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Global existence and boundedness of a certain nonlinear vector integro-differential equation of second order with multiple deviating arguments

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Abstract. In this study, we consider a vector integro-differential equation with multiple deviating arguments. Based on the Lyapunov-Krasovskii functional approach, the global existence and boundedness of all solutions are discussed. We give an example to illustrate the theoretical analysis made in this study and to show the effectiveness of the method used here.
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1. Introduction

The complexity of delay differential equations, the difficulties of their theoretical study, and their wide-ranging applications have attracted much interest in the community of applied mathematics, physics and biology. Also, delay differential equations have been used for many years in control theory and only a couple of decades ago they were widely applied to biological models. However, it should be expressed that the number of results related to the qualitative behavior of solutions of certain nonlinear vector delay differential equations is very few in comparison to that on nonlinear scalar delay differential equations. As is well known, the investigation of qualitative properties of solutions (stability, global existence, boundedness, convergence, instability, asymptotic behaviour of solutions and so on) of delay differential equations is a very important problem in the theory and application of differential equations. On the other hand, the global existence and boundedness of solutions of delay differential equations are among the most attractive topics in the qualitative theory of differential equations due to their applications. Numerous research activities are concerned with the qualitative properties of solutions to different ordinary nonlinear scalar and vector differential equations of high order with and without delay. For some related contributions, we refer the reader to the books or the papers of Ahmad and Rama Mohana Rao [2], Burton [3], Burton and Zhang [4], El'sgol'ts [5], Gao and Zhao [6], Hara and Yoneyama [7], Huang and Yu [8], Jitsuro and Yusuke [10], Kato [11], Kolmanovskii and Myshkis [12], Krasovskii [13], Luk [14], Napoles

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Valdes [17], Sugie and Amano [18], Tunç [19]-[25] Tunç and Şevli [26], Tunç and Ayhan [27], Yang [28], Wei and Huang [29], Wiandt [1], Zhang [30], Zhou and Liu [31], and the references therein. Throughout most of the results presented in the books and the papers mentioned above, Lyapunov's second (or direct) method (Lyapunov [15]), has been used as a basic tool to verify the results established in these works.

We give some background details regarding the study of various classes of delay vector differential equations of second order. In this sense, it should be mentioned that, in 2013, for the cases $P = 0$ and $P \neq 0$, respectively, Tunç [24] studied asymptotic stability of the zero solution and boundedness of all solutions of a vector Lienard differential equation with a constant deviating argument, $\tau > 0$,

$$X''(t) + F(X(t), X'(t))X'(t) + H(X'(t-\tau)) = P(t).$$

Later, in 2013, Tunç [25] took into account the following nonlinear vector Lienard differential equation with multiple deviating arguments, $\tau_i > 0$:

$$X''(t) + F(X(t), X'(t))X'(t) + G(X(t)) + \sum_{i=1}^{n} H_i(X(t-\tau_i)) = P(t),$$

for which the author obtained asymptotic stability of the zero solution and boundedness of all solutions of this equation and the main results of this paper were proved by the application of an auxiliary Lyapunov-Krasovskii functional.

In this study, we consider the vector integro-differential equation of second order with multiple constant deviating arguments, $\tau_i > 0$:

$$\begin{aligned}(r(t)X')' + A(t)F(X,X')X' + B(t)E(X') + \sum_{i=1}^{n} C_i(t)H_i(X(t-\tau_i)) \\ = \int_0^t K(t,s)X'(s)ds,\end{aligned} \tag{1}$$

where $t \in \Re^+$, $\Re^+ = [0,\infty)$, $t - \tau_i \geq 0$ and $X \in \Re^n$; r is a positive and continuously differentiable increasing function on $\Re^+$; A, B, C_i and F are $n \times n-$symmetric continuous matrix functions; $E : \Re^n \to \Re^n$, $H_i : \Re^n \to \Re^n$ are continuous and E, H_i are differentiable with $E(0) = H_i(0) = 0$. Also, $K(t,s)$ is an $n \times n-$continuous matrix function for $0 \leq t \leq s < \infty$.

Instead of equation (1), throughout this article we consider a equivalent differential system form

$$\begin{aligned}X' &= Y \\ Y' &= \frac{1}{r(t)}\int_0^t K(t,s)Y(s)ds - \frac{r'(t)}{r(t)}Y - \frac{1}{r(t)}A(t)F(X,Y)Y - \frac{1}{r(t)}B(t)E(Y) \\ &\quad - \frac{1}{r(t)}\sum_{i=1}^{n} C_i(t)H_i(X) + \frac{1}{r(t)}\sum_{i=1}^{n} C_i(t)\int_{t-\tau_i}^{t} J_{H_i}(X(s))Y(s)ds,\end{aligned} \tag{2}$$

which was obtained as usual by setting $X' = Y$ in equation (1), and also where $X(t)$ and $Y(t)$ are abbreviated as X and Y throughout the paper, respectively.

To the best of our knowledge, the global existence and boundedness of solutions of equation (1) have not been discussed in the literature. This case displays the originality of the present paper.

The motivation for the present work has been inspired basically by the papers of Tunç [24, 25], Tunç and Ayhan [27] and the references listed therein. The main aim of this paper is to give some sufficient conditions for the global existence and boundedness of solutions of equation (1) by the construction of a new Lyapunov functional for this equation. This paper is also the first attempt to investigate the global existence and boundedness of solutions of vector integro-differential equations of second order with multiple constant deviating arguments, it is a new improvement and has a contribution to the subject in the literature; on the other hand, this paper may also be beneficial to researchers working on the qualitative behavior of solutions of scalar and vector integro-differential equations. Besides, the result obtained in this investigation improves the existing results on second order nonlinear scalar and vector integro-differential equations in the literature.

Let $J_E(Y)$ and $J_{H_i}(X)$ denote Jacobian matrices corresponding to the functions $E(Y)$ and $H_i(X)$, that is,

$$J_E(Y) = (\frac{\partial e_i}{\partial y_j}),\ J_{H_i}(Y) = (\frac{\partial h_{1i}}{\partial x_j}), \ldots, J_{Hn}(Y) = (\frac{\partial h_{ni}}{\partial x_j}), \quad i, j = 1, 2, \ldots, n,$$

where $(x_1, x_2, \ldots, x_n)$, $(y_1, y_2, \ldots, y_n)$, $(e_1, e_2, \ldots, e_n)$ and $(h_{i1}, h_{i2}, \ldots, h_{in})$ are the components of X, Y, E and H_i, respectively. Otherwise, it supposed that the derivative $\frac{d}{dt}C_i(t) = C_i'(t)$ and the Jacobian matrix $J_{H_i}(X)$ exist and are continous. By the same token, it is also assumed that all matrices given in the pairs $A(t)$, $F(X;Y)$; $B(t)$, $J_E(Y)$; $C_i(t)$, $J_{H_i}(X)$; and $C_i'(t)$, $J_{H_i}(X)$ are symmetric and commute with each other. Furthermore, the symbol $\langle X, Y\rangle$ corresponding to any pair X and Y in $\Re^n$ stands for the usual scalar product $\sum_{i=1}^n x_i y_i$, that is,

$$\langle X, Y\rangle = \sum_{i=1}^n x_i y_i;$$

thus

$$\langle X, X\rangle = \|X\|^2$$

and also $\lambda_i(\Psi)$ $(i = 1, 2, \ldots, n)$, are the eigenvalues of the $n \times n$-matrix Ψ.

In addition to the basic assumptions imposed on A, F, B, E, C_i and H_i, that appear in Eq. (1), we assume that there exist some positive constants a, b, c_i, f, ε, δ_i and β_i such that the following conditions hold:

(A1) The matrices A, B, C_i and C_i' are symmetric, also $\lambda_i(A(t)) \geq a$, $\lambda_i(B(t)) \geq b$, $\lambda_i(C_i(t)) \geq c_i$, and $\lambda_i(C_i'(t)) \leq 0$, for all $t \in [0, \infty)$;

(A2) $F(X, Y)$ is $n \times n-$symmetric and $\lambda_i(F(X, Y)) \geq f$ for all $t \in [0, \infty)$ and $X, Y \in \Re^n$;

(A3) $J_E(Y)$ is symmetric and $\lambda_i(J_E(Y)) \geq \varepsilon$ for all $Y \in \Re^n$;

(A4) $J_{H_i}(X)$ are symmetric and $\delta_i \leq \lambda_i(J_{H_i}(X)) \leq \beta_i$ for all $X \in \Re^n$;

(A5) $\frac{1}{r(t)} \leq 1$;

(A6) $\int\limits_0^t \|K(t,s)\|\, ds + \int\limits_t^\infty \|K(u,t)\|\, du \leq R$;

(A7) $R + 2\sigma\tau^* - \frac{r'(t)}{r(t)} \leq 0$, ($\sigma$ and τ^* to be determined later on p. 9).

2. Preliminaries

Before beginning with our main result, we give some well known preliminary results which will be required in the proof of our main result. Consider a nonautonomous differential system

$$\frac{dx}{dt} = F(t,x), \tag{3}$$

where x is an $n-$vector, $t \in [0, \infty)$. Suppose that $F(t,x)$ is continuous in (t,x) on D , where D is a connected open set in $\Re \times \Re^n$. Now, we shall give the following theorem and the lemmas.

Theorem 1. *Let $F \in C(D)$ and $|F| \leq M$ on D. Suppose that φ is a solution of (3) on the interval $j = (\alpha, \beta)$ such that the following conditions hold:*

- (i) *The two limits $\lim\limits_{t \to \alpha^+} \varphi(t) = \varphi(\alpha^+)$ and $\lim\limits_{t \to \beta^-} \varphi(t) = \varphi(\beta^-)$ exist;*
- (ii) *$(\alpha, \varphi(\alpha^+))$ (i.e., $(\beta, \varphi(\beta^-))$) is in D.*

Then the solution φ can be continued to the right pass the point $t = \beta$ (i.e., to the left pass the point $t = \varphi$).

Proof. See Hsu [9]. □

Lemma 1. *Let S be a real symmetric $n \times n$ matrix and $\overline{s} \geq \lambda_i(S) \geq s > 0$, $(i = 1, 2, \ldots, n)$, where $\overline{s}$ and s are constants. Then*

$$\overline{s}\left\langle X, X\right\rangle \geq \left\langle SX, X\right\rangle \geq s\left\langle X, X\right\rangle$$

and

$$\overline{s}^2\left\langle X, X\right\rangle \geq \left\langle SX, SX\right\rangle \geq s^2\left\langle X, X\right\rangle.$$

Proof. See Mirsky [16]. □

Lemma 2. *Let M, N be any two real $n \times n$ commuting symmetric matrices. Then*

- (i) *The eigenvalues $\lambda_i(MN)$, $(i = 1, 2, \ldots, n)$ of the product matrix MN are real and satisfy*

$$\max_{1 \leq j,k \leq n} \lambda_j(M)\lambda_k(N) \geq \lambda_i(MN) \geq \min_{1 \leq j,k \leq n} \lambda_j(M)\lambda_k(N).$$

(ii) The eigenvalues $\lambda_i(M+N)$, $(i=1,2,\ldots,n)$ *of the sum matrix* M *and* N *are real and satisfy*

$$\left\{\max_{1\leq j,k\leq n}\lambda_j(M)+\max_{1\leq j,k\leq n}\lambda_k(N)\right\}$$
$$\geq \lambda_i(M+N)\geq\left\{\min_{1\leq j,k\leq n}\lambda_j(M)+\min_{1\leq j,k\leq n}\lambda_k(N)\right\},$$

where $\lambda_j(M)$ *and* $\lambda_k(N)$ *are, the eigenvalues* M *and* N, *respectively.*

Proof. See Mirsky [16]. □

3. Main result

Theorem 2. *Suppose that conditions (A1) - (A7) hold. Then every solution of system (2) are continuable and bounded.*

Proof. Now, throughout our main result, as a basic tool we will use a continuously differentiable Lyapunov functional $V=V(t,X,Y)$, defined by:

$$\begin{aligned} V=&\frac{1}{2}\langle Y,Y\rangle+\frac{1}{r(t)}\sum_{i=1}^{n}\int_0^1\langle C_i(t)H_i(\sigma X),X\rangle\,d\sigma+\sum_{i=1}^{n}\sigma_i\int_{-\tau_i}^{0}\int_{t+s}^{t}\|Y(u)\|^2\,du\;ds\\ &+\int_0^t\int_t^\infty\|K(u,s)\|\,\|Y(u)\|^2\,du\;ds,\end{aligned}\tag{4}$$

where s is a real variable such that the integrals

$$\int_{-\tau_i}^{0}\int_{t+s}^{t}\|Y(u)\|^2\,du\;ds$$

and

$$\int_0^t\int_t^\infty\|K(u,s)\|\,\|Y(u)\|^2\,du\;ds$$

are non-negative, and σ_i are certain positive constants to be determined later in the proof.

It is clear that $V(t,0,0)=0$. Next, since $H_i(0)=0$, $\frac{\partial}{\partial\sigma}H_i(\sigma X)=J_{H_i}(\sigma X)X$, then we can write

$$H_i(X)=\int_0^1 J_{H_i}(\sigma X)X\,d\sigma.$$

From assumptions (A1) and (A4) we get

$$\begin{aligned}\sum_{i=1}^{n}\int_{0}^{1}\langle C_i(t)H_i(\sigma X),X\rangle\,d\sigma &= \sum_{i=1}^{n}\int_{0}^{1}\int_{0}^{1}\langle \sigma_1 C_i(t)J_{H_i}(\sigma_1\sigma_2 X)X,X\rangle\,d\sigma_2 d\sigma_1\\ &\geq \sum_{i=1}^{n}\int_{0}^{1}\int_{0}^{1}\langle \sigma_1 c_i\delta_i X,X\rangle\,d\sigma_2 d\sigma_1\\ &= \frac{1}{2}\left(\sum_{i=1}^{n}c_i\delta_i\right)\|X\|^2, \end{aligned} \tag{5}$$

and

$$\begin{aligned}&\frac{d}{dt}\sum_{i=1}^{n}\int_{0}^{1}\langle C_i(t)H_i(\sigma X),X\rangle\,d\sigma\\ &\quad= \sum_{i=1}^{n}\int_{0}^{1}\sigma\langle C_i(t)J_{H_i}(\sigma X)Y,X\rangle\,d\sigma + \sum_{i=1}^{n}\int_{0}^{1}\langle C_i(t)H_i(\sigma X),Y\rangle\,d\sigma\\ &\qquad+\sum_{i=1}^{n}\int_{0}^{1}\langle C_i'(t)H_i(\sigma X),X\rangle\,d\sigma\\ &\quad= \sum_{i=1}^{n}\int_{0}^{1}\sigma\frac{\partial}{\partial\sigma}\langle C_i(t)H_i(\sigma X),Y\rangle\,d\sigma + \sum_{i=1}^{n}\int_{0}^{1}\langle C_i(t)H_i(\sigma X),Y\rangle\,d\sigma\\ &\qquad+\sum_{i=1}^{n}\int_{0}^{1}\langle C_i'(t)H_i(\sigma X),X\rangle\,d\sigma\\ &\quad= \sum_{i=1}^{n}\sigma\langle C_i(t)H_i(\sigma X),Y\rangle\big|_0^1 + \sum_{i=1}^{n}\int_{0}^{1}\langle C_i'(t)H_i(\sigma X),X\rangle\,d\sigma\\ &\quad= \sum_{i=1}^{n}\langle C_i(t)H_i(X),Y\rangle + \sum_{i=1}^{n}\int_{0}^{1}\langle C_i'(t)H_i(\sigma X),X\rangle\,d\sigma.\end{aligned}$$

In view of function (4) and inequality (5) together, it follows that

$$V \geq \frac{1}{2}\|Y\|^2 + \frac{1}{2r(t)}\left(\sum_{i=1}^{n}c_i\delta_i\right)\|X\|^2 \geq 0. \tag{6}$$

Thus, the function V defined by expression (4) is positive definite. Calculating the derivative of the function V along any solution $(X(t),Y(t))$ of system (2) we have that

$$
\begin{aligned}
V' =& \frac{1}{r(t)} \int_0^1 \langle K(t,s)Y(s), Y(t)\rangle \, ds - \frac{r'(t)}{r(t)} \langle Y, Y\rangle - \frac{1}{r(t)} \langle A(t)F(t,X,Y)Y, Y\rangle \\
& - \frac{1}{r(t)} \langle B(t)E(Y), Y\rangle + \frac{1}{r(t)} \sum_{i=1}^{n} C_i(t) \int_{t-\tau_i}^{t} \langle J_{H_i}(X(s))Y(s), Y(t)\rangle \, ds \\
& + \sum_{i=1}^{n} \int_0^1 \langle C_i'(t)H_i(\sigma X), X\rangle \, d\sigma - \frac{r'(t)}{r^2(t)} \sum_{i=1}^{n} \int_0^1 \langle C_i(t)H_i(\sigma X), X\rangle \, d\sigma \\
& + \frac{d}{dt} \sum_{i=1}^{n} \sigma_i \int_{-\tau_i}^{0} \int_{t+s}^{t} \|Y(u)\|^2 \, ds du + \frac{d}{dt} \int_0^t \int_t^{\infty} \|K(t,s)\| \, \|Y(u)\|^2 \, ds du.
\end{aligned} \tag{7}
$$

We also remind that

$$
\frac{d}{dt} \sum_{i=1}^{n} \sigma_i \int_{-\tau_i}^{0} \int_{t+s}^{t} \|Y(u)\|^2 \, du \ ds = \sum_{i=1}^{n} \sigma_i \tau_i \|Y(t)\|^2 - \sum_{i=1}^{n} \sigma_i \int_{t-\tau_i}^{t} \|Y(s)\|^2 \ ds
$$

and

$$
\frac{d}{dt} \int_0^t \int_t^{\infty} \|K(u,s)\| \, \|Y(u)\|^2 \, du \ ds = \|Y(t)\|^2 \int_t^{\infty} \|K(u,t)\| \, du - \int_0^t \|K(u,s)\| \, \|Y(s)\|^2 \, ds.
$$

These equalities and (7) lead to the following

$$
\begin{aligned}
V' = & \frac{1}{r(t)} \int_0^1 \langle K(t,s)Y(s), Y(t)\rangle \, ds - \frac{r'(t)}{r(t)} \langle Y, Y\rangle - \frac{1}{r(t)} \langle A(t)F(t,X,Y)Y, Y\rangle \\
& - \frac{1}{r(t)} \langle B(t)E(Y), Y\rangle + \frac{1}{r(t)} \sum_{i=1}^{n} C_i(t) \int_{t-\tau_i}^{t} \langle J_{H_i}(X(s))Y(s), Y(t)\rangle \, ds \\
& + \frac{1}{r(t)} \sum_{i=1}^{n} \int_0^1 \langle C_i'(t)H_i(\sigma X), X\rangle \, d\sigma - \frac{r'(t)}{r^2(t)} \sum_{i=1}^{n} \int_0^1 \langle C_i(t)H_i(\sigma X), X\rangle \, d\sigma \\
& + \sum_{i=1}^{n} \sigma_i \tau_i \|Y(t)\|^2 - \sum_{i=1}^{n} \sigma_i \int_{t-\tau_i}^{t} \|Y(s)\|^2 \ ds \\
& + \|Y(t)\|^2 \int_t^{\infty} \|K(u,t)\| \, du - \int_0^t \|K(u,s)\| \, \|Y(s)\|^2 \, ds.
\end{aligned}
$$

By assumptions $(A1)-(A5)$, Lemma 1, Lemma 2, and the inequalities $\langle U,V\rangle \leq \|UV\| \leq \|U\|\,\|V\| \leq \frac{1}{2}(\|U\|^2+\|V\|^2)$, the following estimates can be verified:

$$\begin{aligned}\frac{1}{r(t)}\int_0^t \langle Y(t),K(t,s)Y(s)\rangle\,ds &\leq \int_0^t \|Y(t),K(t,s)Y(s)\|\,ds\\ &\leq \int_0^t \|K(t,s)\|\,\|Y(t)\|\,\|Y(s)\|\,ds\\ &\leq \|Y(t)\|^2\int_0^t \|K(t,s)\|\,ds\\ &\quad+\int_0^t \|K(t,s)\|\,\|Y(s)\|^2\,ds,\end{aligned}$$

$$\frac{1}{r(t)}\langle A(t)F(X,Y)Y,Y\rangle \geq \frac{af}{r(t)}\|Y\|^2 \geq 0,$$

$$\langle B(t)E(Y),Y\rangle=\int_0^1\langle B(t)J_E(\sigma Y)Y,Y\rangle\,d\sigma \geq b\varepsilon\|Y\|^2 \geq 0,$$

$$\frac{1}{r(t)}\sum_{i=1}^n C_i(t)\int_{t-\tau_i}^t\langle J_{H_i}(X(s))Y(s),Y(t)\rangle\,ds \leq \sum_{i=1}^n c_i\beta_i\tau_i\|Y(t)\|^2+\sum_{i=1}^n c_i\beta_i\int_{t-\tau_i}^t\|Y(s)\|^2\,ds,$$

$$\frac{r'(t)}{r^2(t)}\sum_{i=1}^n\int_0^1\langle C_i(t)H_i(\sigma X),X\rangle\,d\sigma \geq \frac{r'(t)}{2r^2(t)}\sum_{i=1}^n c_i\delta_i\|X\|^2 \geq 0,$$

and

$$\frac{1}{r(t)}\sum_{i=1}^n\int_0^1\langle C_i'(t)H_i(\sigma X),X\rangle\,d\sigma \leq 0.$$

From these estimates and assumptions $(A6)$ we have that

$$\begin{aligned}V' \leq{}& \left(R-\frac{r'(t)}{r(t)}\right)\|Y(t)\|^2+\sum_{i=1}^n(c_i\beta_i\tau_i+\sigma_i\delta_i)\|Y(t)\|^2+\sum_{i=1}^n(c_i\beta_i\\ &+\sigma_i)\int_{t-\tau_i}^t\|Y(s)\|^2\,ds.\end{aligned}$$

Let

$$\tau^*=\max\{\tau_1,\tau_2,\ldots,\tau_n\}$$

and

$$\sigma = \sum_{i=1}^{n} \sigma_i = \sum_{i=1}^{n} c_i \beta_i.$$

Therefore, in view of the discussion and $(A7)$, we can finalize that

$$V' \leq \left(R + 2\sigma\tau^* - \frac{r'(t)}{r^2(t)} \right) \|Y(t)\|^2 \leq 0.$$

This implies that Lyapunov functional V is decreasing along the trajectories of system (2).

Since all the functions appearing in equation (1) are continuous, then it is obvious that there exists at least a solution of equation (1) defined on $[t_0, t_0 + \delta]$ for some $\delta > 0$. We need to show that the solution can be extended to the entire interval $[t_0, \infty)$. We assume on the contrary that there is a first time $T < \infty$ such that the solution exists on $[t_0, T)$ and

$$\lim_{t \to T^-} (\|X\| + \|Y\|) = \infty. \tag{8}$$

Let $(X(t), Y(t))$ be such a solution of system (2) with the initial condition (X_0, Y_0). Since $V(t)$ is a positive and decreasing function on the trajectories of system (2), then in view of inequality (6), we get

$$\frac{1}{2} \|Y(T)\|^2 + \frac{1}{2r(T)} \left(\sum_{i=1}^{n} c_i \delta_i \right) \|X(T)\|^2 \leq V_0,$$

where $V_0 = V(t_0, X_0, Y_0)$. From the last inequality we have that there exists a positive constant L such that $\|X(T)\| \leq L$ and $\|X(T)\| \leq L$ as $t \to T^-$. Hence, we conclude that $T < \infty$ is not possible, we must have $T = \infty$. This completes the proof of the theorem. □

Example 1. *As a special case of Eq. (1), let us take for $n = 2$ that*

$$r(t) = e^{3t},$$

$$A(t) = \begin{bmatrix} 1 + 2t^2 & t^2 \\ t^2 & 1 + 2t^2 \end{bmatrix}, \; F(X, Y) = \begin{bmatrix} 2 + \frac{1}{1 + x_1^2 + y_1^2} & 0 \\ 0 & 2 + \frac{1}{1 + x_1^2 + y_1^2} \end{bmatrix},$$

$$B(t) = \begin{bmatrix} 2 + t^2 & 0 \\ 0 & 2 + t^2 \end{bmatrix}, \; E(Y) = \begin{bmatrix} y_1^3 + 2y_1 \\ y_2^3 + 2y_2 \end{bmatrix},$$

$$C_1(t) = \begin{bmatrix} 2 + 2e^{-2t} & e^{-2t} \\ e^{-2t} & 2 + 2e^{-2t} \end{bmatrix}, \; H_1(X) = \begin{bmatrix} x_1(t - 0.01) + \arctan(x_1(t - 0.01)) \\ x_2(t - 0.01) + \arctan(x_2(t - 0.01)) \end{bmatrix},$$

$$C_2(t) = \begin{bmatrix} 2 + 2e^{-3t} & e^{-3t} \\ e^{-3t} & 2 + 2e^{-3t} \end{bmatrix}, \; H_2(X) = \begin{bmatrix} x_1(t - 0.02) + \arctan(x_1(t - 0.02)) \\ x_2(t - 0.02) + \arctan(x_2(t - 0.02)) \end{bmatrix},$$

and

$$K(t,s)=\begin{bmatrix}\frac{2t}{(t^2+1)^2} & 0\\ 0 & \frac{2t}{(t^2+1)^2}\end{bmatrix}.$$

It is obvious that

$$\frac{1}{r(t)}=\frac{1}{e^{3t}}\le 1 \text{ and } \frac{r'(t)}{r(t)}=\frac{3e^{3t}}{e^{3t}}=3,$$

also clearly, $A(t)$ and $F(X,Y)$ are symmetric and commute with each other. Hence, by an elementary method, one can easily find eigenvalues of the matrices as follows:

$$\lambda_1(F(X,Y))=\lambda_2(F(X,Y))=2+\frac{1}{1+x_1^2+y_1^2},$$

and

$$\lambda_1(A(t))=1+t^2,\ \lambda_2(A(t))=1+3t^2,$$

so that $\lambda_i(F(X,Y))\ge f=2$ and $\lambda_i(A(t))\ge a=1$, $(i=1,2)$.

The Jacobian matrices of $J_E(Y)$, $J_{H_1}(X)$ and $J_{H_2}(X)$ are given by

$$J_E(Y)=\begin{bmatrix}3y_1^2+2 & 0\\ 0 & 3y_2^2+2\end{bmatrix},$$

$$J_{H_1}(X)=\begin{bmatrix}1+(1+x_1(t-0.01))^{-1} & 0\\ 0 & 1+(1+x_2(t-0.01))^{-1}\end{bmatrix},$$

and

$$J_{H_2}(X)=\begin{bmatrix}1+(1+x_1(t-0.02))^{-1} & 0\\ 0 & 1+(1+x_2(t-0.02))^{-1}\end{bmatrix}.$$

It can be easily seen that $B(t),\ J_E(Y)$; $C_i(t),\ J_{H_i}(X)$; and $C_i'(t),\ J_{H_i}(X)$ are symmetric matrices and commute with each other. Withal, one can easily find eigenvalues of the matrices as follows:

$$\lambda_i(J_E(Y))\ge\varepsilon=2,\ \lambda_i(B(t))\ge b=2,\ 1=\delta_i\le\lambda_i(J_{H_i}(X))\le\beta_i=2,$$

and

$$\lambda_i(C_i(t))\ge c_i=2,\ \lambda_i(C_i'(t))\le 0,\ (i=1,2).$$

Further, it follows that

$$\begin{aligned}\int_0^t\|K(t,s)\|\,ds+\int_t^\infty\|K(u,t)\|\,du&=\frac{2\sqrt{2}t}{(t^2+1)^2}\int_0^t ds+\int_t^\infty\frac{2\sqrt{2}u}{(u^2+1)^2}du\\&=\frac{\sqrt{2}(3t^2+1)}{(t^2+1)^2}\le\sqrt{2}=R.\end{aligned}$$

Thus, all the assumptions of the theorem are satisfied. So, we can conclude that all solutions shown in Figure 1 are continuable and bounded for the special case chosen.

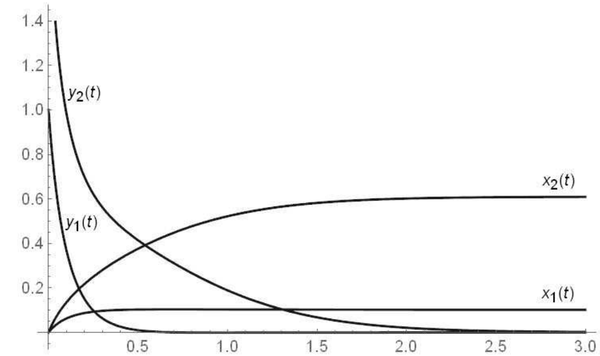

Figure 1: *Time evolutions of the states X(t) and Y(t)*

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