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# Numerical study of unsteady flow with heat transfer due to a rotating disk in porous media

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**SUMMARY**

*The unsteady flow of an incompressible viscous fluid above an infinite rotating disk is studied with heat transfer in a porous medium. The disk is started motion impulsively from rest and rotates with a constant angular velocity $\omega$. Numerical solutions of the nonlinear partial differential equations which govern the hydrodynamics and energy transfer are obtained. The effect of the porosity of the medium on the velocity and temperature distributions is considered.*

***Key words****:* *heat transfer, rotating disk, unsteady flow, porous media, numerical analysis.*

## 1. INTRODUCTION

The pioneering study of fluid flow due to an infinite rotating disk was carried out by von Karman in 1921 [1]. Von Karman gave a formulation of the problem and then introduced his famous transformations which reduced the governing partial differential equations to ordinary differential equations. Cochran [2] obtained asymptotic solutions for the steady hydrodynamic problem formulated by von Karman. Benton [3] improved Cochran's solutions and solved the unsteady problem. The problem of heat transfer from a rotating disk maintained at a constant temperature was first considered by Millsaps and Pohlhausenb [4] for a variety of Prandtl numbers in the steady state. Sparrow and Gregg [5] studied the steady state heat transfer from a rotating disk maintained at a constant temperature to fluids at any Prandtl number. The influence of an external uniform magnetic field on the flow due to a rotating disk was studied in Refs. [6-8]. The effect of uniform suction or injection through a rotating porous disk on the steady hydrodynamic or hydromagnetic flow induced by the disk was investigated in Refs. [9-11].

In the present work, the unsteady laminar flow of a viscous incompressible fluid due to the uniform rotation of a disk of infinite extent in a porous medium is studied with heat transfer. The flow in the porous media deals with the analysis in which the differential equation governing the fluid motion is based on the Darcy's law which accounts for the drag exerted by the porous medium [12-14]. The temperature of the disk is impulsively changed and then maintained at a constant value. The governing nonlinear partial differential equations are integrated numerically using the finite difference approximations with suitable coordinate transformations to remove a discontinuity between the initial and boundary conditions. The effect of the porosity of the medium on the unsteady flow and heat transfer is presented and discussed.

## 2. BASIC EQUATIONS

Let the disk lie in the plane $z = 0$ and the space $z > 0$ is equiped by a viscous incompressible fluid. The motion is due to the rotation of an insulated disk of infinite extent about an axis perpendicular to its plane with constant angular speed $\omega$ through a porous medium where the Darcy model is assumed [14]. Otherwise the fluid is at rest under pressure $p_\infty$. The equations of unsteady motion are given by:

$$\frac{\partial u}{\partial r}+\frac{u}{r}+\frac{\partial w}{\partial z}=0 \tag{1}$$

$$\rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial r}+w\frac{\partial u}{\partial z}-\frac{v^2}{r}\right)+\frac{\partial p}{\partial r}=$$
$$=\mu\left(\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}-\frac{u}{r^2}+\frac{\partial^2 u}{\partial z^2}\right)-\frac{\mu}{K}u \tag{2}$$

$$\rho\left(\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial r}+w\frac{\partial v}{\partial z}+\frac{uv}{r}\right)=$$
$$=\mu\left(\frac{\partial^2 v}{\partial r^2}+\frac{1}{r}\frac{\partial v}{\partial r}-\frac{v}{r^2}+\frac{\partial^2 v}{\partial z^2}\right)-\frac{\mu}{K}v \tag{3}$$

$$\rho\left(\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial r}+w\frac{\partial w}{\partial z}\right)+\frac{\partial p}{\partial z}=$$
$$=\mu\left(\frac{\partial^2 w}{\partial r^2}+\frac{1}{r}\frac{\partial w}{\partial r}+\frac{\partial^2 w}{\partial z^2}\right)-\frac{\mu}{K}w \tag{4}$$

where $u$, $v$ and $w$ are velocity components in the directions of increasing $r$, $\varphi$ and $z$ respectively, $p$ is denoting the pressure, $\mu$ is the coefficient of viscosity, $\rho$ is the density of the fluid, and $K$ is the Darcy permeability [12-14]. We introduce von Karman transformations [1] of the form:

$$u=r\omega F,\quad v=r\omega G,\quad w=\sqrt{\omega\nu}H,$$
$$z=\sqrt{\nu/\omega}\zeta,\quad p-p_\infty=-\rho\nu\omega P$$

where $\zeta$ is a non-dimensional distance measured along the axis of rotation, $F$, $G$, $H$ and $P$ are non-dimensional functions of $\zeta$ and $t$, and $\nu$ is the kinematic viscosity of the fluid, $\nu = \mu/\rho$. With these definitions, Eqs. (1) to (4) take the form:

$$\frac{\partial H}{\partial \zeta}+2F=0 \tag{5}$$

$$\frac{\partial F}{\partial t}-\frac{\partial^2 F}{\partial \zeta^2}+H\frac{\partial F}{\partial \zeta}+F^2-G^2+MF=0 \tag{6}$$

$$\frac{\partial G}{\partial t}-\frac{\partial^2 G}{\partial \zeta^2}+H\frac{\partial G}{\partial \zeta}+2FG+MG=0 \tag{7}$$

$$\frac{\partial H}{\partial t}-\frac{\partial^2 H}{\partial \zeta^2}+H\frac{\partial H}{\partial \zeta}-\frac{dP}{d\zeta}+MH=0 \tag{8}$$

where $M=\nu/K\omega$ is the porosity parameter. The initial and boundary conditions for the velocity problem are given by:

$$t=0,\quad F=0,\quad G=0,\quad H=0, \tag{9a}$$

$$\zeta=0,\quad F=0,\quad G=1,\quad H=0, \tag{9b}$$

$$\zeta\to\infty,\quad F\to 0,\quad G\to 0,\quad P\to 0. \tag{9c}$$

The initial conditions are given by Eq. (9a). Equation (9b) indicates the no-slip condition of viscous flow applied at the surface of the disk. Far from the surface of the disk, all fluid velocities must vanish aside the induced axial component as indicated in Eq. (9c). The above system of Eqs. (5) to (7) with the prescribed initial and boundary conditions given by Eq. (9) are sufficient to solve the three components of the flow velocity. Equation (8) can be used to solve the pressure distribution if required.

Due to the difference in temperature between the wall and the ambient fluid, heat transfer takes place. The energy equation without the dissipation terms takes the form [4-5]:

$$\rho c_p\left(u\frac{\partial T}{\partial r}+w\frac{\partial T}{\partial z}\right)-k\left(\frac{\partial^2 T}{\partial z^2}+\frac{\partial^2 T}{\partial r^2}+\frac{1}{r}\frac{\partial T}{\partial r}\right)=0 \tag{10}$$

where $T$ is the temperature of the fluid, $c_p$ is the specific heat at constant pressure of the fluid, and $k$ is the thermal conductivity of the fluid. The initial and boundary conditions for the energy problem are that the temperature is changed impulsively from rest and, by continuity considerations, it equals $T_w$ at the surface of the disk. At large distances from the disk, $T$ tends to $T_\infty$ where $T_\infty$ is the temperature of the ambient fluid. In terms of the non-dimensional variable $\theta = (T-T_\infty)/(T_w-T_\infty)$ and using von Karman transformations, Eq. (10) takes the form:

$$\frac{\partial \theta}{\partial t}-\frac{1}{Pr}\frac{\partial^2 \theta}{\partial \zeta^2}+H\frac{\partial \theta}{\partial \zeta}=0 \tag{11}$$

where $Pr$ is the Prandtl number, $Pr = c_p\,\mu/k$. The initial and boundary conditions in terms of $\theta$ are expressed as:

$$\theta(0,\zeta)=0,\quad \theta(t,0)=1,\quad \theta(t,\infty)=0 \tag{12}$$

The significant velocity and temperature variations in the fluid are confined to the region adjacent to the disk which constructs viscous as well as thermal boundary layers. We define the thickness of these layers by certain standard measures [5]. The first of these is the displacement thickness. Since the radial flow is zero both at the disk surface and at infinity, a radial displacement thickness would have very little meaning. Then, for the tangential direction, we define a displacement thickness as [5]:

$$\delta_\varphi = \int_0^\infty G d\zeta$$

In physical terms, $\delta_\varphi$ gives the thickness of a fictitious layer of fluid which is rotating at a uniform tangential velocity $r\omega$ and is carrying a tangential mass flow equal to that carried by the actual tangential velocity distribution.

Also, as a measure of the extent of the thermal layer, we may introduce a thermal thickness based on the temperature excess ($T$ - $T_\infty$) above the ambient fluid. Then:

$$\delta_\theta = \int_0^\infty \theta d\zeta$$

Physically speaking, $\delta_\theta$ is the thickness of a fictitious fluid layer at temperature $T_w$ whose integrated temperature excess over $T_\infty$ is identical to that of the actual temperature distribution.

The heat transfer from the disk surface to the fluid is computed by application of Fourier's law:

$$Q = -k\left(\frac{dT}{dz}\right)_w$$

Introducing the transformed variables, the expression for $Q$ becomes:

$$Q = -k(T_w - T_\infty)\sqrt{\frac{\omega}{\nu}}\frac{d\theta(t,0)}{d\zeta}$$

By rephrasing the heat transfer results in terms of a Nusselt number defined as:

$$N_u = Q\sqrt{\omega/\nu}\,/\,k(T_w - T_\infty)$$

the last equation becomes:

$$N_u = -\frac{d\theta(t,0)}{d\zeta}$$

Numerical solution for the governing nonlinear Eqs. (5) to (7) with the conditions given by Eq. (9), using the finite differences, leads to a numerical oscillation problem resulting from the discontinuity between the initial and boundary conditions (9a) and (9b). The same discontinuity occurs between the initial and boundary conditions for the energy problem (see Eq. (12)). A solution for this numerical problem is achieved by using proper coordinate transformations, as suggested by Ames [15] for similar problems. Expressing Eqs. (5) to (7) and (11) in terms of the modified coordinate $\eta = \zeta / 2\sqrt{t}$ we get:

$$\frac{\partial H}{\partial \eta} + 4\sqrt{t}F = 0 \tag{13}$$

$$\frac{\partial F}{\partial t} - \frac{\eta}{2t}\frac{\partial F}{\partial \eta} - \frac{1}{4t}\frac{\partial^2 F}{\partial \eta^2} + \frac{1}{2\sqrt{t}}H\frac{\partial F}{\partial \eta} + F^2 - G^2 + MF = 0 \tag{14}$$

$$\frac{\partial G}{\partial t} - \frac{\eta}{2t}\frac{\partial G}{\partial \eta} - \frac{1}{4t}\frac{\partial^2 G}{\partial \eta^2} + \frac{1}{2\sqrt{t}}H\frac{dG}{d\zeta} + 2FG + MG = 0 \tag{15}$$

$$\frac{\partial \theta}{\partial t} - \frac{\eta}{2t}\frac{\partial \theta}{\partial \eta} - \frac{1}{4t\,Pr}\frac{\partial^2 \theta}{\partial \eta^2} + \frac{1}{2\sqrt{t}}H\frac{\partial \theta}{\partial \eta} = 0 \tag{16}$$

Equations (13) to (16) represent a coupled system of non-linear partial differential equations which are solved numerically under the initial and boundary conditions (9) and (12) using the finite difference approximations. A linearization technique is first applied to replace the nonlinear terms at a linear stage, with the corrections incorporated in subsequent iterative steps until convergence is reached. Then the Crank-Nicolson implicit method is used at two successive time levels [15]. An iterative scheme is used to solve the linearized system of difference equations. The solution at a certain time step is chosen as an initial guess for next time step and the iterations are continued till convergence, within a prescribed accuracy. Finally, the resulting block tri-diagonal system is solved using the generalized Thomas-algorithm [15]. Finite difference equations relating the variables are obtained by writing the equations at the mid point of the computational cell and then replacing the different terms by their second order central difference approximations in the $\eta$-direction. The diffusion terms are replaced by the average of the central differences at two successive time-levels. The computational domain is divided into meshes each of dimension $\Delta t$ and $\Delta\eta$ in time and space respectively. The modified Eqs. (13) to (16) are integrated from $t = 0$ to $t = 1$. Then, the solution obtained at $t = 1$ is used as the initial condition for integrating Eqs. (5) to (7) and (11) from $t = 1$ towards the steady state.

The resulting system of equations has to be solved in the infinite domain $0 < \zeta < \infty$. A finite domain in the $\zeta$-direction can be used instead with $\zeta$ chosen large enough to ensure that the solutions are not affected by imposing the asymptotic conditions at a finite distance. The independence of the results from the length of the finite domain as well as the grid density was ensured and successfully checked by various trial and error numerical experimentations. Computations are carried out for $\zeta_\infty = 10$ and step size $\Delta\zeta = 0.04$ which are found adequate for the ranges of the parameters studied here. Larger finite distance or smaller step size do not show any significant change in the results. Convergence of the scheme is assumed when all of the variables $F$, $G$, $H$, $\theta$, $\delta F/\delta\zeta$, $\delta G/\delta\zeta$, and $\delta\theta/\delta\zeta$ for the last two approximations differs from unity by less than $10^{-6}$ for all values of $\zeta$ in $0 < \zeta < 10$ and all $t$.

## 3. RESULTS AND DISCUSSION

Figures 1 and 2 present the time growth of the azimuthal, radial, and vertical velocity profiles towards the steady state, respectively, in the cases *M = 0* and *M = 1*. The figures indicate that the vertical velocity component reaches the steady state slower than the radial velocity component and much slower than the azimuthal velocity component. This is due to the fact that the centrifugal effect is the source of the radial motion which is the source of the vertical motion. Comparison between Figures 1 and 2 shows the restraining effect of the porosity of the medium on the flow and its influence on reducing the time required for the velocity profiles to approach their steady state profiles. It is also clear from Figure 1 that the velocity components *F* and *H* do not reach their steady state profiles monotonically with time. With time progress, both *F* and *H* decrease for small values of $\zeta$ (near the disk) and increase for large $\zeta$. This accounts for the crossing of each of the *F* and *H* profiles with time which is more apparent for *H* than for *F*. Figure 2 indicates the marked effect of the porosity on pushing the crossover occuring in both *F* and *H* profiles far from the disk.

Figure 3 presents the growth of the profile of the temperature $\theta$ with time for the cases *M = 0* and *M = 1*, respectively and for *Pr = 0.7*. It is shown in the figure that $\theta$ reaches the steady state monotonically. Also the figure indicates the influence of increasing the porosity parameter on increasing $\theta$ as a result of the effect of the porosity in preventing the fluid at near-ambient temperature from reaching the surface of the disk.

Figure 4 presents the time variation of the displacement thickness $\delta_\varphi$ and the thermal thickness $\delta_\theta$, the Nusselt number *Nu* respectively, for various values of the porosity parameter *M* and for *Pr = 0.7*. It is concluded from Figure 4a that increasing *M* decreases $\delta_\varphi$ and its steady state time due to its restraining effect on *G*. On the other hand, as shown in Figure 4b, increasing *M* increases $\delta_\theta$, and its steady state time as a result of increasing the temperature $\theta$. It is clear from Figure 4c that increasing *M*, which decreases the axial flow towards the disk, decreases *Nu* since the absence of the fluid at near-ambient temperature close to the surface of the disk increases the heat transfer. For small values of *t*, *Nu* increases with time until it reaches a maximum value which does not depend greatly on *M* (for small values of *M*) due to the very small variation in $\theta$. With time progress, the variation in $\theta$ with *M* increases and then *Nu* decreases.

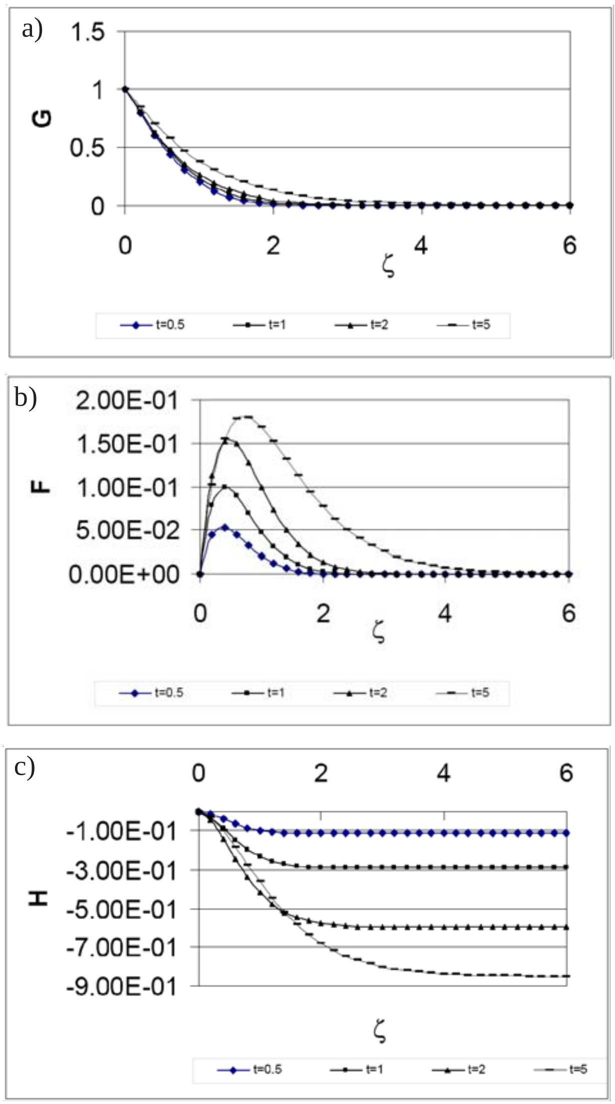


*Fig. 1 Time variation of the velocity profiles for M = 0*

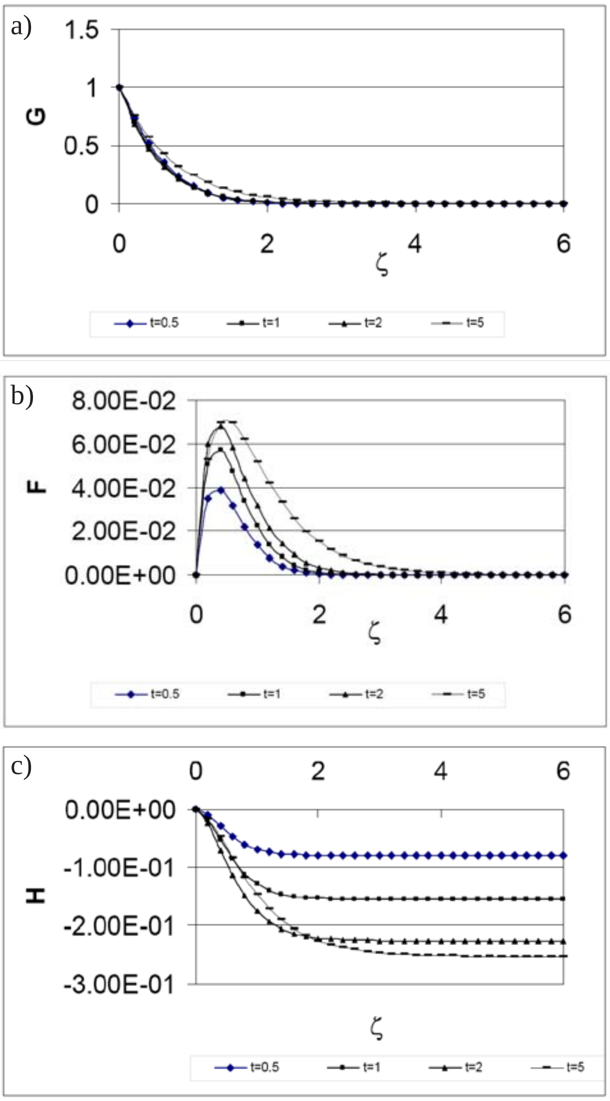


Fig. 2 Time variation of the velocity profiles for *M* = 1

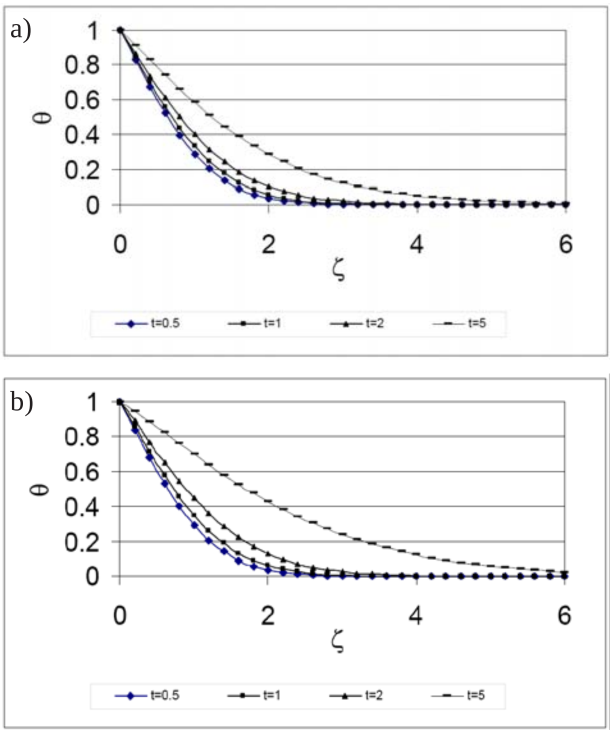


Fig. 3 Time variation of the profiles *θ* for: (a) *M* = 0; (b) *M* = 1

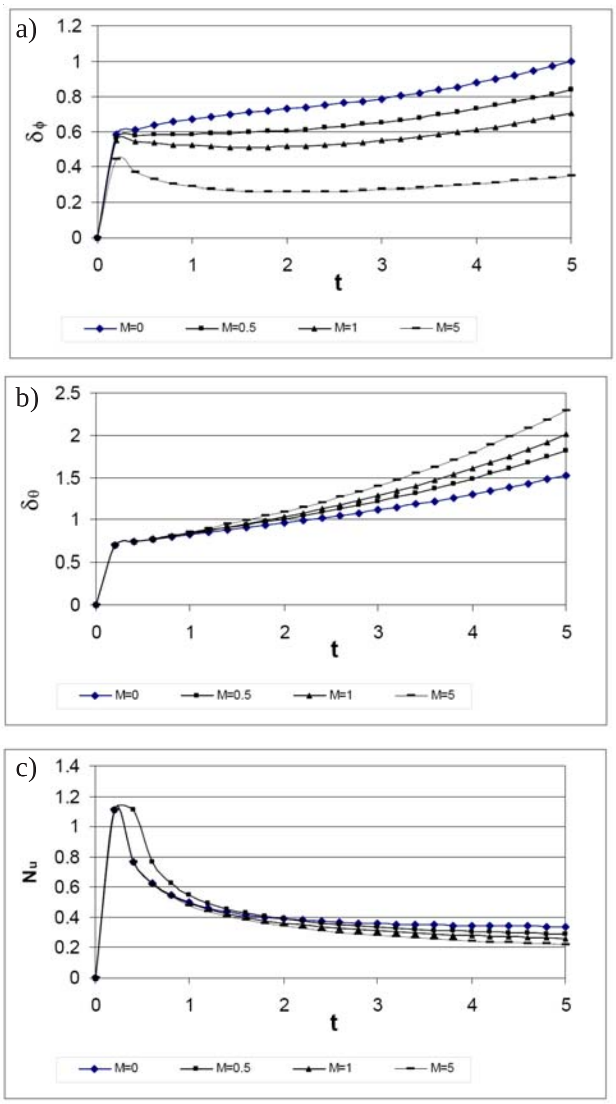


Fig. 4 Effect of *M* on the time variation of: (a) $\delta_\varphi$; (b) $\delta_\theta$; and (c) *Nu*

## 4. CONCLUSION

In this study the unsteady flow induced by a rotating disk with heat transfer in a porous medium was studied. A proper coordinate transformation is used to remove the numerical oscillations resulting from the discontinuity between the initial and boundary conditions. The results indicate the restraining effect of the porosity on the unsteady flow. On the other hand, increasing the porosity parameter increases the temperature of the fluid. It is observed that the radial and vertical components of the velocity do not reach their steady state profiles monotonically which accounts for crossing of the charts of these velocity components with time. The porosity of the medium has an interesting effect on pushing the crossing points far from the disk.

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## NUMERIČKA STUDIJA NERAVNOMJERNOG TOKA S PRIJENOSOM TOPLINE ZBOG ROTIRAJUĆEG DISKA U POROZNOM MEDIJU


### SAŽETAK

*Proučava se neravnomjeran tok nestlačive viskozne tekućine iznad beskonačnog rotirajućeg diska s prijenosom topline u poroznom mediju. Disk se počeo naglo gibati iz mirnog stanja, a okreće se konstantnom kutnom brzinom $\omega$. Dobila su se numerička rješenja nelinearnih parcijalnih diferencijalnih jednadžbi koja opisuju hidrodinamiku i prijenos energije. Razmatra se i djelovanje poroznosti medija na brzinu i raspodjelu temperature.*


**Ključne riječi**: *prijenos topline, rotirajući disk, neravnomjerni tok, porozni medij, numerička analiza.*