Math. Commun. **22**(2017), 177–192

# Computation of constants in multiparametric quon algebras. A twisted group algebra approach

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Received August 3, 2016; accepted November 27, 2016

**Abstract.** This paper describes the constants in generic weight subspaces $\mathcal{B}_Q$ of multiparametric quon algebra $\mathcal{B}$, where it is shown that one can perform calculations of constants in terms of certain iterated $\boldsymbol{q}$-commutators. In order to simplify some algebraic manipulations, here we use a twisted group algebra approach.
**AMS subject classifications**: 16S32, 16A03, 05E10, 05E15

**Key words**: multiparametric quon algebras, iterated $\boldsymbol{q}$-commutators

## 1. Introduction

One of fundamental problems in multiparametric quon algebra $\mathcal{B} = \mathcal{B}^{\boldsymbol{q}}$ equipped with a multiparametric $\boldsymbol{q}$-differential structure is a determination of the space $\mathcal{C}$ of all constants. The algebra $\mathcal{B}$ is naturally graded by the total degree and more generally it can be considered as multigraded, because it has a finer decomposition into multigraded components $\mathcal{B}_Q$ called weight subspaces. Thus the fundamental problem can be reduced to simpler problems of determining all finite dimensional spaces $\mathcal{C}_Q$ of all constants belonging to $\mathcal{B}_Q$. Of particular interest are generic weight subspaces $\mathcal{C}_Q$, where $Q$ is a set (see Section 2 and also [7]).

To solve this problem, one needs some special matrices and their factorizations in terms of simpler matrices. In order to simplify these algebraic manipulations, first, we have introduced a twisted group algebra $\mathcal{A}(S_n)$ of the symmetric group $S_n$ with coefficients in the polynomial ring $R_n$ in $n^2$ commuting variables $X_{a\,b}$, where we have studied nontrivial factorization of certain canonically defined elements (see (8), Section 3 and also [8]). Then we have used a natural representation of $\mathcal{A}(S_n)$ on the generic weight subspaces $\mathcal{B}_Q$ of the algebra $\mathcal{B}$. This approch is used because in this representation some factorizations of certain canonical elements from $\mathcal{A}(S_n)$ immediately give the corresponding matrix factorizations and also determinant factorizations.

Similar factorizations in a one-parameter case are given in [11] and in the multiparameter case in [5], where the factorizations were given on the matrix level. More general factorizations in braid group algebra can be found in [1]. In this paper, we are motivated to solve the problem of computing the constants in multiparametric quon algebra, therefore the factorizations here are more suitable and algebraically much

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simpler. Note that the algebra $\mathcal{B}$ has a direct sum decomposition into the generic subspace $\mathcal{B}^{\text{gen}}$ spanned by all multilinear monomials and the degenerate subspace $\mathcal{B}^{\text{deg}}$ spanned by all monomials which are nonlinear in at least one variable, that can be written by $\mathcal{B} = \mathcal{B}^{\text{gen}} \oplus \mathcal{B}^{\text{deg}}$ with

$$\mathcal{B}^{\text{gen}} = \bigoplus_{Q \text{ a set}} \mathcal{B}_Q, \mathcal{B}^{\text{deg}} = \bigoplus_{Q \text{ a multiset (not set)}} \mathcal{B}_Q.$$

Thus we distinguish between generic and degenerate subspaces of $\mathcal{B}$. Therefore, the computation of constants in $\mathcal{B}$ indicates the calculations of constants in all generic and also degenerate subspaces of $\mathcal{B}$, but here we use the fact that it is enough to compute the constants in generic subspaces, because the constants in degenerated subspaces can be constructed from those in the generic case by a certain specialization procedure (elaborated in the forthcoming paper [9]). In what follows, we will give nice formulas to describe the constants in every generic weight subspace of $\mathcal{B}$, where we will show that every nontrivial (basic) constant can be expressed in terms of certain iterated $\boldsymbol{q}$-commutators defined by (7) (see [7]).

## 2. The algebra $\mathcal{B}$

We recall that the free unital associative complex algebra $\mathcal{B}$ is generated by $N$ generators $\{e_i\}_{i\in\mathcal{N}}$, each of degree one, where $\mathcal{N} = \{i_1, \ldots, i_N\}$ is a fixed subset of the set of nonnegative integers. The $\boldsymbol{q}$-differential structure on $\mathcal{B}$ is given by $\boldsymbol{q}$-differential operators $\{\partial_i\}_{i\in\mathcal{N}}$ that act on $\mathcal{B}$ according to the twisted Leibniz rule

$$\partial_i(e_j x) = \delta_{ij} x + q_{ij} e_j \partial_i(x) \quad \text{for each} \quad x \in \mathcal{B}, i, j \in \mathcal{N} \tag{1}$$

with $\partial_i(1) = 0$ and $\partial_i(e_j) = \delta_{ij}$, where $\delta_{ij}$ is a standard Kronecker delta and $q_{ij}$'s are given complex numbers. Every weight subspace $\mathcal{B}_Q$, corresponding to a multiset $Q = \{l_1 \le \cdots \le l_n\}$ (of cardinality $n$), is given by

$$\mathcal{B}_Q = span_{\mathbb{C}} \left\{ e_{j_1\ldots j_n} = e_{j_1} \cdots e_{j_n} \mid j_1 \ldots j_n \in \widehat{Q} \right\}, \tag{2}$$

where $\widehat{Q}$ denotes the set of all distinct permutations of $Q$. Thus, $\dim \mathcal{B}_Q = \text{Card}\, \widehat{Q}$. Let $\underline{j} := j_1 \ldots j_n$ and let us denote by $\mathfrak{B}_Q = \{e_{\underline{j}} \mid \underline{j} \in \widehat{Q}\}$ the monomial basis of $\mathcal{B}_Q$; then by applying the formula (1) to $e_{\underline{j}} \in \mathfrak{B}_Q$ we obtain

$$\partial_i(e_{\underline{j}}) = \sum_{1\le k\le n, j_k = i} q_{ij_1} \cdots q_{ij_{k-1}} e_{j_1\ldots \widehat{j_k} \ldots j_n}, \tag{3}$$

an explicit formula for the action of $\partial_i$ on a typical monomial in $\mathfrak{B}_Q$. Here $\widehat{j_k}$ denotes the omission of the corresponding index $j_k$.

**Example 1.** $\partial_1(e_{1321212}) = e_{321212} + q_{11}q_{12}q_{13}e_{132212} + q_{11}^2 q_{12}^2 q_{13} e_{132122}$.

It is obvious that if $Q$ is a set (sometimes called the generic case), then (3) reduces to

$$\partial_{j_k}(e_{\underline{j}}) = q_{j_k j_1} \cdots q_{j_k j_{k-1}} e_{j_1\ldots \widehat{j_k} \ldots j_n}. \tag{4}$$

Motivated by the idea to treat better the matrices of $\partial_i|_{\mathcal{B}_Q}$, we introduce a multidegree operator $\partial\colon \mathcal{B} \to \mathcal{B}$ with

$$\partial = \sum_{i\in\mathcal{N}} e_i\,\partial_i,$$

where the operators $e_i\colon \mathcal{B} \to \mathcal{B}$ are considered as multiplication by $e_i$. Let $\partial^Q$ denote the restriction of $\partial$ to the subspace $\mathcal{B}_Q$ and let us denote by $\mathbf{B}_Q$ the matrix of the operator $\partial^Q$ with respect to the basis $\mathfrak{B}_Q$ (totally ordered by the Johnson-Trotter ordering on permutations, c.f. [10]). Then we can write

$$\mathbf{B}_Q\left(e_{\underline{j}}\right) = \sum_{1\le k\le n} q_{j_k j_1}\cdots q_{j_k j_{k-1}}\, e_{j_k j_1\ldots \widehat{j_k}\ldots j_n} \tag{5}$$

for each $\underline{j} \in \widehat{Q}$. The entries of this matrix are polynomials in $q_{ij}$'s, that are reduced to monomials in the generic case. Clearly, in this case, the size of $\mathbf{B}_Q$ is equal to $n!$. In the algebra $\mathcal{B}$ the elements called *constants* are of particular interest. A constant $C$ in $\mathcal{B}$ is defined as an element in $\mathcal{B}$ annihilated by all multiparametric partial derivatives $\partial_i$ (equivalent to $\partial C = 0$). Note that every linear combination of some constants is constant. Then linearly independent constants called basic constants (and sometimes constants) are of particular interest. We denote by $\mathcal{C}$ the space of all (basic) constants in $\mathcal{B}$ and by $\mathcal{C}_Q$ the space of all (basic) constants belonging to $\mathcal{B}_Q$. Then $\mathcal{C} = \ker\partial$ (the kernel of the multidegree operator $\partial$) and similar by $\mathcal{C}_Q = \ker\partial^Q$.

In what follows, we will consider only basic constants in generic weight subspaces $\mathcal{B}_Q$ of $\mathcal{B}$ (because the basic constants in degenerated subspaces can be constructed from those in the generic case by a certain specialization procedure, c.f. [9]). The existence of nontrivial basic constants depends on $\det\mathbf{B}_Q$ (c.f. (5)) which, in the generic case, is given explicitly as the product of binomial factors $(1-\sigma_T)$, i.e.,

$$\det\mathbf{B}_Q = \prod_{\substack{T\subseteq Q\\ 2\le|T|\le n}} (1-\sigma_T)^{(|T|-2)!(n-|T|)!}, \tag{6}$$

where $\sigma_T = \prod_{a\neq b\in T} q_{ab}$ (see also [5] and Theorem 4.12 in [6]). Here $|T| = \text{Card}\,T$ denotes the cardinality of the set $T$. If $\det\mathbf{B}_Q = 0$, i.e., if there is at least one $\sigma_T = 1$, then we say that $q_{ij}$'s are singular parameters, otherwise they are regular (or in general position c.f. [2]). In other words, the space $\mathcal{C}_Q$ is nonzero only for singular parameters that can be classified into two types, satisfying:

- *Type 1:* $Q$-cocycle condition (i.e., *the top cocycle condition*, see [2], [3]),
- *Type 2:* $(Q;T)$-cocycle condition for fixed $T \subsetneq Q$,

(see [7] for details). In the generic case, they take the form

- *Type 1:* $\sigma_Q = 1$, $\sigma_T \neq 1$ for all $T \subsetneq Q$,
- *Type 2:* $\sigma_Q = 1$, $\sigma_T = 1$, $\sigma_S \neq 1$ for all $S \subsetneq Q$, $S \neq T$.

Here we consider only Type 1 singular parameters because Type 2 could be obtained from Type 1 by a certain specialization procedure (c.f. [7], Section 4 for the special cases $2 \leq |Q| \leq 4$; more details will be given in [9]). Thus, in this paper, we will study only nontrivial basic constants in generic weight subspaces $\mathcal{B}_Q \subseteq \mathcal{B}$ under the $Q$-cocycle condition (sometimes written as $\sigma_Q = 1$).

In hat follows, we will use an important result of Frønsdal and Galindo (c.f. [3, Theorem 4.1.2]) that can be interpreted as follows: *in the generic case under the $Q$-cocycle condition the space $\mathcal{C}_Q$ has dimension $(n-2)!$, where $n = Card\ Q$.* It is easy to see that if $n = 1$, then zero is the only constant in $\mathcal{B}_Q$. Hence nontrivial constants might exist only in the spaces $\mathcal{B}_Q$, $n \geq 2$.

We use the following abbreviations $Y_{j_1 \dots j_p}$ for the iterated $\boldsymbol{q}$-commutators defined recursively by

$$Y_{j_1} := e_{j_1}, \quad Y_{j_1 \dots j_p} := \left[Y_{j_1 \dots j_{p-1}}, e_{j_p}\right]_{q_{j_p j_1} \cdots q_{j_p j_{p-1}}}, \tag{7}$$

where $Y_{j_1 j_2} = [e_{j_1}, e_{j_2}]_{q_{j_2 j_1}} = e_{j_1 j_2} - q_{j_2 j_1} e_{j_2 j_1}$ (see [7] and also [4] for details).

## 3. The algebra $\mathcal{A}(S_n)$

Recall that $\mathcal{A}(S_n) = R_n \rtimes \mathbb{C}[S_n]$ denotes a twisted group algebra of the symmetric group $S_n$ with coefficients in the polynomial algebra $R_n$ in $n^2$ commuting variables $X_{a\,b}$ $(1 \leq a, b \leq n)$ over the set of complex numbers (c.f. [8]) with $1 \in R_n$ as a unit element of $R_n$. Here $\rtimes$ denotes the semidirect product. The multiplication in $\mathcal{A}(S_n)$ is given by

$$\begin{aligned}&(p_1(\dots, X_{a\,b}, \dots)g_1) \cdot (p_2(\dots, X_{c\,d}, \dots)g_2)\\ &\qquad = p_1(\dots, X_{a\,b}, \dots) \cdot p_2(\dots, X_{g_1(c)\,g_1(d)}, \dots)g_1 g_2.\end{aligned}$$

The following canonically defined elements (c.f. [8], page 7) are of particular interest:

$$\alpha_n^* = \sum_{g \in S_n} \left( \prod_{(a,b) \in I(g^{-1})} X_{a\,b} \right) g \tag{8}$$

in the algebra $\mathcal{A}(S_n)$, where $I(g) = \{(a,b) \mid 1 \leq a < b \leq n,\ g(a) > g(b)\}$ denotes the set of all inversions $(a,b)$ of the permutation $g$.

To recapitulate from [8], first we have considered the cyclic permutation $t_{b,a} \in S_n$ which maps $a$ to $a+1$ to $a+2$ $\cdots$ to $b$ to $a$ and fixes all $1 \leq k \leq a-1$ and $b+1 \leq k \leq n$ (c.f. [5]), and then we have decomposed $g \in S_n$ into cycles from the left (this is more appropriate for determination of constants in the algebra $\mathcal{B}$) as follows: $g = t_{k_n,n} \cdot t_{k_{n-1},n-1} \cdots t_{k_j,j} \cdots t_{k_2,2} \cdot t_{k_1,1}$, where $k_j \geq j$. The corresponding elements in the algebra $\mathcal{A}(S_n)$ were given by

$$t_{b,a}^* = \left( \prod_{a+1 \leq j \leq b} X_{a\,j} \right) t_{b,a} \quad \text{for each} \quad 1 \leq a \leq b \leq n. \tag{9}$$

Moreover, in [8] we have defined

$$\begin{aligned} \beta^*_{n-k+1} &= t^*_{n,k} + t^*_{n-1,k} + \cdots + t^*_{k+1,k} + t^*_{k,k}, \\ \gamma^*_{n-k+1} &= \left(id - t^*_{n,k}\right) \cdot \left(id - t^*_{n-1,k}\right) \cdots \left(id - t^*_{k+1,k}\right), \\ \delta^*_{n-k+1} &= \left(id - (t^*_k)^2\, t^*_{n,k+1}\right) \cdot \left(id - (t^*_k)^2\, t^*_{n-1,k+1}\right) \cdots \left(id - (t^*_k)^2\, t^*_{k+1,k+1}\right), \end{aligned}$$

$1 \le k \le n-1$ with $t^*_{a,a} = id$ and $(t^*_k)^2 = X_{\{k,\,k+1\}}\, id$, where

$$t^*_k := t^*_{k+1,k}, \quad X_{\{k,\,k+1\}} := X_{k\,k+1} \cdot X_{k+1\,k}.$$

Note that $k = n$ implies: $\beta^*_1 = id$. Then we have obtained

$$\alpha^*_n = \beta^*_2 \cdot \beta^*_3 \cdots \beta^*_n \quad \text{with} \quad \beta^*_k = \delta^*_k \cdot (\gamma^*_k)^{-1}, \quad 2 \le k \le n.$$

So, $\alpha^*_n$ has a nontrivial factorization. It is firstly expressed as the product of simpler elements $\beta^*_k$ over all $1 \le k \le n$ and then $\beta^*_k$ in terms of yet simpler products $\gamma^*_k$ and $\delta^*_k$.

## 4. A representation of $\mathcal{A}(S_n)$ on the generic subspaces $\mathcal{B}_Q$

Since $\mathcal{A}(S_n) = R_n \rtimes \mathbb{C}[S_n]$, firstly we consider a representation $\varrho_1$ of $R_n$ and then a representation $\varrho_2$ of $\mathbb{C}[S_n]$ as follows:

- $\varrho_1 \colon R_n \to \mathrm{End}(\mathcal{B}_Q), \quad \varrho_1(X_{a\,b}) := Q_{a\,b}, \quad 1 \le a, b \le n,$
- $\varrho_2 \colon \mathbb{C}[S_n] \to \mathrm{End}(\mathcal{B}_Q), \quad \varrho_2(g)\, e_{j_1 \dots j_n} := e_{j_{g^{-1}(1)} \cdots j_{g^{-1}(n)}},$

for every $X_{a\,b} \in R_n$ and $g \in S_n$; here $Q_{a\,b}$ denotes a diagonal operator on $\mathcal{B}_Q$ defined by

$$Q_{a\,b}\, e_{j_1 \dots j_n} = q_{j_a j_b}\, e_{j_1 \dots j_n}.$$

Note that $Q_{a\,b} \cdot Q_{c\,d} = Q_{c\,d} \cdot Q_{a\,b}$.

**Proposition 1.** *Suppose that a map $\varrho \colon \mathcal{A}(S_n) \to End(\mathcal{B}_Q)$ is defined on decomposable elements by $\varrho(pg) := \varrho_1(p) \cdot \varrho_2(g)$ for every $p \in R_n$ and $g \in S_n$ and extended by additivity. Then $\varrho$ is a representation*

To prove this proposition it is enough to check that $\varrho$ preserves the following two types of basic relations of the multiplication in $\mathcal{A}(S_n)$:

$$X_{a\,b} \cdot X_{c\,d} = X_{c\,d} \cdot X_{a\,b}, \qquad g.X_{a\,b} = X_{g(a)\,g(b)}\, g$$

(see [6, Proposition 4.5] for details). Note that then the basic instance of the multiplication in $\mathcal{A}(S_n)$ can be written as $(X_{a\,b}\, g_1) \cdot (X_{c\,d}\, g_2) = X_{a\,b} \cdot X_{g_1(c)\, g_1(d)}\, g_1 g_2$. In what follows, we will consider the twisted regular representation $\varrho \colon \mathcal{A}(S_n) \to \mathrm{End}(\mathcal{B}_Q)$ in the generic case, where $\mathcal{B}_Q$ is the generic weight subspace of $\mathcal{B}$. We have

$$\varrho(t^*_{b,a})\, e_{j_1 \dots j_a j_{a+1} \dots j_b \dots j_n} = \prod_{a \le i \le b-1} q_{j_b j_i}\, e_{j_1 \dots j_b j_a \dots j_{b-1} \dots j_n}$$

and in the special case

$$\varrho(t_a^*)\, e_{j_1\ldots j_a j_{a+1}\ldots j_n} = q_{j_{a+1}j_a}\, e_{j_1\ldots j_{a+1} j_a\ldots j_n},$$

and

$$\varrho((t_a^*)^2)\, e_{j_1\ldots j_n} = \sigma_{j_a j_{a+1}}\, e_{j_1\ldots j_n}$$

where we use the abbreviation $\sigma_{j_a j_{a+1}} = q_{j_a j_{a+1}} q_{j_{a+1} j_a}$. Then the element $\varrho(\alpha_n^*) \in \mathrm{End}(\mathcal{B}_Q)$ is given by

$$\varrho(\alpha_n^*)\, e_{\underline{j}} = \sum_{g\in S_n} \left( \prod_{(a,b)\in I(g)} q_{j_b j_a}\, e_{\underline{k}} \right),$$

where $g$ satisfies $\underline{k} = g.\underline{j}$; see (8), (9). Similarly, the elements $\varrho(\beta_{n-k+1}^*) \in \mathrm{End}(\mathcal{B}_Q)$, $1 \le k \le n-1$ are given by

$$\varrho(\beta_{n-k+1}^*)\, e_{\underline{j}} = \sum_{k+1\le m\le n} \varrho(t_{m,k}^*)\, e_{\underline{j}} + e_{\underline{j}}, \tag{10}$$

with $\varrho(\beta_1^*)\, e_{\underline{j}} = e_{\underline{j}}$. In order to write the given elements in the matrix notation, we introduce the abbreviations $\mathbf{T}_{b,a} := \varrho(t_{b,a}^*)$, $\mathbf{T}_a := \varrho(t_a^*)$ with $\mathbf{T}_{a,a} = \mathbf{I}$ and similarly $\mathbf{B}_{Q,k} := \varrho(\beta_k^*)$, $2 \le k \le n$. Then identity (10) can be written in the matrix notation as

$$\mathbf{B}_{Q,n-k+1} = \sum_{k+1\le m\le n} \mathbf{T}_{m,k} + \mathbf{I}. \tag{11}$$

Then, its factorization is given by

$$\mathbf{B}_{Q,n-k+1} = \prod_{k+1\le m\le n}^{\leftarrow} \left(\mathbf{I} - (\mathbf{T}_k)^2\, \mathbf{T}_{m,k+1}\right) \prod_{k+1\le m\le n}^{\rightarrow} \left(\mathbf{I} - \mathbf{T}_{m,k}\right)^{-1},$$

where $1 \le k \le n-1$, or shorter

$$\mathbf{B}_{Q,k} = \mathbf{D}_{Q,k} \cdot \left(\mathbf{C}_{Q,k}\right)^{-1}, \quad 2 \le k \le n, \tag{12}$$

where $\mathbf{C}_{Q,k} := \varrho(\gamma_k^*)$, $\mathbf{D}_{Q,k} := \varrho(\delta_k^*)$ (see Section 3). Similarly, we get

$$\mathbf{A}_Q = \prod_{1\le k\le n-1}^{\leftarrow} \left( \prod_{k+1\le m\le n}^{\leftarrow} \left(\mathbf{I} - (\mathbf{T}_k)^2\, \mathbf{T}_{m,k+1}\right) \cdot \prod_{k+1\le m\le n}^{\rightarrow} \left(\mathbf{I} - \mathbf{T}_{m,k}\right)^{-1} \right),$$

where $\mathbf{A}_Q := \varrho(\alpha_n^*)$. Here we have used

$$\left(\mathbf{T}_{b,a}\right)_{\underline{k},\underline{j}} = \begin{cases} \prod_{a\le i\le b-1} q_{j_b j_i} & \text{if } \underline{k} = t_{b,a}.\underline{j} \\ 0 & \text{otherwise} \end{cases}, \quad \left(\mathbf{T}_a\right)_{\underline{k},\underline{j}} = \begin{cases} q_{j_{a+1} j_a} & \text{if } \underline{k} = t_a.\underline{j} \\ 0 & \text{otherwise} \end{cases} \tag{13}$$

with $t_{b,a}.\underline{j} = j_1 \ldots j_b j_a \ldots j_{b-1} \ldots j_n$ and $t_a.\underline{j} = j_1 \ldots j_{a+1} j_a \ldots j_n$ and also that $(\mathbf{T}_a)^2$ is a diagonal matrix with $\sigma_{j_a j_{a+1}}$ as its $\underline{j}$-th diagonal entry.

Of particular interest is the study of $\det(\mathbf{B}_{Q,k})$, $2 \le k \le n$ and also $\det(\mathbf{A}_Q)$. From the above formulas it follows that these determinants can be calculated if one finds formulas for computing $\det(\mathbf{I}-\mathbf{T}_{b,a})$, $1 \le a < b \le n$ and $\det(\mathbf{I}-(\mathbf{T}_{a-1})^2\mathbf{T}_{b,a})$, $1 < a \le b \le n$, see [6, Lemma 4.11] for details and compare with [5, Lemma 1.9.1]. Then we obtain the following formulas

$$\det(\mathbf{B}_{Q,n-k+1}) = \prod_{2\le m\le n-k+1} \prod_{T\in(Q;m)} (1-\sigma_T)^{(m-2)!\cdot(n-m)!}, \quad 1 \le k \le n-1,$$
$$\det(\mathbf{A}_Q) = \prod_{2\le m\le n} \prod_{T\in(Q;m)} (1-\sigma_T)^{(m-2)!\cdot(n-m+1)!}$$

(c.f. [6, Theorem 4.12]; compare with [5, Theorem 1.9.2]), where we have used the following notations

$$(Q;m) = \{T \subseteq Q \mid \text{Card } T = m\}, \quad \sigma_T = \prod_{i\ne j\in T} q_{ij}.$$

An important special case of $\mathbf{B}_{Q,n-k+1}$ arises for $k=1$. Then the $(\underline{k},\underline{j})$-entry of the matrix $\mathbf{B}_{Q,n}$ (c.f. (11) and (13)) is given by

$$(\mathbf{B}_{Q,n})_{\underline{k},\underline{j}} = \begin{cases} q_{j_m j_1}\cdots q_{j_m j_{m-1}} & \text{if } \underline{k} = t_{m,1}.\underline{j}, \quad 1 \le m \le n \\ 0 & \text{otherwise} \end{cases}$$

with $t_{m,1}.\underline{j} = j_m j_1 \dots j_{m-1} j_{m+1} \dots j_n$. Hence we can write

$$\mathbf{B}_{Q,n}\, e_{\underline{j}} = \sum_{1\le m\le n} q_{j_m j_1}\cdots q_{j_m j_{m-1}}\, e_{j_m j_1 \dots j_{m-1} j_{m+1} \dots j_n}. \tag{14}$$

Now it is obvious that

$$\det(\mathbf{B}_{Q,n}) = \prod_{2\le m\le n} \prod_{T\in(Q;m)} (1-\sigma_T)^{(m-2)!\cdot(n-m)!}. \tag{15}$$

By comparing (14) with the matrix $\mathbf{B}_Q$ (c.f. (5)) of the operator $\partial^Q$ with respect to the basis $\mathfrak{B}_Q$ it follows that $\mathbf{B}_{Q,n} = \mathbf{B}_Q$. Consequently, their determinants must be equal (c.f. (15) with (6)).

## 5. The constants in the generic weight subspaces $\mathcal{B}_Q \subseteq \mathcal{B}$

We recall that $\mathbf{B}_Q$ denotes the matrix of the operator $\partial^Q$ with respect to the monomial basis of $\mathcal{B}_Q$ and also that there exist nontrivial constants in $\mathcal{B}_Q$ only for singular parameters $q_{ij}$'s for which $\det \mathbf{B}_Q = 0$ (c.f. (6) and (15)). Therefore, $\mathbf{B}_Q = \mathbf{B}_{Q,n}$ leads us to the conclusion that the matrix $\mathbf{B}_Q$ can be factorized by applying identity (12) for $k=n$.

In this section we will compute nontrivial basic constants in every generic weight subspace $\mathcal{B}_Q \subseteq \mathcal{B}$ (Card $Q \ge 2$) under the $Q$-cocycle condition (see Section 2). Note that this is equivalent to determine the kernel of the operator $\partial^Q = \partial|_{\mathcal{B}_Q}$. Then in what follows, we will rewrite the operator $\partial^Q$ in terms of simpler operators $T_{m,1}$ $(2 \le m \le n)$ acting on $\mathcal{B}_Q$ and replace the matrix notation with the corresponding operator notation. Therefore, for each $\underline{j} \in \widehat{Q}$ we get

$$T_{m,1}\, e_{\underline{j}} = q_{j_m j_1} \cdots q_{j_m j_{m-1}}\, e_{j_m j_1 \dots j_{m-1} j_{m+1} \dots j_n} \tag{16}$$

(c.f. (13)) with $T_{1,1} = id$, so the identity (11), for $k = 1$, takes the form

$$\partial^Q = \sum_{1 \le m \le n} T_{m,1}.$$

Moreover, we obtain

$$\partial^Q \cdot C_{Q,n} = D_{Q,n} \tag{17}$$

(c.f. (12)) with

$$C_{Q,n} = (id - T_{n,1}) \cdots (id - T_{2,1}) = \overleftarrow{\prod_{2 \le m \le n}} (id - T_{m,1})\,, \tag{18}$$

$$D_{Q,n} = \left(id - (T_1)^2\, T_{n,2}\right) \cdots \left(id - (T_1)^2\, T_{2,2}\right) = \overleftarrow{\prod_{2 \le m \le n}} \left(id - (T_1)^2\, T_{m,2}\right), \tag{19}$$

where

$$(T_1)^2\, T_{m,2}\, e_{\underline{j}} = \sigma_{j_1 j_m} q_{j_m j_2} \cdots q_{j_m j_{m-1}}\, e_{j_1 j_m \dots j_{m-1} j_{m+1} \dots j_n}\,. \tag{20}$$

Observe that (17) is a special case of the braid factorization from [1, Proposition 4.7] (c.f. with [5]).

**Proposition 2.** *Suppose that* $U \in \ker\left(id - (T_1)^2\, T_{n,2}\right)$. *Then the corresponding vector* $X \in \ker \partial^Q$ *is given by*

$$X = C_{Q,n} \prod_{2 \le m \le n-1} \left(id - (T_1)^2\, T_{m,2}\right)^{-1} \cdot U. \tag{21}$$

**Proof**. Observe that (17) can be rewritten as

$$\partial^Q \cdot C_{Q,n} = \left(id - (T_1)^2\, T_{n,2}\right) \overleftarrow{\prod_{2 \le m \le n-1}} \left(id - (T_1)^2\, T_{m,2}\right),$$

i.e.,

$$\partial^Q \cdot C_{Q,n} \prod_{2 \le m \le n-1} \left(id - (T_1)^2\, T_{m,2}\right)^{-1} = \left(id - (T_1)^2\, T_{n,2}\right)$$

$$\partial^Q \cdot C_{Q,n} \prod_{2 \le m \le n-1} \left(id - (T_1)^2\, T_{m,2}\right)^{-1} \cdot U = \left(id - (T_1)^2\, T_{n,2}\right) \cdot U,$$

for every $U \in \mathcal{B}$. Note that the operators $(id - (T_1)^2\, T_{m,2})$ for $m = 2, \dots, n-1$ are invertible because $\sigma_T \ne 1$ for all $T \subsetneq Q$ (i.e., the $Q$-cocycle condition is satisfied).

Therefore, we can relate $\ker\big(id-(T_1)^2\,T_{n,2}\big)\subset\mathcal{B}_Q$ to $\ker\partial^Q$ (the space of all (basic) constants in $\mathcal{B}_Q$). Then for each $U\in\ker\big(id-(T_1)^2\,T_{n,2}\big)$ the right-hand side of the last formula is equal to zero, hence for every $X\in\ker\partial^Q$ it follows that $X$ is given by (21). □

**Remark 1.** *Similarly, one can show that if* $U_j\in\ker\big(id-(T_1)^2\,T_{j+1,2}\big)$, $2\le j\le n-1$, *then the corresponding vector* $X_j\in\ker\partial^Q$ *is given by*

$$X_j=C_{Q,n}\prod_{2\le m\le j}\big(id-(T_1)^2\,T_{m,2}\big)^{-1}\cdot U_j.$$

*In the special case, if* $U_1\in\ker\big(id-(T_1)^2\,T_{2,2}\big)$, *then* $X_1=C_{Q,n}\cdot U_1\in\ker\partial^Q$, *where* $C_{Q,n}$ *is given by (18).*

The vectors in the kernel of the operator $(id-(T_1)^2\,T_{n,2})$ are of particular interest. Now we can raise two questions, first how one can write the vectors spanning the kernel $\ker\big(id-(T_1)^2\,T_{n,2}\big)$ and then how to find a basis?

By considering the proof of Lemma 4.11 from [6] (see also [5, Lemma 1.9.1], where the matrix factorizations from the right is used) we can write

$$\big((T_1)^2\,T_{n,2}\big)^{n-1}\;e_{\underline{j}}=\sigma_Q\,e_{\underline{j}},\tag{22}$$

i.e.,

$$\big(id-((T_1)^2\,T_{n,2})^{n-1}\big)\;e_{\underline{j}}=(1-\sigma_Q)\;e_{\underline{j}},$$

where $\sigma_Q=\prod_{\{i,j\}\subset Q}\sigma_{ij}=\prod_{i\neq j\in Q}q_{ij}$ and $Q=\{l_1,\ldots,l_n\}$ is a set of cardinality $n$. Recall that here we have used the factorizations from the left. By applying the property

$$\begin{aligned}&\big(id-((T_1)^2\,T_{n,2})^{n-1}\big)\;e_{\underline{j}}\\&\quad=\big(id-(T_1)^2\,T_{n,2}\big)\Big(id+\big((T_1)^2\,T_{n,2}\big)+\cdots+\big((T_1)^2\,T_{n,2}\big)^{n-2}\Big)\;e_{\underline{j}}\end{aligned}$$

it follows that the last formula can be written as

$$\big(id-(T_1)^2\,T_{n,2}\big)\Big(id+\big((T_1)^2\,T_{n,2}\big)+\cdots+\big((T_1)^2\,T_{n,2}\big)^{n-2}\Big)\;e_{\underline{j}}=(1-\sigma_Q)\;e_{\underline{j}}.\tag{23}$$

Now it is easy to see that if $\sigma_Q=1$, then $U_{\underline{j}}\in\ker\big(id-(T_1)^2\,T_{n,2}\big)$, where

$$U_{\underline{j}}:=\Big(id+\big((T_1)^2\,T_{n,2}\big)+\cdots+\big((T_1)^2\,T_{n,2}\big)^{n-2}\Big)\;e_{\underline{j}}.\tag{24}$$

This leads us to the conclusion that the corresponding vector $U_{\underline{j}}$ belongs to the kernel of the operator $\big(id-(T_1)^2\,T_{n,2}\big)$ if the $Q$-cocycle condition is satisfied.

**Remark 2.** *Under the $Q$-cocycle condition we have:*

$$\dim\Big(\ker\big(id-(T_1)^2\,T_{n,2}\big)\Big)=n\cdot(n-2)!$$

*(see also long orbits treated in [2]), where the corresponding linearly independent vectors can be taken in the form*

$$U_{l_1 l_2 j_3 \ldots j_n} = \Big(id + \big((T_1)^2\, T_{n,2}\big) + \cdots + \big((T_1)^2\, T_{n,2}\big)^{n-2}\Big)\, e_{l_1 l_2 j_3 \ldots j_n},$$
$$U_{l_k l_1 i_3 \ldots i_n} = \Big(id + \big((T_1)^2\, T_{n,2}\big) + \cdots + \big((T_1)^2\, T_{n,2}\big)^{n-2}\Big)\, e_{l_k l_1 i_3 \ldots i_n},$$

*for each* $2 \le k \le n$*, where* $Q = \{l_1 < l_2 < \cdots < l_n\}$ *and*

$$j_3 \ldots j_n \in \widehat{Q'}, \quad Q' = Q \backslash \{l_1, l_2\} = \{l_3, \ldots, l_n\}$$
$$i_3 \ldots i_n \in \widehat{Q''}, \quad Q'' = Q \backslash \{l_1, l_k\} = \{l_2, \ldots, \widehat{l_k}, \ldots l_n\}, \;\; 2 \le k \le n.$$

$\widehat{Q'}$ *(resp.* $\widehat{Q''}$*) denotes the set of all distinct permutations of the set* $Q'$ *(resp.* $Q''$*).*

More generally, one can show more identities like (22)

$$\big((T_1)^2\, T_{m,2}\big)^{m-1}\, e_{\underline{j}} = \sigma_T\, e_{\underline{j}}, \quad 2 \le m \le n,$$

where $T = \{j_1, \ldots, j_m\} \subseteq Q$, Card $T = m$ and $\sigma_T = \prod_{\{a,b\} \subset T} \sigma_{ab} = \prod_{a \ne b \in T} q_{ab}$.

**Example 2.** *Let* $\mathcal{B}_Q$ *correspond to* $Q = \{1,2,3,4\}$ *and suppose that* $\sigma_{1234} = 1$*. Then* $\dim\big(\ker\big(id - (T_1)^2\, T_{4,2}\big)\big) = 8$ *and the appropriate linearly independent vectors can be given by*

$$\begin{aligned}
U_{1234} &= e_{1234} + q_{42}q_{43}\sigma_{14}\, e_{1423} + q_{32}q_{42}\sigma_{134}\, e_{1342},\\
U_{1243} &= e_{1243} + q_{32}q_{34}\sigma_{13}\, e_{1324} + q_{42}q_{32}\sigma_{134}\, e_{1432},\\
U_{2134} &= e_{2134} + q_{41}q_{43}\sigma_{24}\, e_{2413} + q_{31}q_{41}\sigma_{234}\, e_{2341},\\
U_{2143} &= e_{2143} + q_{31}q_{34}\sigma_{23}\, e_{2314} + q_{41}q_{31}\sigma_{234}\, e_{2431},\\
U_{3124} &= e_{3124} + q_{41}q_{42}\sigma_{34}\, e_{3412} + q_{21}q_{41}\sigma_{234}\, e_{3241},\\
U_{3142} &= e_{3142} + q_{21}q_{24}\sigma_{23}\, e_{3214} + q_{41}q_{21}\sigma_{234}\, e_{3421},\\
U_{4123} &= e_{4123} + q_{31}q_{32}\sigma_{34}\, e_{4312} + q_{21}q_{31}\sigma_{234}\, e_{4231},\\
U_{4132} &= e_{4132} + q_{21}q_{23}\sigma_{24}\, e_{4213} + q_{31}q_{21}\sigma_{234}\, e_{4321},
\end{aligned}$$

*where* $U_{\underline{j}} = \Big(id + \big((T_1)^2\, T_{4,2}\big) + \big((T_1)^2\, T_{4,2}\big)^2\Big)\, e_{\underline{j}}$ *for every* $\underline{j} \in \widehat{Q}$*. It is easy to check that the remaining linearly dependent vectors are related as follows*

$$\begin{aligned}
U_{1324} &= q_{23}q_{43}\sigma_{124}\, U_{1243}, & U_{1342} &= q_{23}q_{24}\sigma_{12}\, U_{1234},\\
U_{1423} &= q_{24}q_{34}\sigma_{123}\, U_{1234}, & U_{1432} &= q_{23}q_{24}\sigma_{12}\, U_{1243},\\
U_{2314} &= q_{13}q_{43}\sigma_{124}\, U_{2143}, & U_{2341} &= q_{13}q_{14}\sigma_{12}\, U_{2134},\\
U_{2413} &= q_{14}q_{34}\sigma_{123}\, U_{2134}, & U_{2431} &= q_{13}q_{14}\sigma_{12}\, U_{2143},\\
U_{3214} &= q_{12}q_{42}\sigma_{134}\, U_{3142}, & U_{3241} &= q_{12}q_{14}\sigma_{13}\, U_{3124},\\
U_{3412} &= q_{14}q_{24}\sigma_{123}\, U_{3124}, & U_{3421} &= q_{12}q_{14}\sigma_{13}\, U_{3142},\\
U_{4213} &= q_{12}q_{32}\sigma_{134}\, U_{4132}, & U_{4231} &= q_{12}q_{13}\sigma_{14}\, U_{4123},\\
U_{4312} &= q_{13}q_{23}\sigma_{124}\, U_{4123}, & U_{4321} &= q_{12}q_{13}\sigma_{14}\, U_{4132}.
\end{aligned}$$

Now we can determine the vectors in $\ker \partial^Q$ under the $Q$-cocycle condition, i.e., we can compute nontrivial basic constants in every such generic weight subspace $\mathcal{B}_Q$ of the algebra $\mathcal{B}$. In view of an important result of Frønsdal and Galindo (c.f. [3, Theorem 4.1.2]), it turns out that in the generic case

$$\dim(\ker \partial^Q) = (n-2)! \quad \text{if} \quad \sigma_Q = 1. \tag{25}$$

We recall that $\ker \partial^Q$ denotes the space $\mathcal{C}_Q$ of all constants in $\mathcal{B}_Q$. Let $\sigma_Q = 1$ (where $Q$ is a set of cardinality $n$) and let $U_{\underline{j}} \in \ker\left(id - (T_1)^2\, T_{n,2}\right)$ for each $\underline{j} \in \widehat{Q}$. Then, by temporarily working under condition $\sigma_Q - 1 \neq 0$, by using identities (24) and (23), it follows $U_{\underline{j}} = \left(id - (T_1)^2\, T_{n,2}\right)^{-1} (1-\sigma_Q)\; e_{\underline{j}}$. Considering the definition of diagonal operator $Q_{\{1,\dots,n\}}$ on $\mathcal{B}_Q$:

$$Q_{\{1,\dots,n\}}\, e_{\underline{j}} = \prod_{\{a,b\}\subset\{1,\dots,n\}} Q_{\{a,b\}}\, e_{\underline{j}}$$

and

$$Q_{\{a,b\}}\, e_{\underline{j}} = Q_{a\,b}\cdot Q_{b\,a}\, e_{\underline{j}} = q_{j_a j_b} q_{j_b j_a}\, e_{\underline{j}} = \sigma_{j_a j_b}\, e_{\underline{j}}$$

we can write

$$U_{\underline{j}} = \left(id - (T_1)^2\, T_{n,2}\right)^{-1} \left(id - Q_{\{1,\dots,n\}}\right)\, e_{\underline{j}}.$$

Then the vector $X_{\underline{j}} \in \ker \partial^Q$ (c.f. Proposition 2) is given by

$$X_{\underline{j}} = C_{Q,n} \prod_{2\le m\le n} \left(id - (T_1)^2\, T_{m,2}\right)^{-1} \left(id - Q_{\{1,\dots,n\}}\right)\, e_{\underline{j}}$$

i.e.,

$$X_{\underline{j}} = C_{Q,n} (D_{Q,n})^{-1} \left(id - Q_{\{1,\dots,n\}}\right)\, e_{\underline{j}} \tag{26}$$

for each $\underline{j} \in \widehat{Q}$, see also (19). Note that an additional problem of determining the basis of $\ker \partial^Q$ arises from (26), where first we must determine the inverse of the operator $D_{Q,n}$. This problem is directly linked to a more general problem of determining the inverse of the elements $\delta^*_{n-k+1} \in \mathcal{A}(S_n)$, $1 \le k \le n-1$. Here we will consider a special case of a solution of this problem, where we will use Proposition 3.10 from [8] for $k = 1$ (see also [5]). By applying a twisted regular representation $\varrho$ on the elements from $\mathcal{A}(S_n)$ treated in [8, Proposition 3.10] as well as the previously introduced matrix notations, we can use the following labels, $\mathbf{\Delta_n} = \varrho(\Delta_n)$, $\mathbf{E}_{Q,n} = \varrho(\varepsilon^*_n)$ with $\mathbf{W}_n(g) = \varrho(\omega_n(g))$ and $\mathbf{G} = \varrho(g^*)$. Then we replace the matrix notation with the corresponding operator notation, such that these labels replace with appropriate without bold tags. Here is meant that the operator $\mathcal{Q}_n$ corresponds to the matrix $\mathbf{\Delta_n}$. Let us denote by

$$Des(\sigma) := \{1 \le i \le n-1 \mid \sigma(i) > \sigma(i+1)\}$$

the descent set of a permutation $\sigma \in S_n$. Now we can formulate the following theorem that is a direct consequence of Proposition 3.10 from [8].

**Theorem 1.** *Suppose that the parameters $q_{ij}$'s are in general position, i.e., $\sigma_T \neq 1$ for all $T \subseteq Q$. Then the inverse of the operator $D_{Q,n}$ is given by the formula*

$$(D_{Q,n})^{-1} = (\mathcal{Q}_n)^{-1} E_{Q,n}, \tag{27}$$

*where*

$$\mathcal{Q}_n = \left(id - Q_{\{1,2\}}\right)\left(id - Q_{\{1,2,3\}}\right)\cdots\left(id - Q_{\{1,2,\ldots,n\}}\right),$$
$$E_{Q,n} = \sum_{g \in S_1 \times S_{n-1}} W_n(g) \cdot g$$

*and*

$$W_n(g) = \prod_{i \in Des(g^{-1})} Q_{\{1,2,\ldots,i\}}$$

*with $Des(g^{-1}) = \{1 \leq i \leq n-1 \mid g^{-1}(i) > g^{-1}(i+1)\}$.*

Note that $g \in S_1 \times S_{n-1}$ fixes the first index.

**Theorem 2.** *Let $\mathcal{B}_Q$ correspond to a set $Q = \{l_1, \ldots, l_n\}$ and let the $Q$-cocycle condition be satisfied. If*

$$X_{\underline{j}} = \left(C_{Q,n}(\mathcal{Q}_{n-1})^{-1} E_{Q,n}\right) e_{\underline{j}}, \tag{28}$$

*then* $X_{\underline{j}} \in \mathcal{C}_Q$.

**Proof**. In view of the facts

$$\mathcal{Q}_n = \left(id - Q_{\{1,2\}}\right)\left(id - Q_{\{1,2,3\}}\right)\cdots\left(id - Q_{\{1,2,\ldots,n\}}\right),$$
$$\mathcal{Q}_{n-1} = \left(id - Q_{\{1,2\}}\right)\left(id - Q_{\{1,2,3\}}\right)\cdots\left(id - Q_{\{1,2,\ldots,n-1\}}\right),$$

it follows $(\mathcal{Q}_{n-1})^{-1} = \left(id - Q_{\{1,2,\ldots,n\}}\right)(\mathcal{Q}_n)^{-1}$, so (28) can be rewritten as

$$X_{\underline{j}} = \left(C_{Q,n}\left(id - Q_{\{1,2,\ldots,n\}}\right)(D_{Q,n})^{-1}\right) e_{\underline{j}}.$$

Here we have applied identity (27) from Theorem 1. Note that the product of $\left(id - Q_{\{1,2,\ldots,n\}}\right)$ and $(D_{Q,n})^{-1}$ commutes because $\left(id - Q_{\{1,2,\ldots,n\}}\right)$ is a diagonal operator. By assuming that the $Q$-cocycle condition is satisfied, from the last formula it follows $X_{\underline{j}} \in \ker \partial^Q$, c.f. (26). We recall that $\mathcal{C}_Q = \ker \partial^Q$ denotes the space of all (basic) constants belonging to $\mathcal{B}_Q$. □

Note that under the $Q$-cocycle condition, there are $n!$ (nontrivial) vectors $X_{\underline{j}}$ in the kernel of the operator $\partial^Q$ (c.f. Theorem 2), but they are not linearly independent. By Remark 2 it follows that the number of the vectors $X_{\underline{j}} \in \ker \partial^Q$ can be reduced to $n \cdot (n-2)!$ with

$$X_{l_1 l_2 j_3 \ldots j_n} = \left(C_{Q,n}(\mathcal{Q}_{n-1})^{-1} E_{Q,n}\right) e_{l_1 l_2 j_3 \ldots j_n},$$
$$X_{l_k l_1 i_3 \ldots i_n} = \left(C_{Q,n}(\mathcal{Q}_{n-1})^{-1} E_{Q,n}\right) e_{l_k l_1 i_3 \ldots i_n},$$

for each $2 \le k \le n$, where the indices of $X$ (resp. corresponding generators $e$) are also given in Remark 2. By using the identity (similar to (23)),

$$(id - T_{n,1})\left(id + T_{n,1} + \cdots + (T_{n,1})^{n-1}\right) e_{\underline{j}} = (1 - \sigma_Q)\, e_{\underline{j}},$$

one can show that for each $2 \le k \le n$ the vectors $X_{l_k l_1 i_3 \ldots i_n}$ depend on the linearly independent vectors $X_{l_1 l_2 j_3 \ldots j_n}$ (recall that $j_3 \ldots j_n \in \widehat{Q'}$, $Q' = \{l_3, \ldots, l_n\}$). Then we can conclude, the dimension of $\ker \partial^Q (= \mathcal{C}_Q)$ is equal to $(n-2)!$ that explains more directly a result of Frønsdal and Galindo. Thus, we can state the following proposition.

**Proposition 3.** *Let $\mathcal{B}_Q$ correspond to a set $Q = \{l_1, \ldots, l_n\}$ and let us denote $Q' = Q \backslash \{l_1, l_2\} = \{l_3, \ldots, l_n\}$. Then under the Q-cocycle condition*

$$\dim(\mathcal{C}_Q) = (n-2)!$$

*and the nontrivial basic constants in the space $\mathcal{C}_Q$ are given by*

$$C_{l_1 l_2 j_3 \ldots j_n} = \left(C_{Q,n} (\mathcal{Q}_{n-1})^{-1} E_{Q,n}\right) e_{l_1 l_2 j_3 \ldots j_n}$$

*for all $j_3 \ldots j_n \in \widehat{Q'}$.*

We recall that $C_{Q,n}$, $\mathcal{Q}_{n-1}$ and $E_{Q,n}$ are also given in Theorem 1 and $\widehat{Q'}$ denotes the set of all distinct permutations of the set $Q'$.

In what follows, we will apply the iterated $\boldsymbol{q}$-commutators (c.f. (7)) defined by

$$Y_{i_1 \ldots i_p} = Y_{i_1 \ldots i_{p-1}}\, e_{i_p} - q_{i_p j_1} \cdots q_{i_p i_{p-1}}\, e_{i_p} Y_{i_1 \ldots i_{p-1}} \quad \text{with} \quad Y_{i_1} = e_{i_1}.$$

**Proposition 4.** *Let $Q = \{l_1, \ldots, l_n\} \subseteq \mathcal{N}$ and $\underline{j} = j_1 \ldots j_n \in \widehat{Q}$. Then*

$$Y_{\underline{j}} = C_{Q,n}\, e_{\underline{j}},$$

*where $C_{Q,n}$ is given by (18).*

**Proof**. Here we use (16) for every $2 \le m \le n$. If $m = 2$, then it follows

$$\begin{aligned}(id - T_{2,1})\, e_{j_1 \ldots j_n} &= e_{j_1 j_2 j_3 \ldots j_n} - q_{j_2 j_1}\, e_{j_2 j_1 j_3 \ldots j_n} \\ &= (e_{j_1 j_2} - q_{j_2 j_1}\, e_{j_2 j_1})\, e_{j_3 \ldots j_n} = [e_{j_1}, e_{j_2}]_{q_{j_2 j_1}}\, e_{j_3 \ldots j_n} \\ &= Y_{j_1 j_2}\, e_{j_3 \ldots j_n}.\end{aligned}$$

Similarly, we obtain

$$\begin{aligned}&(id - T_{3,1})\,(id - T_{2,1})\, e_{j_1 \ldots j_n} \\ &\quad = (id - T_{3,1})\,(e_{j_1 j_2} - q_{j_2 j_1}\, e_{j_2 j_1})\, e_{j_3 \ldots j_n} \\ &\quad = (id - T_{3,1})\,(e_{j_1 j_2 j_3} - q_{j_2 j_1}\, e_{j_2 j_1 j_3})\, e_{j_4 \ldots j_n} \\ &\quad = (e_{j_1 j_2 j_3} - q_{j_2 j_1}\, e_{j_2 j_1 j_3} - q_{j_3 j_1} q_{j_3 j_2}\, e_{j_3 j_1 j_2} + q_{j_3 j_2} q_{j_3 j_1} q_{j_2 j_1}\, e_{j_3 j_2 j_1}) \\ &\quad = Y_{j_1 j_2 j_3}\, e_{j_4 \ldots j_n}.\end{aligned}$$

As described above, one can show that $(id - T_{n,1}) \cdots (id - T_{2,1})\, e_{j_1 \ldots j_n} = Y_{j_1 \ldots j_n}$. Thus it follows $C_{Q,n}\, e_{j_1 \ldots j_n} = Y_{j_1 \ldots j_n}$. □

**Theorem 3.** *Let the weight subspace $\mathcal{B}_Q \subseteq \mathcal{B}$ correspond to a set $Q = \{l_1, \ldots, l_n\}$ of cardinality $n \geq 2$ and let $Q' = \{l_3, \ldots, l_n\}$. If $\sigma_Q = 1$, then*

$$C_{l_1 l_2 j_3 \ldots j_n} = \left((\mathcal{Q}_{n-1})^{-1} E_{Q,n}\right) Y_{l_1 l_2 j_3 \ldots j_n} \tag{29}$$

*for every $j_3 \ldots j_n \in \widehat{Q'}$.*

**Proof**. This Theorem is a direct consequence of Proposition 3 and Proposition 4. □

Consequently, in the space $\mathcal{C}_Q$ (of all constants belonging to $\mathcal{B}_Q$) there are $(n-2)!$ (nontrivial) basic constants that can be rewritten as

$$C_{l_1 l_2 j_3 \ldots j_n} = \frac{\sum\limits_{g \in S_1 \times S_{n-1}} \left( \prod\limits_{i \in Des(g^{-1})} Q_{\{1,2,\ldots,i\}} \right) \cdot g}{\left(id - Q_{\{1,2\}}\right)\left(id - Q_{\{1,2,3\}}\right) \cdots \left(id - Q_{\{1,2,\ldots,n-1\}}\right)} Y_{l_1 l_2 j_3 \ldots j_n},$$

where $g \in S_1 \times S_{n-1}$ fixes the first index. The right-hand side of the last formula is composed in terms of $(n-1)!$ iterated $\boldsymbol{q}$-commutators $Y_{l_1 \xi}$ such that the first index $l_1 \in Q$ is fixed and the remaining $n-1$ indices $\xi = l_2 j_3 \ldots j_n$ vary. Let us denote

$$x^* := \frac{1}{1-x} \quad x^+ := \frac{x}{1-x}. \tag{30}$$

**Example 3.** *By applying formula (29) under the Q-cocycle condition, in what follows, we will show basic constants in $\mathcal{B}_Q$ for Card $Q = 2, 3, 4$.*

- *If $\sigma_{l_1 l_2} = 1$, then in the generic weight subspace $\mathcal{B}_{l_1 l_2}$ there is one nontrivial basic constant given by $C_{l_1 l_2} = Y_{l_1 l_2}$.*
- *Let $\sigma_{l_1 l_2 l_3} = 1$. Then in the generic weight subspace $\mathcal{B}_{l_1 l_2 l_3}$ there is one nontrivial basic constant, given by*

$$C_{l_1 l_2 l_3} = \frac{1}{1 - \sigma_{l_1 l_2}} Y_{l_1 l_2 l_3} + \frac{q_{l_3 l_2} \sigma_{l_1 l_3}}{1 - \sigma_{l_1 l_3}} Y_{l_1 l_3 l_2},$$

  *which can be rewritten by using (c.f. (30)) as*

$$C_{l_1 l_2 l_3} = \sigma^*_{l_1 l_2} Y_{l_1 l_2 l_3} + q_{l_3 l_2} \sigma^+_{l_1 l_3} Y_{l_1 l_3 l_2}.$$

- *If $\sigma_{l_1 l_2 l_3 l_4} = 1$, then in $\mathcal{B}_{l_1 l_2 l_3 l_4}$ there are two nontrivial basic constants*

$$\begin{aligned} C_{l_1 l_2 l_3 l_4} = {} & \sigma^*_{l_1 l_2} \sigma^*_{l_1 l_2 l_3} Y_{l_1 l_2 l_3 l_4} + q_{l_4 l_3} \sigma^*_{l_1 l_2} \sigma^+_{l_1 l_2 l_4} Y_{l_1 l_2 l_4 l_3} \\ & + q_{l_3 l_2} \sigma^+_{l_1 l_3} \sigma^*_{l_1 l_2 l_3} Y_{l_1 l_3 l_2 l_4} + q_{l_3 l_2} q_{l_4 l_2} \sigma^*_{l_1 l_3} \sigma^+_{l_1 l_3 l_4} Y_{l_1 l_3 l_4 l_2} \\ & + q_{l_4 l_2} q_{l_4 l_3} \sigma^+_{l_1 l_4} \sigma^*_{l_1 l_2 l_4} Y_{l_1 l_4 l_2 l_3} + q_{l_3 l_2} q_{l_4 l_2} q_{l_4 l_3} \sigma^+_{l_1 l_4} \sigma^+_{l_1 l_3 l_4} Y_{l_1 l_4 l_3 l_2}, \\ C_{l_1 l_2 l_4 l_3} = {} & q_{l_3 l_4} \sigma^*_{l_1 l_2} \sigma^+_{l_1 l_2 l_3} Y_{l_1 l_2 l_3 l_4} + \sigma^*_{l_1 l_2} \sigma^*_{l_1 l_2 l_4} Y_{l_1 l_2 l_4 l_3} \\ & + q_{l_3 l_2} q_{l_3 l_4} \sigma^+_{l_1 l_3} \sigma^*_{l_1 l_2 l_3} Y_{l_1 l_3 l_2 l_4} + q_{l_3 l_2} q_{l_3 l_4} q_{l_4 l_2} \sigma^+_{l_1 l_3} \sigma^+_{l_1 l_3 l_4} Y_{l_1 l_3 l_4 l_2} \\ & + q_{l_4 l_2} \sigma^+_{l_1 l_4} \sigma^*_{l_1 l_2 l_4} Y_{l_1 l_4 l_2 l_3} + q_{l_3 l_2} q_{l_4 l_2} \sigma^*_{l_1 l_4} \sigma^+_{l_1 l_3 l_4} Y_{l_1 l_4 l_3 l_2}. \end{aligned}$$

  *Here we have used the lexicographical ordering.*

**Example 4.** *If $\sigma_{l_1l_2l_3l_4l_5} = 1$, then in $\mathcal{B}_{l_1l_2l_3l_4l_5}$ there are six nontrivial basic constants, each consisting of 24 terms. Accordingly, here we will show only the first constant, where we use abbreviations (30) and the lexicographical ordering.*

$$
\begin{aligned}
C_{l_1l_2l_3l_4l_5} = {} & \sigma^*_{l_1l_2}\sigma^*_{l_1l_2l_3}\sigma^*_{l_1l_2l_3l_4}\, Y_{l_1l_2l_3l_4l_5} + q_{l_5l_4}\sigma^*_{l_1l_2}\sigma^*_{l_1l_2l_3}\sigma^+_{l_1l_2l_3l_5}\, Y_{l_1l_2l_3l_5l_4} \\
& + q_{l_4l_3}\sigma^*_{l_1l_2}\sigma^+_{l_1l_2l_4}\sigma^*_{l_1l_2l_3l_4}\, Y_{l_1l_2l_4l_3l_5} + q_{l_4l_3}q_{l_5l_3}\sigma^*_{l_1l_2}\sigma^*_{l_1l_2l_4}\sigma^+_{l_1l_2l_4l_5}\, Y_{l_1l_2l_4l_5l_3} \\
& + q_{l_5l_3}q_{l_5l_4}\sigma^*_{l_1l_2}\sigma^+_{l_1l_2l_5}\sigma^*_{l_1l_2l_3l_5}\, Y_{l_1l_2l_5l_3l_4} \\
& + q_{l_4l_3}q_{l_5l_3}q_{l_5l_4}\sigma^*_{l_1l_2}\sigma^+_{l_1l_2l_5}\sigma^+_{l_1l_2l_4l_5}\, Y_{l_1l_2l_5l_4l_3} \\
& + q_{l_3l_2}\sigma^+_{l_1l_3}\sigma^*_{l_1l_2l_3}\sigma^*_{l_1l_2l_3l_4}\, Y_{l_1l_3l_2l_4l_5} + q_{l_3l_2}q_{l_5l_4}\sigma^+_{l_1l_3}\sigma^*_{l_1l_2l_3}\sigma^+_{l_1l_2l_3l_5}\, Y_{l_1l_3l_2l_5l_4} \\
& + q_{l_3l_2}q_{l_4l_2}\sigma^*_{l_1l_3}\sigma^+_{l_1l_3l_4}\sigma^*_{l_1l_2l_3l_4}\, Y_{l_1l_3l_4l_2l_5} \\
& + q_{l_3l_2}q_{l_4l_2}q_{l_5l_2}\sigma^*_{l_1l_3}\sigma^*_{l_1l_3l_4}\sigma^+_{l_1l_3l_4l_5}\, Y_{l_1l_3l_4l_5l_2} \\
& + q_{l_3l_2}q_{l_5l_2}q_{l_5l_4}\sigma^*_{l_1l_3}\sigma^+_{l_1l_3l_5}\sigma^*_{l_1l_2l_3l_5}\, Y_{l_1l_3l_5l_2l_4} \\
& + q_{l_3l_2}q_{l_4l_2}q_{l_5l_2}q_{l_5l_4}\sigma^*_{l_1l_3}\sigma^+_{l_1l_3l_5}\sigma^+_{l_1l_3l_4l_5}\, Y_{l_1l_3l_5l_4l_2} \\
& + q_{l_4l_2}q_{l_4l_3}\sigma^+_{l_1l_4}\sigma^*_{l_1l_2l_4}\sigma^*_{l_1l_2l_3l_4}\, Y_{l_1l_4l_2l_3l_5} \\
& + q_{l_4l_2}q_{l_4l_3}q_{l_5l_3}\sigma^+_{l_1l_4}\sigma^*_{l_1l_2l_4}\sigma^+_{l_1l_2l_4l_5}\, Y_{l_1l_4l_2l_5l_3} \\
& + q_{l_3l_2}q_{l_4l_2}q_{l_4l_3}\sigma^+_{l_1l_4}\sigma^+_{l_1l_3l_4}\sigma^*_{l_1l_2l_3l_4}\, Y_{l_1l_4l_3l_2l_5} \\
& + q_{l_3l_2}q_{l_4l_2}q_{l_4l_3}q_{l_5l_2}\sigma^+_{l_1l_4}\sigma^*_{l_1l_3l_4}\sigma^+_{l_1l_3l_4l_5}\, Y_{l_1l_4l_3l_5l_2} \\
& + q_{l_4l_2}q_{l_4l_3}q_{l_5l_2}q_{l_5l_3}\sigma^*_{l_1l_4}\sigma^+_{l_1l_4l_5}\sigma^*_{l_1l_2l_4l_5}\, Y_{l_1l_4l_5l_2l_3} \\
& + q_{l_3l_2}q_{l_4l_2}q_{l_4l_3}q_{l_5l_2}q_{l_5l_3}\sigma^*_{l_1l_4}\sigma^+_{l_1l_4l_5}\sigma^+_{l_1l_3l_4l_5}\, Y_{l_1l_4l_5l_3l_2} \\
& + q_{l_5l_2}q_{l_5l_3}q_{l_5l_4}\sigma^+_{l_1l_5}\sigma^*_{l_1l_2l_5}\sigma^*_{l_1l_2l_3l_5}\, Y_{l_1l_5l_2l_3l_4} \\
& + q_{l_4l_3}q_{l_5l_2}q_{l_5l_3}q_{l_5l_4}\sigma^+_{l_1l_5}\sigma^*_{l_1l_2l_5}\sigma^+_{l_1l_2l_4l_5}\, Y_{l_1l_5l_2l_4l_3} \\
& + q_{l_3l_2}q_{l_5l_2}q_{l_5l_3}q_{l_5l_4}\sigma^+_{l_1l_5}\sigma^+_{l_1l_3l_5}\sigma^*_{l_1l_2l_3l_5}\, Y_{l_1l_5l_3l_2l_4} \\
& + q_{l_3l_2}q_{l_4l_2}q_{l_5l_2}q_{l_5l_3}q_{l_5l_4}\sigma^+_{l_1l_5}\sigma^*_{l_1l_3l_5}\sigma^+_{l_1l_3l_4l_5}\, Y_{l_1l_5l_3l_4l_2} \\
& + q_{l_4l_2}q_{l_4l_3}q_{l_5l_2}q_{l_5l_3}q_{l_5l_4}\sigma^+_{l_1l_5}\sigma^+_{l_1l_4l_5}\sigma^*_{l_1l_2l_4l_5}\, Y_{l_1l_5l_4l_2l_3} \\
& + q_{l_3l_2}q_{l_4l_2}q_{l_4l_3}q_{l_5l_2}q_{l_5l_3}q_{l_5l_4}\sigma^+_{l_1l_5}\sigma^+_{l_1l_4l_5}\sigma^+_{l_1l_3l_4l_5}\, Y_{l_1l_5l_4l_3l_2}.
\end{aligned}
$$

*The remaining five constants $C_{l_1l_2l_3l_5l_4}$, $C_{l_1l_2l_4l_3l_5}$, $C_{l_1l_2l_4l_5l_3}$, $C_{l_1l_2l_5l_3l_4}$, $C_{l_1l_2l_5l_4l_3}$ can be obtained from $C_{l_1l_2l_3l_4l_5}$ by replacing the same indices in each of its terms. In particular, the constant $C_{l_1l_2l_3l_5l_4}$ can be obtained if we take $\sigma^*_{l_1l_2}\sigma^*_{l_1l_2l_3}\sigma^*_{l_1l_2l_3l_5}Y_{l_1l_2l_3l_5l_4}$ as its first term and then we permute the indices in the remaining 23 terms, as given in the first constant. Similarly, we can obtain constant $C_{l_1l_2l_4l_3l_5}$, $C_{l_1l_2l_4l_5l_3}$, and $C_{l_1l_2l_5l_3l_4}$; $C_{l_1l_2l_5l_4l_3}$ by taking $\sigma^*_{l_1l_2}\sigma^*_{l_1l_2l_4}\sigma^*_{l_1l_2l_3l_4}\, Y_{l_1l_2l_4l_3l_5}$, $\sigma^*_{l_1l_2}\sigma^*_{l_1l_2l_4}\sigma^*_{l_1l_2l_4l_5}\, Y_{l_1l_2l_4l_5l_3}$, $\sigma^*_{l_1l_2}\sigma^*_{l_1l_2l_5}\sigma^*_{l_1l_2l_3l_5}\, Y_{l_1l_2l_5l_3l_4}$, and $\sigma^*_{l_1l_2}\sigma^*_{l_1l_2l_5}\sigma^*_{l_1l_2l_4l_5}\, Y_{l_1l_2l_5l_4l_3}$, respectively, as its first term.*

To recapitulate, in generic weight subspaces $\mathcal{B}_Q$ of the algebra $\mathcal{B}$ there exist nontrivial (basic) constants if and only if parameters $q_{ij}$'s are singular, i.e., if there is at least one $\sigma_T = 1$ such that $\det \mathbf{B}_Q = 0$, where $\mathbf{B}_Q$ denotes the matrix of the operator $\partial^Q$ with respect to the basis of $\mathcal{B}_Q$. Singular parameters that satisfy the $Q$-cocycle condition (or in Frønsdal's terminology *the top cocycle condition*, see [2, 3]) are of particular interest. Then under the $Q$-cocycle condition, by applying an explicit formula (29), one can compute basic constants in every generic weight subspace $\mathcal{B}_Q$. If Card $Q = n$, then in $\mathcal{B}_Q$ there exist $(n-2)!$ distinct basic constants, each consisting of $(n-1)!$ terms. Computation of basic constants in degenerated weight subspaces is more complicated because there is no single formula to describe the constants in all degenerate subspaces. By studying in detail the basic constants in generic as well as in degenerate subspaces of algebra $\mathcal{B}$, we have concluded that the basic constants in degenerated subspaces can be constructed from those in the generic case by a certain specialization procedure (c.f. [9]). In this way, we have solved explicitly, under the top cocycle condition, the fundamental problem of determining the constants in the algebra $\mathcal{B}$.

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