.

5. A RECTANGULAR PLATE WITH SIMPLY SUPPORTED EDGES

Potential functions are assumed in the trigonometric form:

$$\begin{aligned} &\Phi(x,y) = A\sin\alpha_m x\sin\beta_n y,\\ &\Psi(x,y) = B\cos\alpha_m x\cos\beta_n y,\\ &\alpha_m = \frac{m\pi}{a},\ \ \beta_n = \frac{n\pi}{b},\ \ m,n = 1,2,.... \end{aligned} \qquad (23)$$

According to Eq. (4) displacements read:

$$\begin{aligned} &U(x,y) = C\cos\alpha_m x\sin\beta_n y,\\ &V(x,y) = D\sin\alpha_m x\cos\beta_n y, \end{aligned} \qquad (24)$$

where $C = (A\alpha_m - B\beta_n)$ and $D = (A\beta_m + B\alpha_n)$. Displacements, Eq. (24), satisfy boundary conditions SS1 at all four edges, Table 1.

Substituting Eq. (24) into Eq. (3), the following system of two algebraic equations is obtained:

$$\begin{bmatrix} d_{11} & d_{12}\\ d_{21} & d_{22}\end{bmatrix}\begin{Bmatrix} C\\ D\end{Bmatrix} = \begin{Bmatrix} 0\\ 0\end{Bmatrix}, \qquad (25)$$

where:

$$\begin{aligned} &d_{11} = \frac{\rho}{E^*}\omega^2 - \left[\alpha_m^2 + \frac{1}{2}(1-\nu)\beta_n^2\right],\\ &d_{12} = d_{21} = -\frac{1}{2}(1+\nu)\alpha_m\beta_n,\\ &d_{22} = \frac{\rho}{E^*}\omega^2 - \left[\frac{1}{2}(1-\nu)\alpha_m^2 + \beta_n^2\right]. \end{aligned} \qquad (26)$$

The determinant of system (25) represents a frequency equation which after some manipulation takes the following simple form:

$$Det(\omega) = d_{11}d_{22} - d_{12}^2 = \Lambda^2 - p_{mn}\Lambda + q_{mn} = 0, \qquad (27)$$

where:

$$\begin{aligned} &\Lambda = \frac{\rho}{E^*}\omega^2, \quad p_{mn} = \frac{1}{2}(3-\nu)c_{mn},\\ &q_{mn} = \frac{1}{2}(1-\nu)c_{mn}^2, \quad c_{mn} = \alpha_m^2 + \beta_n^2. \end{aligned} \qquad (28)$$

Solutions of Eq. (27), $\Lambda_1 = c_{mn}$ and $\Lambda_2 = (1-\nu)c_{mn}/2$, give:

$$\omega_{mn}^{(1)} = \sqrt{\frac{G}{\rho}c_{mn}}, \quad \omega_{mn}^{(2)} = \sqrt{\frac{E^*}{\rho}c_{mn}}, \qquad (29)$$

or in the non-dimensional form $\Omega = \frac{\omega a}{\pi}\sqrt{\frac{\rho}{E^*}}$:

$$\begin{aligned} &\Omega_{mn}^{(1)} = \sqrt{\frac{1-\nu}{2}\left[m^2 + \left(\frac{a}{b}\right)^2 n^2\right]},\\ &\Omega_{mn}^{(2)} = \sqrt{m^2 + \left(\frac{a}{b}\right)^2 n^2}. \end{aligned} \qquad (30)$$

Additional displacement fields for simply supported plate can be specified, i.e.:

SS1-SS2-SS1-SS2:

$$\begin{aligned} &U(x,y) = C\cos\alpha_m x\cos\beta_n y,\\ &V(x,y) = D\sin\alpha_m x\sin\beta_n y, \end{aligned} \qquad (31)$$

SS2-SS1-SS2-SS1:

$$\begin{aligned} &U(x,y) = C\sin\alpha_m x\sin\beta_n y,\\ &V(x,y) = D\cos\alpha_m x\cos\beta_n y, \end{aligned} \qquad (32)$$

SS2-SS2-SS2-SS2:

$$\begin{aligned} &U(x,y) = C\sin\alpha_m x\cos\beta_n y,\\ &V(x,y) = D\cos\alpha_m x\sin\beta_n y. \end{aligned} \qquad (33)$$

In all cases the same expressions, Eqs. (30), for natural frequencies are obtained. Some additional considerations of vibrations of simply supported plates can be found in Refs. [7] and [8].

6. VIBRATIONS OF A CIRCULAR PLATE

Differential equations of motion derived for a rectangular plate can be directly applied for a circular plate by transforming orthogonal coordinates x and y into polar coordinates r and φ, see Figure 2.

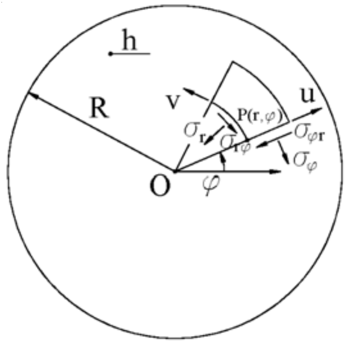

Fig. 2 Circular plate for in-plane vibrations

Hence, Eqs. (6) take the following form:

$$\begin{aligned} & E^{*}\Delta_r \Phi + \rho\omega^2 \Phi = 0, \\ & G\Delta_r \Psi + \rho\omega^2 \Psi = 0, \end{aligned} \tag{34}$$

where: $\Delta_r(.) = \frac{\partial^2(.)}{\partial r^2} + \frac{1}{r}\frac{\partial(.)}{\partial r} + \frac{\partial^2(.)}{r^2\partial\varphi^2}$, [14].

Assuming harmonic circular variation of the potential functions:

$$\begin{aligned} & \Phi(r,\varphi) = \Phi_r(r)\sin n\varphi, \\ & \Psi(r,\varphi) = \Psi_r(r)\cos n\varphi, \end{aligned} \tag{35}$$

Eqs. (34) are transformed into Bessel's equations:

$$\begin{aligned} & \frac{d^2\Phi_r}{d\xi^2} + \frac{1}{\xi}\frac{d\Phi_r}{d\xi} + \left(1 - \frac{n^2}{\xi^2}\right)\Phi_r = 0, \\ & \frac{d^2\Psi_r}{d\eta^2} + \frac{1}{\eta}\frac{d\Psi_r}{d\eta} + \left(1 - \frac{n^2}{\eta^2}\right)\Psi_r = 0, \end{aligned} \tag{36}$$

where:

$$\begin{aligned} & \xi = \alpha_C r, \quad \alpha_C = \omega\sqrt{\frac{\rho}{E^*}}, \\ & \eta = \beta_C r, \quad \beta_C = \omega\sqrt{\frac{\rho}{G}}. \end{aligned} \tag{37}$$

Solutions of Eqs. (36) are expressed with special functions:

$$\begin{aligned} & \Phi_r = A_1 J_n(\xi) + A_2 Y_n(\xi), \\ & \Psi_r = B_1 J_n(\eta) + B_2 Y_n(\eta), \end{aligned} \tag{38}$$

where J_n is a Bessel function of the first kind of order n, and Y_n is a Bessel function of the second kind of order n.

According to Eq. (4) radial and circumferential displacement read:

$$\begin{aligned} & U = \frac{\partial\Phi}{\partial r} + \frac{\partial\Psi}{r\partial\varphi} = U_r \sin n\varphi, \\ & V = \frac{\partial\Phi}{r\partial\varphi} - \frac{\partial\Psi}{\partial r} = V_r \cos n\varphi, \end{aligned} \tag{39}$$

where:

$$U_r = \frac{d\Phi_r}{dr} - \frac{n}{r}\Psi_r, \quad V_r = \frac{n}{r}\Phi_r - \frac{d\Psi_r}{dr}. \tag{40}$$

Substituting Eq. (38) into Eq. (40) one arrives at:

$$\begin{aligned} U_r = {} & A_1\alpha_C \frac{dJ_n(\xi)}{d\xi} + A_2\alpha_C \frac{dY_n(\xi)}{d\xi} - \\ & - B_1 \frac{n}{r} J_n(\eta) - B_2 \frac{n}{r} Y_n(\eta), \\ V_r = {} & A_1 \frac{n}{r} J_n(\xi) + A_2 \frac{n}{r} Y_n(\xi) - \\ & - B_1\beta_C \frac{dJ_n(\eta)}{d\eta} - B_2\beta_C \frac{dY_n(\eta)}{d\eta}. \end{aligned} \tag{41}$$

Stresses in polar coordinate system, [9], read:

$$\begin{aligned} & \sigma_r = E^*\left(\frac{\partial U}{\partial r} + \nu\frac{\partial V}{r\partial\varphi} + \frac{\nu}{r}U\right) = \Sigma_r \sin n\varphi, \\ & \sigma_\varphi = E^*\left(\frac{\partial V}{r\partial\varphi} + \frac{U}{r} + \nu\frac{\partial U}{\partial r}\right) = \Sigma_\varphi \sin n\varphi, \\ & \sigma_{r\varphi} = G\left(\frac{\partial V}{\partial r} + \frac{\partial U}{r\partial\varphi} - \frac{V}{r}\right) = \Sigma_{r\varphi} \cos n\varphi, \end{aligned} \tag{42}$$

where:

$$\begin{aligned} \Sigma_r = {} & E^*\left[\frac{d^2\Phi_r}{dr^2} + \frac{\nu}{r}\left(\frac{d\Phi_r}{dr} - \frac{n^2}{r}\Phi_r\right) + \right. \\ & \left. + \frac{(1-\nu)n}{r}\left(-\frac{d\Psi_r}{dr} + \frac{\Psi_r}{r}\right)\right], \\ \Sigma_\varphi = {} & E^*\left[\nu\frac{d^2\Phi_r}{dr^2} + \frac{1}{r}\left(\frac{d\Phi_r}{dr} - \frac{n^2}{r}\Phi_r\right) + \right. \\ & \left. + \frac{(1-\nu)n}{r}\left(\frac{d\Psi_r}{dr} - \frac{\Psi_r}{r}\right)\right], \\ \Sigma_{r\varphi} = {} & G\left[\frac{2n}{r}\left(\frac{d\Phi_r}{dr} - \frac{1}{r}\Phi_r\right) - \frac{d^2\Psi_r}{dr^2} + \right. \\ & \left. + \frac{1}{r}\frac{d\Psi_r}{dr} - \frac{n^2}{r^2}\Psi_r\right]. \end{aligned} \tag{43}$$

Substitution of Eq. (38) into Σ_r and $\Sigma_{r\phi}$, which are used in specification of boundary conditions, yields:

$$\begin{aligned}\Sigma_r = E^* \Bigg\{ & A_1 \left[{\alpha_C}^2 \frac{d^2 J_n(\xi)}{d\xi^2} + \frac{\nu \alpha_C}{r} \frac{dJ_n(\xi)}{d\xi} - \frac{\nu n^2}{r^2} J_n(\xi) \right] + \\ & + A_2 \left[{\alpha_C}^2 \frac{d^2 Y_n(\xi)}{d\xi^2} + \frac{\nu \alpha_C}{r} \frac{dY_n(\xi)}{d\xi} - \frac{\nu n^2}{r^2} Y_n(\xi) \right] - \\ & - B_1 \frac{(1-\nu) n}{r} \left[\beta_C \frac{dJ_n(\eta)}{d\eta} - \frac{1}{r} J_n(\eta) \right] - B_2 \frac{(1-\nu) n}{r} \left[\beta_c \frac{dY_n(\eta)}{d\eta} - \frac{1}{r} Y_n(\eta) \right] \Bigg\}, \\ \Sigma_{r\varphi} = G \Bigg\{ & A_1 \frac{2n}{r} \left[\alpha_C \frac{dJ_n(\xi)}{d\xi} - \frac{1}{r} J_n(\xi) \right] + A_2 \frac{2n}{r} \left[\alpha_C \frac{dY_n(\xi)}{d\xi} - \frac{1}{r} Y_n(\xi) \right] + \\ & + B_1 \left[-{\beta_C}^2 \frac{d^2 J_n(\eta)}{d\eta^2} + \frac{\beta_C}{r} \frac{dJ_n(\eta)}{d\eta} - \frac{n^2}{r^2} J_n(\eta) \right] + B_2 \left[-{\beta_C}^2 \frac{d^2 Y_n(\eta)}{d\eta^2} + \frac{\beta_C}{r} \frac{dY_n(\eta)}{d\eta} - \frac{n^2}{r^2} Y_n(\eta) \right] \Bigg\}.\end{aligned} \tag{44}$$

Integration constants A_i and B_i, $i=1, 2$, in the above formulae for displacements and stresses are determined satisfying boundary conditions at a plate's inner and outer edges, as specified in Table 3.

Table 3. Boundary conditions for in-plane vibrations of a circular plate

	$r = r_0$ or $r = R$
Simply supported, SS1	$V = 0,\ \sigma_r = 0$
Simply supported, SS2	$U = 0,\ \sigma_{r\varphi} = 0$
Clamped, C	$U = 0,\ V = 0$
Free, F	$\sigma_r = 0,\ \sigma_{r\varphi} = 0$

For illustration, let us consider vibrations of a clamped plate. Boundary conditions for a plate without the central hole read: $U_r(R) = 0$ and $V_r(R) = 0$. Constants A_2 and B_2 in Eq. (41) are zero since functions $Y_n(\xi) = 0$ and $Y_n(\eta) = 0$ take infinite values at $r = 0$. Hence, the system of boundary equations read:

$$\begin{bmatrix} \alpha_C \frac{dJ_n(\xi)}{d\xi} & -\frac{n}{R} J_n(\eta) \\ \frac{n}{R} J_n(\xi) & -\beta_C \frac{dJ_n(\eta)}{d\eta} \end{bmatrix} \begin{Bmatrix} A_1 \\ B_1 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}, \tag{45}$$

that leads to the frequency equation:

$$Det(\omega) = \alpha_C \beta_C \frac{dJ_n(\xi)}{d\xi} \frac{dJ_n(\eta)}{d\eta} - \frac{n^2}{R^2} J_n(\xi) J_n(\eta) = 0. \tag{46}$$

This type of equation is also obtained in Ref. [9].

In case of a free plate without a central hole, boundary conditions read: $\Sigma_r(R) = 0$ and $\Sigma_{r\phi}(R) = 0$. Employing Eq. (44), one finds for frequency equation:

$$Det(\omega) = d_{11} d_{22} - d_{12} d_{21} = 0, \tag{47}$$

where:

$$\begin{aligned} & d_{11} = {\alpha_C}^2 \frac{d^2 J_n(\xi)}{d\xi^2} + \frac{\nu \alpha_C}{R} \frac{dJ_n(\xi)}{d\xi} - \frac{\nu n^2}{R^2} J_n(\xi), \quad d_{22} = -{\beta_C}^2 \frac{d^2 J_n(\eta)}{d\eta^2} + \frac{\beta_C}{R} \frac{dJ_n(\eta)}{d\eta} - \frac{n^2}{R^2} J_n(\eta), \\ & d_{12} = -\frac{(1-\nu) n}{R} \left[\beta_C \frac{dJ_n(\eta)}{d\eta} - \frac{1}{R} J_n(\eta) \right], \quad d_{21} = \frac{2n}{R} \left[\alpha_C \frac{dJ_n(\xi)}{d\xi} - \frac{1}{R} J_n(\xi) \right]. \end{aligned} \tag{48}$$

In-plane vibrations of a disk on shaft are another interesting problem. Boundary conditions at inner, $r = r_0$, and outer, $r = R$, edge read: $U_r(r_0) = 0$, $V_r(r_0) = 0$, $\Sigma_r(R) = 0$ and $\Sigma_{r\phi}(R) = 0$. Employing Eqs. (41) and (44), a homogenous system of algebraic equations is formed:

$$\left[A(\omega)\right]\{C\}=\{0\}, \tag{49}$$

where:

$$\{C\}=\begin{Bmatrix} A_1 \\ A_2 \\ B_1 \\ B_2 \end{Bmatrix}, \tag{50}$$

while the elements of the matrix $\left[A(\omega)\right]$ are coefficients of integration constants in Eqs. (41) and (44). Natural frequencies are determined by satisfying conditions for a nontrivial solution, $Det\left[A(\omega)\right]=0$.

7. NUMERICAL EXAMPLES

7.1 Rectangular plate

The application of the presented potential vibration theory on in-plane vibrations is illustrated for a rectangular plate of the following characteristics: $a=1.2\ m$, $b=1\ m$, $h=0.01\ m$, $E=2.110^{11}\ N/m^2$, $\nu=0.3$, $\rho=7850\ kg/m^3$. Three boundary value problems are considered: simply supported longitudinal edges (SS2), combined with clamped, free and clamped-free transverse edges. Value of the frequency parameter $\Omega=\omega a\sqrt{\rho/E^*}/\pi$ is determined by employing the corresponding frequency equation from Table 2. For this purpose arguments of trigonometric functions, Eqs. (13), are expressed in terms of Ω:

$$\begin{aligned} \alpha a &= \pi\sqrt{\Omega^2-\left(\frac{a}{b}\right)^2 n^2}, \\ \beta a &= \pi\sqrt{\frac{2}{1-\nu}\Omega^2-\left(\frac{a}{b}\right)^2 n^2}. \end{aligned} \tag{51}$$

Frequency parameter Ω is assumed and depends on a combination of real and imaginary value of arguments, i.e. α and β, $\bar{\alpha}$ and β, $\bar{\alpha}$ and $\bar{\beta}$; frequency equation of Type 1, 2 and 3 is employed, respectively, as follows:

$$\begin{aligned} &Type\ 1\ (\alpha,\beta):\ \frac{a}{b}n<\Omega, \\ &Type\ 2\ (\bar{\alpha},\beta):\ \sqrt{\frac{1-\nu}{2}}\frac{a}{b}n<\Omega\le\frac{a}{b}n, \\ &Type\ 3\ (\bar{\alpha},\bar{\beta}):\ 0<\Omega\le\sqrt{\frac{1-\nu}{2}}\frac{a}{b}n. \end{aligned} \tag{52}$$

In the considered numerical example ranges of the frequency parameter depending on n are shown in Table 4.

Table 4. Types of solution depending on value of frequency parameter, a/b=1.2

	$n=1$	$n=2$	$n=3$
Type 3, $(\bar{\alpha},\bar{\beta})$	$0\le\Omega\le0.7099$	$0\le\Omega\le1.4199$	$0\le\Omega\le2.1298$
Type 2, $(\bar{\alpha},\beta)$	$0.7099<\Omega\le1.2$	$1.4199<\Omega\le2.4$	$2.1298<\Omega\le3.6$
Type 1, (α,β)	$1.2<\Omega$	$2.4<\Omega$	$3.6<\Omega$

The present analytical solutions are listed in Tables 5, 6 and 7 for the three cases of boundary conditions. The problem is also solved using the finite element method employing NASTRAN package [17] with fine *120 x 100* mesh and membrane elements. Values of the frequency parameters are included in Tables 5, 6 and 7, which indicate a rather good agreement between analytical and numerical results.

Table 5. Frequency parameter $\Omega=\omega a\sqrt{\rho/E^*}/\pi$ of a rectangular plate, a/b=1.2, C-SS2-C-SS2

Mode no.	n	Analytical		FEM
1	0	1	Eq. (23)	1.0000
2	1	1.0936	Type 2	1.0936
3	1	1.2669	Type 1	1.2668
4	2	1.5968	Type 2	1.5964
5	1	1.6977	Type 1	1.6974
6	1	1.8269	Type 1	1.8265
7	2	1.9717	Type 2	1.9710
8	0	2	Eq. (23)	1.9998
9	3	2.2371	Type 2	2.2361
10	1	2.3634	Type 1	2.3627

Table 6. Frequency parameter $\Omega = \omega a\sqrt{\rho / E^*} / \pi$ *of a rectangular plate, a/b=1.2, F-SS2-F-SS2*

Mode no.	n	Analytical		FEM
1	1	0.5660	Type 3	0.5661
2	1	0.9052	Type 2	0.9051
3	0	1	Eq. (23)	1.0000
4	1	1.2270	Type 1	1.2270
5	1	1.2495	Type 1	1.2494
6	2	1.2635	Type 3	1.2634
7	2	1.3665	Type 3	1.3662
8	1	1.7185	Type 1	1.7181
9	1	1.8038	Type 1	1.8033
10	2	1.8356	Type 2	1.8355

Table 7. Frequency parameter $\Omega = \omega a\sqrt{\rho / E^*} / \pi$ *of a rectangular plate, a/b=1.2, C-SS2-F-SS2*

Mode no.	n	Analytical		FEM
1	0	0.5	Eq. (23)	0.5000
2	1	0.7078	Type 3	0.7078
3	1	1.1287	Type 2	1.1287
4	2	1.3053	Type 3	1.3052
5	1	1.4547	Type 1	1.4545
6	1	1.4754	Type 1	1.4752
7	0	1.5	Eq. (23)	1.4999
8	2	1.7021	Type 2	1.7016
9	3	1.9516	Type 3	1.9510
10	1	2.0105	Type 1	2.0100

The corresponding 10 natural modes, determined by a coarse mesh (*24 x 20* elements) are, for transparency of deformations, shown in Figures 3, 4 and 5. The bar natural modes, *n=0*, can be easily noticed.

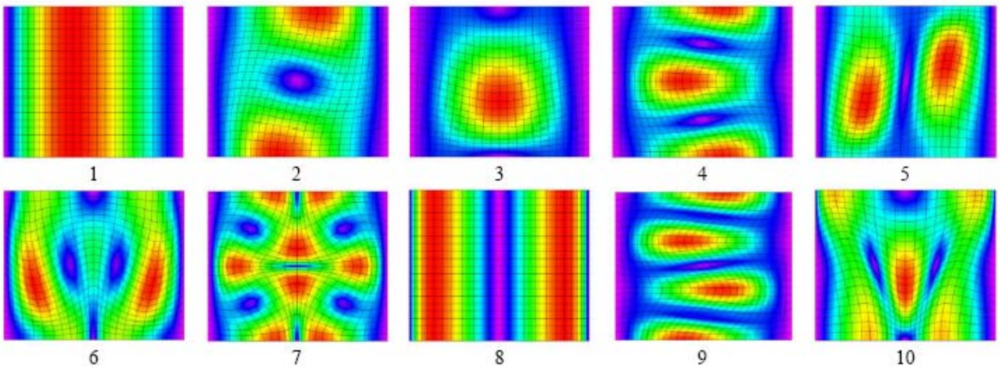

Fig. 3 Natural modes of in-plane vibrations of a rectangular plate, a/b=1.2, C-SS2-C-SS2

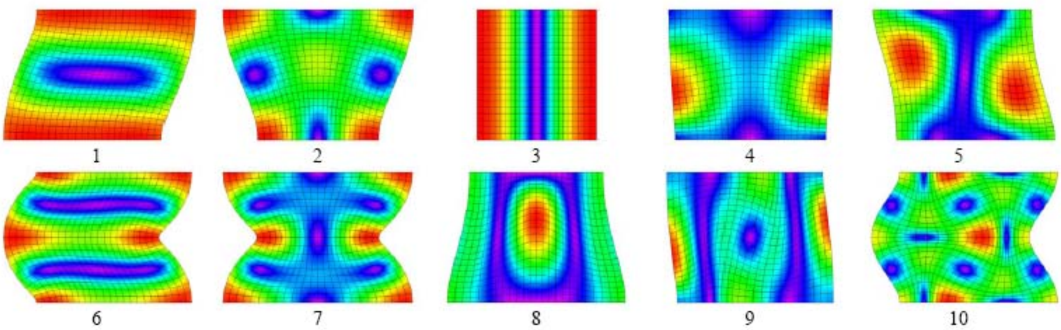

Fig. 4 Natural modes of in-plane vibrations of a rectangular plate, a/b=1.2, F-SS2-F-SS2

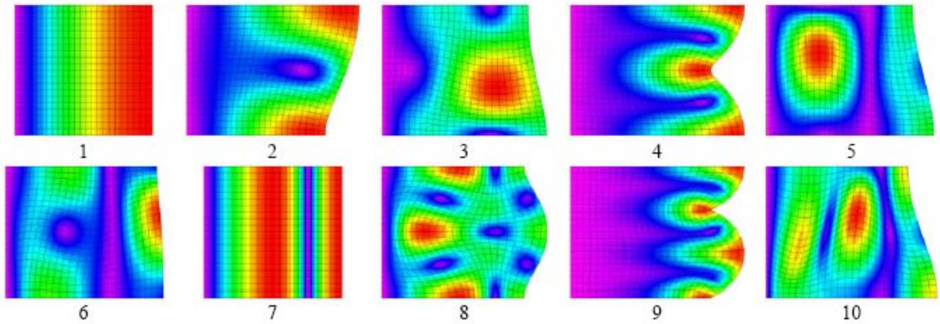

Fig. 5 Natural modes of in-plane vibrations of a rectangular plate, a/b=1.2, C-SS2-F-SS2

The preceding examples are also analyzed in Ref. [7], where frequency parameter is specified in a different way, i.e. $\Omega = \omega b\sqrt{\rho/G}/\pi$. The present results (PR) can be converted to the norm presented in Ref. [7] by factor $C = b/a\sqrt{2/(1-\nu)}$, as it is done in Table 8 for boundary conditions C-SS2-C-SS2. In both cases, analytical solutions are almost identical. However, there are small differences between FEM values, in spite of the fact that the same software and the same mesh density are used in both analyses. Numerical results from Ref. [7] and the present results respectively converge from upper and lower side to the exact solution. It seems that in the meantime non-conforming membrane element in NASTRAN has been substituted with conforming one. The abovementined conclusions are also related to vibration analyses for other cases of boundary conditions.

Table 8. Comparison of frequency parameter $\Omega = \omega b\sqrt{\rho/G}/\pi$ of a rectangular plate, a/b=1.2, C-SS2-C-SS2

Mode no.	n	Analytical		FEM	
		Ref. [7]	PS	Ref. [7]	PS
1	0	1.4086	1.4086	1.4087	1.4086
2	1	1.5406	1.5405	1.5406	1.5404
3	1	1.7846	1.7845	1.7847	1.7844
4	2	2.2493	2.2492	2.2496	2.2487
5	1	2.3915	2.3914	2.3917	2.3909
6	1	2.5735	2.5734	2.5739	2.5728
7	2	2.7774	2.7773	2.7779	2.7763

7.2 Circular plate

Vibrations of clamped and free circular plates are analyzed according to the theory presented in Section 6; natural frequencies are determined employing Eqs. (46) and (47), respectively. Values of vibration parameter $\Omega = 2R\omega\sqrt{\rho/E^*}/\pi$, which is analogous to the one used for a bar, Eq. (22), are listed in Tables 9 and 10, and compared with the FEM results. Quite a good agreement between analytical and numerical results is achieved. The first 10 natural modes are shown in Figures 6 and 7.

Table 9. Frequency parameter $\Omega = 2R\omega\sqrt{\rho/E^*}/\pi$ of a clamped circular plate

Mode no.	n	PS	FEM
1	1	1.2459	1.2457
2	0	1.4430	1.4409
3	2	1.9400	1.9361
4	1	2.0234	2.0171
5	0	2.4393	2.4344
6	3	2.5229	2.5120
7	2	2.6011	2.5876
8	0	2.6422	2.6285
9	4	3.0409	3.0186
10	1	3.1881	3.1649

Table 10. Frequency parameter $\Omega = 2R\omega\sqrt{\rho/E^*}/\pi$ of a free circular plate

Mode no.	n	PS	FEM
1	2	0.8833	0.8832
2	1	1.0297	1.0282
3	0	1.3043	1.3043
4	3	1.3562	1.3537
5	2	1.5986	1.5937
6	4	1.7658	1.7588
7	0	1.9342	1.9291
8	5	2.1500	2.1345
9	3	2.1974	2.1871
10	1	2.2466	2.2436

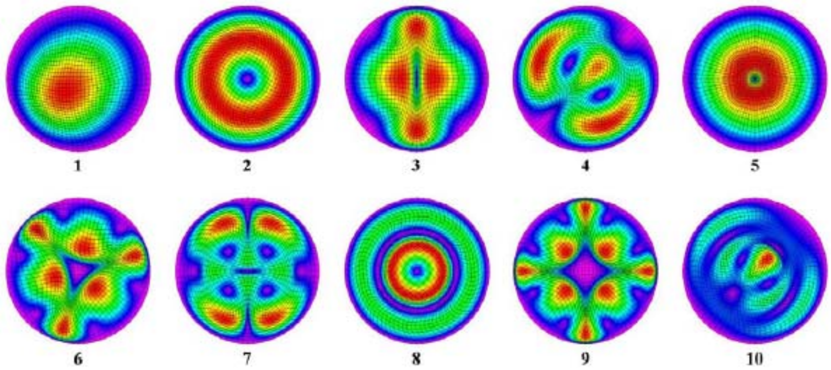

Fig. 6 Natural modes of a clamped circular plate

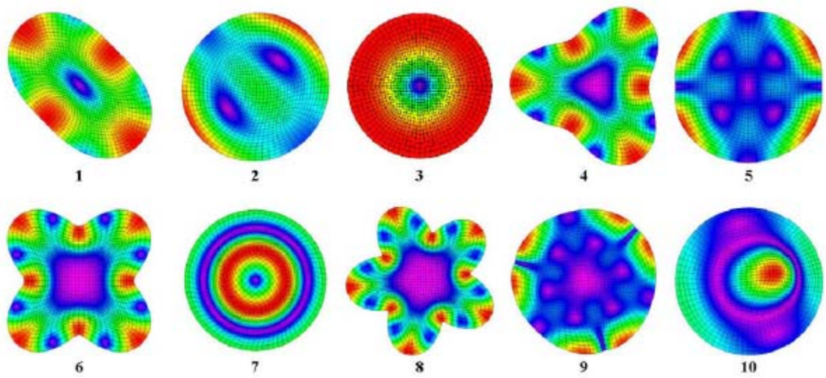

Fig. 7 Natural modes of a free circular plate

A practical problem of annular plate, clamped at the inner and free at the outer edge, $r_0 / R = 0.2$, is considered. The eigenvalue problem is formulated in Section 6. The obtained results of frequency parameter are given in Table 11 and compared to the FEM results. There are very small differences between the analytical and numerical solution, except in case of the second frequency parameter. For a circular plate, discrepancy is reduced for small values of ratio r_0 / R and it disappears for $r_0 = 0$, as can be seen in Table 10. All analytical results are determined by the same algorithm and the same code. Also, the FEM results are determined by the same model. Currently, the causes of discrepancy of frequency parameter only in case of the first mode ($n=1$) for the annular plate cannot be explained. Therefore, the issue remains open for further investigation. The corresponding natural modes for natural frequencies listed in Table 11 are shown in Figure 8. They are similar to those of the free plate, Figure 7, but their ordering is changed.

Table 11. Frequency parameter $\Omega = 2R\omega\sqrt{\rho/E^*}/\pi$ *of a clamped – free annular plate,* $r_0/R=0.2$

Mode no.	n	PS	FEM
1	0	0.2243	0.2246
2	1	0.6824	0.5872
3	2	0.9922	0.9821
4	1	1.3137	1.3465
5	3	1.3728	1.3658
6	0	1.4030	1.4034
7	2	1.6304	1.6447
8	4	1.7679	1.7474
9	5	2.1502	2.1087
10	0	2.1937	2.1831

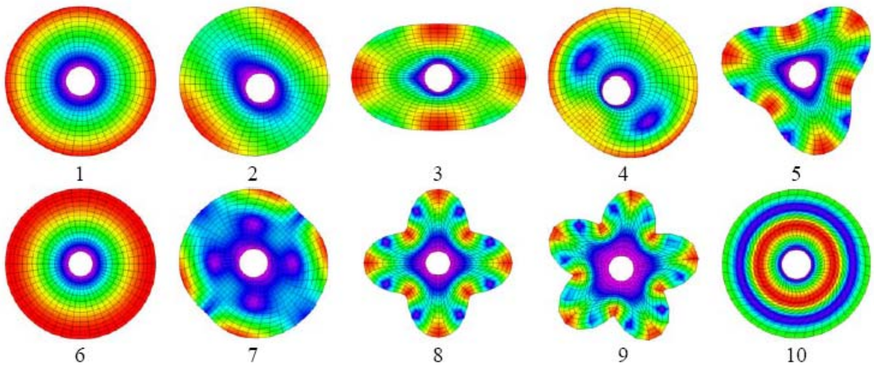

Fig. 8 Natural modes of a clamped-free annular plate, $r_0/R=0.2$

8. CONCLUSION

In-plane vibrations of rectangular and circular plates are interesting problems from a practical point of view. A general approach to the displacement potentials has been used for a rectangular plate. Two coupled differential equations of motion have been decomposed into two mutually independent equations. Employing the method of separation of variables, the vibration problem has been solved in a relatively simple and transparent way. It has been shown that an analytical solution of vibrations can be obtained for a rectangular plate with two simply supported opposite edges, and for any combination of boundary conditions for the remaining two edges. Rather simple frequency equations have been given for clamped, free and combined boundary conditions of a rectangular plate.

Uncoupled equations of displacement potentials derived for rectangular plate have been directly applied for circular plates. Circumferential variation of potentials has been assumed in the form of trigonometric series, while radial variation has been obtained solving governing Bessel equations. In that way vibrations of annular plates can be analyzed with any combination of boundary conditions at the inner and outer edges. Frequency equations have been derived for clamped and free circular, as well for annular plates.

The application of the presented potential theory has been illustrated on a number of boundary value problems of rectangular and circular plates. The obtained results have been validated by the finite element analysis. Analytical solutions can be used as a benchmark for numerical methods.

ACKNOWLEDGMENT

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10. APPENDIX: DERIVATION OF THE FREQUENCY EQUATION FOR RECTANGULAR PLATES, C-SS2-C-SS2

Displacement functions of x variable are considered, $U(x)$ and $V(x)$, Eqs. (16). The boundary conditions read: $U(0) = 0$, $V(0) = 0$, $U(a) = 0$, $V(a) = 0$. The first two conditions give:

$$B_1 = -\frac{\alpha}{\gamma} A_2, \quad A_1 = -\frac{\beta}{\gamma} B_2. \tag{A1}$$

Substituting B_1 and A_1 into the second two conditions, one arrives at the eigenvalue problem:

$$\begin{bmatrix} \alpha\left(\cos\alpha a - \cos\beta a\right) & \dfrac{\alpha\beta}{\gamma}\sin\alpha a + \gamma\sin\beta a \\ -\left(\gamma\sin\alpha a + \dfrac{\alpha\beta}{\gamma}\sin\beta a\right) & \beta\left(\cos\alpha a - \cos\beta a\right) \end{bmatrix} \begin{Bmatrix} A_2 \\ B_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}. \tag{A2}$$

Determinant of the above matrix leads to the frequency equation:

$$Det(\omega) = 2\alpha\beta\left(1 - \cos\alpha a\cos\beta a\right) + \left[\gamma^2 + \left(\frac{\alpha\beta}{\gamma}\right)^2\right]\sin\alpha a\sin\beta a = 0. \tag{A3}$$

It can be presented in the form shown in Table 2, Type 1, Eq. (a).

In case when the origin of the coordinate system is located in the middle of the axial edge $y = 0$, the boundary conditions at $x = a/2$ and $x = -a/2$, read:

$$\begin{aligned}
U\left(\frac{a}{2}\right) &= -A_1\alpha\sin\alpha\frac{a}{2}+A_2\alpha\cos\alpha\frac{a}{2}+B_1\gamma\cos\beta\frac{a}{2}+B_2\gamma\sin\beta\frac{a}{2}=0,\\
U\left(-\frac{a}{2}\right) &= A_1\alpha\sin\alpha\frac{a}{2}+A_2\alpha\cos\alpha\frac{a}{2}+B_1\gamma\cos\beta\frac{a}{2}-B_2\gamma\sin\beta\frac{a}{2}=0,\\
V\left(\frac{a}{2}\right) &= -A_1\gamma\cos\alpha\frac{a}{2}-A_2\gamma\sin\alpha\frac{a}{2}+B_1\beta\sin\beta\frac{a}{2}-B_2\beta\cos\beta\frac{a}{2}=0,\\
V\left(-\frac{a}{2}\right) &= -A_1\gamma\cos\alpha\frac{a}{2}+A_2\gamma\sin\alpha\frac{a}{2}-B_1\beta\sin\beta\frac{a}{2}-B_2\beta\cos\beta\frac{a}{2}=0.
\end{aligned} \tag{A4}$$

Summation and subtraction of the above pairs of expressions give:

$$\begin{aligned}
U\left(\frac{a}{2}\right)+U\left(-\frac{a}{2}\right) &= 2A_2\alpha\cos\alpha\frac{a}{2}+2B_1\gamma\cos\beta\frac{a}{2}=0,\\
U\left(\frac{a}{2}\right)-U\left(-\frac{a}{2}\right) &= 2A_1\alpha\sin\alpha\frac{a}{2}-2B_2\gamma\sin\beta\frac{a}{2}=0,\\
V\left(\frac{a}{2}\right)+V\left(-\frac{a}{2}\right) &= -2A_1\gamma\cos\alpha\frac{a}{2}-2B_2\beta\cos\beta\frac{a}{2}=0,\\
V\left(\frac{a}{2}\right)-V\left(-\frac{a}{2}\right) &= -2A_2\gamma\sin\alpha\frac{a}{2}+2B_1\beta\sin\beta\frac{a}{2}=0.
\end{aligned} \tag{A5}$$

The first and the last, and the second and the third expressions of Eqs. (A5), represent two uncoupled eigenvalue problems:

$$\begin{bmatrix} \alpha\cos\alpha\frac{a}{2} & \gamma\cos\beta\frac{a}{2} \\ -\gamma\sin\alpha\frac{a}{2} & \beta\sin\beta\frac{a}{2} \end{bmatrix} \begin{Bmatrix} A_2 \\ B_1 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}, \tag{A6}$$

$$\begin{bmatrix} \alpha\sin\alpha\frac{a}{2} & -\gamma\sin\beta\frac{a}{2} \\ \gamma\cos\alpha\frac{a}{2} & \beta\cos\beta\frac{a}{2} \end{bmatrix} \begin{Bmatrix} A_1 \\ B_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}. \tag{A7}$$

Their determinants read:

$$Det(\omega)=\alpha\beta\cos\alpha\frac{a}{2}\sin\beta\frac{a}{2}+\gamma^2\sin\alpha\frac{a}{2}\cos\beta\frac{a}{2}=0, \tag{A8}$$

$$Det(\omega)=\alpha\beta\sin\alpha\frac{a}{2}\cos\beta\frac{a}{2}+\gamma^2\cos\alpha\frac{a}{2}\sin\beta\frac{a}{2}=0. \tag{A9}$$

The above frequency equations can be transformed into a simpler form as shown in Table 2, Type 1, Eqs. (b) and (c).

Relative values of the integration constants can be determined from the first or the second two equations of Eq. (45). The former gives:

$$\begin{aligned}
A_2 &= -\gamma\cos\beta\frac{a}{2}, \quad & B_1 &= \alpha\cos\alpha\frac{a}{2},\\
A_1 &= \gamma\sin\beta\frac{a}{2}, \quad & B_2 &= \beta\sin\alpha\frac{a}{2}.
\end{aligned} \tag{A10}$$

Hence, expressions for displacement, Eq. (16), are reduced to the forms:

$$\begin{aligned}
U(x) &= A_2\alpha\cos\alpha x+B_1\gamma\cos\beta x,\\
V(x) &= -A_2\gamma\sin\alpha x+B_1\beta\cos\beta x,
\end{aligned} \tag{A11}$$

and:

$$U(x) = -A_1\alpha \sin\alpha x + B_2\gamma \sin\beta x,$$
$$V(x) = -A_1\gamma \cos\alpha x - B_2\beta \cos\beta x, \tag{A12}$$

where $-\frac{a}{2} \leq x \leq \frac{a}{2}$. Natural frequencies obtained by frequency equations (A8) and (A9) are related to modes (A11) and (A12), respectively. Displacement function $U(x)$ is symmetric and $V(x)$ is antisymmetric for the former modes, and vice versa.

POTENCIJALNA TEORIJA RAVNINSKIH VIBRACIJA PRAVOKUTNIH I OKRUGLIH PLOČA

SAŽETAK

Prikazane su osnovne jednadžbe ravninskih vibracija ploča. Diferencijalne jednadžbe gibanja riješene su analitički pretpostavljajući potencijale pomaka i koristeći metodu separacije varijabli. Izvedene su frekventne jednadžbe za pravokutnu ploču slobodno oslonjenu na dvije suprotne stranice te dvije preostale stranice upete, slobodne odnosno kombinirano upeto slobodne. Dobivena su tri rješenja ovisno o rasponu vrijednosti argumenata funkcija. Diferencijalne jednadžbe gibanja okrugle ploče dobivene su transformacijom izvedenih jednadžbi za pravokutnu ploču. Pretpostavljena je harmonijska promjena potencijala pomaka u cirkularnom smjeru dok je radijalna promjena određena rješenjem formiranih Besselovih diferencijalnih jednadžbi. Izvedene su frekventne jednadžbe za upetu i slobodnu okruglu ploču, a isti postupak je korišten za analizu vibracija ploče s kružnim otvorom. Primjena razvijene potencijalne teorije je ilustrirana na primjerima pravokutne i okrugle ploče s gore navedenim rubnim uvjetima. Analitičke vrijednosti prirodnih frekvencija uspoređene su s rezultatima proračuna metodom konačnih elemenata i dobivena su vrlo dobra poklapanja rezultata analize.

Ključne riječi: *pravokutna ploča, okrugla ploča, ravninske vibracije, potencijalna teorija, analitičko rješenje, MKE rješenje.*